The Braess Paradox in Electric Power Systems

Seth Blumsack and Marija Ilić

***Abstract*-- Braess' Paradox describes a situation in which constructing a Wheatstone bridge causes or worsens congestion in a network, thus increasing the cost to users. While this behavior has been extensively studied in other network industries, its implications for power systems have not. The steady-state conditions under which Braess' Paradox holds in a simple symmetric unbalanced Wheatstone network are derived. While these conditions are more stringent than in other types of networks, Wheatstone structures are quite common in actual power networks and can sometimes provide reliability benefits to the system. The price paid for this reliability benefit is increased congestion throughout the network; eliminating congestion in a Wheatstone network also eliminates the reliability benefit of the meshed network structure.**

***Index Terms*—Braess Paradox, Wheatstone Network, DC Power Flow, Locational Marginal Price**

I. Nomenclature

NL = Number of lines in the network
NB = Number of buses in the network
S_{ij} = Transmission line connecting buses i and j
B_{ij} = Susceptance of the link connecting buses i and j
Y_{ij} = Complex admittance of the link connecting buses i and j
X_{ij} = Reactance of the link connecting buses i and j
θ_i = Phase angle at the ith bus
P_i = Net real power injection at the ith bus; positive for net generation and negative for net withdrawal
P_{Li} = Real power demand at the ith bus
P_{Gi} = Real power demand at the ith bus
δ_{ij} = Phase angle difference between buses i and j
F_{ij} = Real power flow between buses i and j
π_i = Nodal price at bus i
μ_{ij} = Shadow price of transmission between buses i and j
C_i = Total cost function at the ith bus.
MC_i = Marginal cost function at the ith bus, equal to dC_i/dP_i.
d_i = Number of buses connected to bus i (that is, the degree of bus i).
$\mathbf{B}$ = ($NB \times NB$) system susceptance matrix
$\mathbf{B}^{\mathbf{diag}}$ = ($NL \times NL$) diagonal matrix of line susceptances
$\mathbf{X}$ = ($NB \times NB$) system complex reactance matrix
$\mathbf{Y}$ = ($NB \times NB$) system complex admittance matrix
$\mathbf{Z}$ = ($NB \times NB$) system complex resistance matrix
$\mathbf{N}$ = ($NB \times NB$) node-node adjacency matrix
$\mathbf{A}$ = ($NB \times NL$) system node-line adjacency matrix
$\mathbf{P}$ = ($NB \times 1$) vector of bus injections
$\mathbf{F}$ = ($NL \times 1$) vector of line flows
$\boldsymbol{\theta}$ = ($NB \times 1$) vector of bus angles
$\boldsymbol{\delta}$ = ($NL \times 1$) vector of bus angle differences
$\mathbf{p}_k$ = ($NB \times 1$) vector of net bus injections by the kth grid participant
$\boldsymbol{\pi}$ = ($NB \times 1$) vector of nodal prices

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II. Introduction: Wheatstone Networks and the Braess Paradox

The Wheatstone network describes a graph consisting of four nodes, with four corresponding edges on the boundary creating a diamond or circular shape. A fifth edge connects two of the nodes across the interior of the network, thus splitting the network into two triangular (or semicircular) subsystems. This fifth edge is aptly named the "Wheatstone bridge." Although the network is named for Charles Wheatstone, who was the first to publish the network topology in 1843, the network design was apparently the work of Samuel Christie some ten years earlier [1].

The original motivation for the Wheatstone network was the precise measurement of resistances, as shown in Figure 1. In the network, resistances R_1, R_2, and R_3 are known to very high precision, and R_2 is adjustable. The problem is to measure R_x with similar precision. The voltage V across the bridge is equal to:

$$V = \frac{R_2}{R_1 + R_2} V_s + \frac{R_x}{R_3 + R_x} V_s \qquad (1)$$

where V_s is the voltage source. Assuming that $V_s \neq 0$, then the voltage drop across the bridge will be zero at the value of R_x where $R_2 / R_1 = R_x / R_3$. If this condition is satisfied, then the Wheatstone network is said to be *balanced*. If this condition is not satisfied, then there will be a voltage drop across the bridge and the network is said to be *unbalanced*.

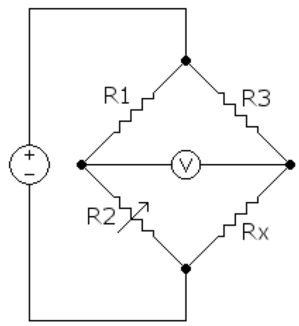

Figure 1. Wheatstone circuit example

As a fairly general topology, Wheatstone networks have arisen as structures of interest in other network situations such as traffic, pipes, and computer networks. Much of the attention paid to Wheatstone structures has centered around the network's seemingly paradoxical behavior. Under certain conditions, connecting a Wheatstone bridge to a formerly parallel network (or, in the context of the circuit in Figure 1, adjusting the boundary resistances so that the network is unbalanced) can actually increase the total user cost of the network. First studied by [2] in the context of traffic networks, this behavior has come to be known as Braess's Paradox.

The exact meaning of the "user cost" of the network has various interpretations depending on the network of interest. In Braess's original example, and in [3], the user cost of highways is the time it takes motorists to reach their final destination. An increase in the user cost, therefore, corresponds to wasted time and irritation from sitting in larger traffic jams. Costs incurred through internet routing networks, as in [3] – [5], arise through increased latency and possibly lost information, as in [6]. Even in circuits, "user cost" can be interpreted as the voltage drop across the circuit as a whole. [7] describe an example in which the addition of a Wheatstone bridge lowers the voltage drop across the network (assuming the network is unbalanced to begin with); thus the "cost" incurred by the Wheatstone bridge is reduced voltage over the circuit as a whole. Braess's Paradox suggests that user costs may increase for reasons independent of the amount of traffic on the network. The network itself, and not its users, may be the ultimate problem, and managing flows or disconnecting certain network links may actually serve to decrease congestion costs for all users.[1]

[8] has studied whether Braess's Paradox is unique to the Wheatstone network. Using a result from [9] that every network topology can be decomposed into purely series-parallel subnetworks and Wheatstone subnetworks, Milchtaich concludes that (apart from uninteresting situations such as simple bottlenecks) the paradoxical behavior cannot occur outside the Wheatstone structure. Thus, observation of the paradox serves as proof of an embedded Wheatstone subnetwork. [4] – [6], [8], and [10] offer the following technical and policy implications of Braess's Paradox:

1. Braess's Paradox occurs in any network that is not purely series-parallel;
2. Local network upgrades (that is, upgrading only congested links) will not resolve Braess's Paradox. Upgrades must be made throughout the system in order to reduce the user cost of the network;
3. System upgrades should focus on connecting "sources" as close as possible to "sinks."

Underlying the policy recommendations is the assumption that flow networks all behave similarly, at least on the surface. While there are good analogies between the behavior in electric power networks and other networks, the analogies are ultimately flawed. Kirchoff's Laws do not hold in other networks.[2] In traffic and some internet systems, routing is determined by user preference rather than by physical laws (e.g., current flows follow Ohm's Law), although installation of FACTS devices could change this for those paths outfitted with devices. Congestion costs in systems with nodal pricing are discontinuous, while in other networks the cost of additional traffic can be described as a continuous function of current traffic. Despite these differences, power networks do exhibit some of the behavior described in other networks; in particular, Braess's Paradox can hold in simple systems or in subsets of more complex systems.

III. A Simple Wheatstone Test System

The four-bus test system used in this discussion is shown in Figure 3.2. There is one generator located at bus 1, an additional generator at bus 4, and one load at bus 4. Buses 2 and 3 are merely tie-points; power is neither injected at nor withdrawn from these two buses. From the analogy to Figure 3.1, the Wheatstone bridge is the link connecting buses 2 and 3. The test system is assumed to be symmetric, in the sense that $B_{12} = B_{34}$ and $B_{13} = B_{24}$. The susceptance of the Wheatstone bridge is given by B_{23} and will be a variable of interest in the discussion that follows. The symmetry assumption implies, among other things, that in the DC load flow, $F_{12} = F_{34}$ and $F_{13} = F_{24}$.[3]

[1] Viewing network traffic as a routing game, Braess's Paradox does not seem all that paradoxical. Each user choosing a network path to minimize their private costs easily lends itself to coordination failures such as the Prisoner's Dilemma. All users would benefit through coordination and cooperation, but no individual user has the incentive to initiate (or perhaps even sustain) this coordination.

[2] In the case of laminar flow, a version of Kirchoff's Law does hold in piping networks. However, real flows through pipes are almost a combination of turbulent and laminar flow.

[3] In the DC load flow, the current magnitude is identical to the admittance (since the voltage magnitudes are all set to 1 per-unit). The symmetry of the admittance matrix implies that the two cut sets in the system (buses 1, 2, and 3;

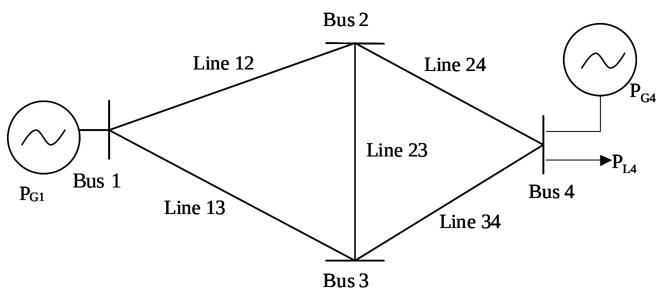

Figure 2: The Wheatstone network. The network is defined to be symmetric if the resistances are equal on lines S_{12} and S_{34}, and if the resistances are equal on lines S_{13} and S_{24}.

The following definitions will help solidify concepts:

Definition 1: A four-node network is said to be a Wheatstone network if its topology is the same as that in Figure 2.

Definition 2: A four-node network is said to be a symmetric Wheatstone network if it is a Wheatstone network, and if the susceptance conditions $B_{12} = B_{34}$ and $B_{13} = B_{24}$ hold.

Definition 3: A four-node network is said to be a symmetric unbalanced Wheatstone network if it is a symmetric Wheatstone network, and the magnitude of the flow across link S_{23} is nonzero.

Although the Wheatstone network shown in Figure 2 is simplistic, the Wheatstone structure is actually quite common in actual systems. [11] provides a graph-theoretic search algorithm for detecting embedded Wheatstone structures. The algorithm follows in part from a theorem proved separately by [8] and [9] that any non-radial network can be decomposed into series-parallel components and components that contain an embedded Wheatstone sub-network.

IV. Conditions for the Paradox to Hold

Of particular interest here is how the addition of the Wheatstone bridge affects the flows on the boundary lines relative to the "base case" with no Wheatstone bridge. Here we are implicitly assuming that the generator injections, load withdrawals, and line susceptances are such that there is no congestion in the system prior to the addition of the bridge. If we take lines S_{12} and S_{34} and combine them in series to form a line with equivalent susceptance B_a, and we combine lines S_{13} and S_{24} in a similar fashion to construct a line with equivalent susceptance B_b, the network is free of congestion if and only if:

$$\frac{B_k}{(B_a + B_b)} P_{G1} < F_k^{\max} \text{, for k = \{a, b\}}^{4} \tag{2}$$

To derive an explicit expression for the new network flows following the addition of the Wheatstone bridge, we will use the method derived by [12] and [13], which compares steady-state line flows in the network before and after the network modification. Such modifications are represented as changes in the susceptance matrix **B**. Although the method in [12] was originally designed to model the effects of contingencies (so that the susceptance change in a given line, ΔB_k, is simply equal to $-B_k$) it is easily adaptable to the construction of a new line.

Since we are using the DC load flow approximation, the admittance matrix consists solely of susceptances:

$$B_{ij} = \begin{cases} -\dfrac{1}{X_{ij}} & i \neq j \\ \displaystyle\sum_{i=0, i\neq j} \dfrac{1}{X_{ij}} & i = j \\ 0 & X_{ij} = 0 \end{cases} . \tag{3}$$

We start with the DC model: [5]

$$\mathbf{P} = \mathbf{B}\,\boldsymbol{\theta} \tag{4}$$

Note that (4) represents the system prior to the addition of the Wheatstone bridge. After the Wheatstone bridge is connected, the load flow equations become:

$$\mathbf{P} = (\mathbf{B} + \mathbf{A}'\,\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}}\mathbf{A})\,\boldsymbol{\theta}^{new}, \tag{4'}$$

where $\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}}$ is a diagonal matrix of changes to the line susceptances. $\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}}$ has dimensionality ($NL \times NL$). Solving (4') for the vector of phase angles yields:

$$\boldsymbol{\theta}^{new} = (\mathbf{B} + \mathbf{A}'\,\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}}\mathbf{A})^{-1}\,\mathbf{P}. \tag{5}$$

Using the Sherman-Morrison-Woodbury matrix inversion lemma and substituting (4), we get:

$$\boldsymbol{\theta}^{new} = (\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}^{-1})\mathbf{B}\boldsymbol{\theta}^{old}. \tag{6}$$

Distributing terms,

buses 2, 3, and 4) are also symmetric, and Kirchoff's Current Law must hold for each cut set.

[4] A similar condition also holds in AC networks, but uses the complex admittance instead of the susceptance.

[5] We could also start with the distribution-factor representation of the DC model, $\mathbf{F} = \mathbf{A}'\mathbf{B}^{\mathbf{diag}}\mathbf{A}\boldsymbol{\theta}$, where $\mathbf{B}^{\mathbf{diag}}$ is a ($NL \times NL$) diagonal matrix of line susceptances. However, starting with the injection equations will allow us to write the new flows in the form $\mathbf{F}^{new} = \mathbf{F}^{old}$ + {adjustment factor}.

$$\boldsymbol{\theta}^{new} = \boldsymbol{\theta}^{old} - \mathbf{B}^{-1}\mathbf{A}(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}, \qquad (7)$$

where $\boldsymbol{\delta}$ is the ($NL \times 1$) vector of phase angle differences.

Following the network modification, the DC flow equations can be written

$$\mathbf{F}^{new} = (\mathbf{A}'(\mathbf{B}^{\mathbf{diag}} + \boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\mathbf{A})\boldsymbol{\theta}^{new}. \qquad (8)$$

Inserting (7) into (8) and distributing terms yields:

$$\begin{aligned}\mathbf{F}^{new} &= \mathbf{A}'(\mathbf{B}^{\mathbf{diag}})\mathbf{A}\boldsymbol{\theta}^{old} - \mathbf{A}'(\mathbf{B}^{\mathbf{diag}})\mathbf{A}\mathbf{B}^{-1}\mathbf{A} \\ &\quad \times(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old} + \mathbf{A}'(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\mathbf{A}\boldsymbol{\theta}^{old} \\ &\quad - \mathbf{A}'(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\mathbf{A}\mathbf{B}^{-1}\mathbf{A}(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old} \\ &= \mathbf{F}^{old} + \mathbf{A}'(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\boldsymbol{\delta}^{old} - (\mathbf{A}'(\mathbf{B}^{\mathbf{diag}} + \boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\mathbf{A}) \\ &\quad \times\mathbf{B}^{-1}\mathbf{A}(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}.\end{aligned}$$

The adjustment is:

$$\begin{aligned}&\mathbf{A}'(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\boldsymbol{\delta}^{old} - (\mathbf{A}'(\mathbf{B}^{\mathbf{diag}} + \boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}})\mathbf{A}) \\ &\quad \times\mathbf{B}^{-1}\mathbf{A}(\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}.\end{aligned}$$

In the special case where the susceptance of only one line changes (as is the case with the Wheatstone bridge example), we can replace the $\boldsymbol{\Delta}\mathbf{B}^{\mathbf{diag}}$ matrix with a scalar ΔB_k (where k indicates the line whose susceptance has been altered), and we can replace the incidence matrix with its kth column, denoted $\mathbf{A_k}$. [12] show that the equivalent of (6) for a single line is:

$$\begin{aligned}F_l^{new} &= B_l\mathbf{A'_l}\left[\boldsymbol{\theta}^{old} - (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{B}^{-1}\mathbf{A_k}\delta_k^{old}\right] \\ &= F_l^{old} + (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{A'_l}\mathbf{B}^{-1}\mathbf{A_k}B_l\delta_k^{old} \\ &= F_l^{old} + b_k^{-1}\mathbf{A'_l}\mathbf{B}^{-1}\mathbf{A_k}B_l\delta_k^{old}.\end{aligned} \qquad (9)$$

In the special case where $l = k$, (9) becomes:

$$F_l^{new} = \left(F_l^{old} - \Delta B_l\delta_l^{old}\right)(1 - b_l^{-1}\mathbf{A'_l}\mathbf{B}^{-1}\mathbf{A_l}). \qquad (10)$$

The sensitivity of flows in the Wheatstone network to the bridge susceptance, as described by equations (6) and (9), are shown in Figure 3, assuming that the boundary links have susceptances $B_{12} = B_{34} = 30$ per-unit (p.u.) and $B_{13} = B_{24} = 15$ p.u.. The flow limit on each line is assumed to be 55 MW; hence the flow on lines S_{12} and S_{34} plateau at this upper limit. Note also that the flow on the Wheatstone bridge and the flow on the remaining two boundary links appear to converge. One implication of this behavior (which will be of importance in the steady-state analysis of congestion in the network) is that if the Wheatstone network is symmetric in the sense of Definition 2, then a maximum of two lines can be congested at a given time.

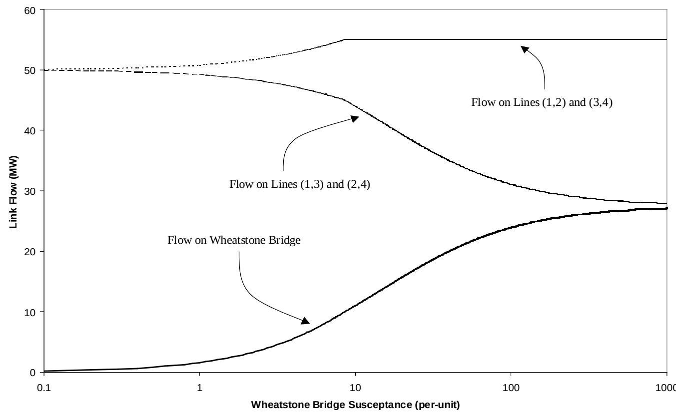

Figure 3: Senstivity of flows in the Wheatstone network to changes in the susceptance of the Wheatstone bridge. The *x*-axis has a logarithmic scale. The flows on lines S_{12} and S_{34} hit the capacity constraint when the susceptance of the bridge reaches 8.6 per-unit.

Of particular interest here are the conditions under which any of the boundary links will become congested with the addition of the Wheatstone bridge (congestion occurs when the generator at bus 4 either does not exist or is not turned on). Without loss of generality, assume that $B_{12} > B_{13}$. Thus, once the bridge is added, more power will flow over link S_{12} than over link S_{23}.[6] We are interested in the conditions under which link S_{12} will become congested. The symmetry assumption implies that a similar condition will hold for link S_{34} to become congested.

Link S_{12} becomes congested if $F_{12}^{new} \geq F_{12}^{\max}$. An equivalent condition is:

$$\begin{aligned}&F_{12}^{old} + b_{23}^{-1}\mathbf{A'}_{12}\mathbf{B}^{-1}\mathbf{A}_{23}B_{12}\delta_{23}^{old} \geq F_{12}^{\max} \\ &\Rightarrow \Delta B_{23}^{-1} \geq \frac{\mathbf{A'}_{12}\mathbf{B}^{-1}\mathbf{A}_{23}B_{12}\delta_{23}^{old}}{F_{12}^{\max} - F_{12}^{old}} - \mathbf{A}_{23}'\mathbf{B}^{-1}\mathbf{A}_{23}.\end{aligned} \qquad (11)$$

This "feasible region" for the susceptance of the Wheatstone bridge is shown in Figure 4 for the configuration where $B_{12} = B_{34} = 30$ p.u., $B_{13} = B_{24} = 15$ p.u., $F_{12}^{\max} = 55$ MW, and $P_{G1} = P_{L4} = 100$ MW. From the DC power flow on this network we get $F_{12}^{old} = 50$ MW and $\delta_{23}^{old} = 1.5$ degrees.

Figure 4 shows that unlike other networks such as internet communications [8], the existence of a Wheatstone configuration is not in itself sufficient for the network to exhibit Braess's Paradox. Equation (11) thus provides two

[6] The symmetry assumption implies that equal amounts of power will flow over both paths in the absence of the Wheatstone bridge.

"rules of thumb" for transmission planning. First, it shows conditions under which parallel networks can become more interconnected without causing congestion in the modified system. Second, it provides a condition on the line limit $F_{12}^{\max}$ under which a conversion of a parallel network to a Wheatstone network would be socially beneficial.

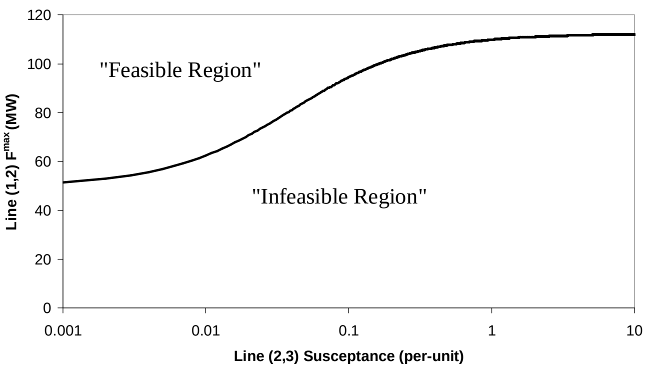

Figure 4: Whether the Wheatstone bridge causes congestion on line S_{12} (and also congestion on line S_{34} in the case of a symmetric Wheatstone network) depends on the susceptance of the Wheatstone bridge and the stability limit of line S_{12}. The "feasible region" above the line indicates susceptance-stability limit combinations that will not result in congestion on the network. The "infeasible region" below the line represents susceptance-stability limit combinations for which the network will become congested. Note that the x-axis has a logarithmic scale.

Why would the Wheatstone bridge ever be installed in a parallel system? One answer is that in some circumstances the bridge may provide reliability benefits. In the parameterization of the network represented in Figure 4, suppose that the load at bus 4 represents a customer with a high demand for reliability, that link S_{24} had an abnormally high outage rate, and that the generator at bus 4 did not exist. In this case, if the remainder of the links had sufficiently small stability limits, the network would not meet (N – 1) reliability criteria with respect to transmission outages. With the addition of the Wheatstone bridge, the reliability criteria might be satisfied, but at the cost of a certain amount of congestion during those times in which link S_{24} was operating normally.[7] Thus, in the Wheatstone network, a tradeoff likely exists between the cost of congestion and the benefit of reliability [14].

V. Optimal Power Flow on the Wheatstone Network

Assume that the cost curves for the two generators in the symmetric unbalanced Wheatstone network are quadratic with the following parameterization:

$$C(P_{G1}) = 200 + 10.3P_{G1} + 0.008P_{G1}^2 \qquad (12)$$

$$C(P_{G4}) = 300 + 50P_{G4} + 0.1P_{G4}^2. \qquad (13)$$

Also assume that every line in the network has a stability limit of 55 MW. Prior to the addition of the Wheatstone bridge, the DC optimal power flow results show that 50 MW flows on each line towards bus 4; thus there is no congestion in the system. The nodal prices are all equal to \$12.11/MWh, and the total system cost is \$1,620 per hour.

Following the addition of the Wheatstone bridge, lines S_{12} and S_{34} become congested, as shown in Figure 5. The total system cost rises to \$1,945 per hour as the economic dispatch is forced to run the expensive generator located at bus 4. Among other things, this implies that the value of reliability to the load is at least \$325 per hour that link S_{24} remains functional.

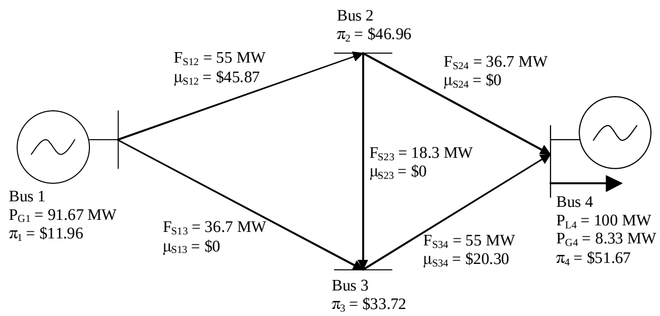

Figure 5: The addition of the Wheatstone bridge connecting buses 2 and 3 causes congestion along links S_{12} and S_{34}. The total system cost rises from \$1,620 per hour without the Wheatstone bridge to \$1,945 per hour with the bridge.

VI. Implications of the Braess Paradox

Equations (9) and (11) have a number of implications for grid management and investment. Some of these implications mirror results described in Section 1 for other types of networks, while some appear to be unique to electric power networks.

Result 1: A symmetric Wheatstone network is balanced (that is, F_{23} = 0) if and only if $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Before proving the result, we note that under the DC power flow approximation, F_{23} = 0 is equivalent to $\theta_2 = \theta_3$, so another way of stating the result is that $\theta_2 = \theta_3$ if and only if $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Proof of Result 3.1: The first part of the proof is to show that $\theta_2 = \theta_3 \Rightarrow \frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

[7] Ideally, controllers would be installed on the system to prevent power from flowing over the Wheatstone bridge except during contingencies on link S_{24}. The congestion cost thus represents the value of such a controller to the system.

Suppose that $\theta_2 = \theta_3$, and thus F_{23} = 0. Because all the power is flowing towards bus 4, and since there are no losses, this condition is equivalent to stating that $F_{12} = F_{24}$ and $F_{13} = F_{34}$. From the DC load flow equations, we see that

$$F_{12} = F_{24} \Rightarrow \frac{1}{X_{12}}(\theta_1 - \theta_2) = \frac{1}{X_{24}}(\theta_2 - \theta_4)$$
$$\Rightarrow \frac{X_{24}}{X_{12}} = \frac{(\theta_2 - \theta_4)}{(\theta_1 - \theta_2)},$$

and

$$F_{13} = F_{34} \Rightarrow \frac{1}{X_{13}}(\theta_1 - \theta_3) = \frac{1}{X_{34}}(\theta_3 - \theta_4)$$
$$\Rightarrow \frac{X_{34}}{X_{13}} = \frac{(\theta_3 - \theta_4)}{(\theta_1 - \theta_3)}.$$

Since $\theta_2 = \theta_3$, we see that $\frac{X_{24}}{X_{12}} = \frac{(\theta_2 - \theta_4)}{(\theta_1 - \theta_2)}$ and $\frac{X_{34}}{X_{13}} = \frac{(\theta_2 - \theta_4)}{(\theta_1 - \theta_2)}$; thus, $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

The second part of the proof is to show that $\theta_2 = \theta_3 \Leftarrow \frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Suppose that $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$. From the DC load flow equations, we see that

$$\frac{X_{24}}{X_{12}} = \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)}$$

and

$$\frac{X_{34}}{X_{13}} = \frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)}.$$

Since $\frac{X_{24}}{X_{12}} = \frac{X_{34}}{X_{13}}$, it must be true that $\frac{X_{24}}{X_{12}} \div \frac{X_{34}}{X_{13}} = 1$, and thus it must also be true that:

$$\frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)} \div \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)} = 1. \quad (14)$$

By the symmetry of the network, we have $F_{12} = F_{34}$ and $F_{13} = F_{24}$. Thus,

$$\frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)} \div \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)} = \frac{F_{13}^2}{F_{34}^2} \frac{(\theta_3 - \theta_4)(\theta_1 - \theta_2)}{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}. \quad (15)$$

For (14) and (15) to hold, it must be true that $\theta_2 = \theta_3$ and thus, $F_{13} = F_{34}$.

Result 2: In a symmetric unbalanced Wheatstone network, suppose that links S_{12} and S_{34} are congested following the construction of the Wheatstone bridge, as in Figure 5. The congestion will be relieved, and the total system cost will decline, only to the extent that upgrades are performed on both lines.

Proof of Result 3.2: Using (9) and (11), for the case in which the Wheatstone bridge causes congestion:

$$F_{12}^{\max} - F_{12}^{old} \le F_{12}^{new} - F_{12}^{old} = B_{23}^{-1}\mathbf{A'}_{12}\,\mathbf{B^{-1}}\mathbf{A}_{23}B_{12}\delta_{23}^{old} \quad (16)$$

and

$$F_{34}^{\max} - F_{34}^{old} \le F_{34}^{new} - F_{34}^{old} = B_{23}^{-1}\mathbf{A'}_{34}\,\mathbf{B^{-1}}\mathbf{A}_{23}B_{34}\delta_{23}^{old}. \quad (17)$$

Since $B_{12} = B_{34}$, we get that $F_{34}^{\max} = F_{12}^{\max} \le F_{12}^{new} = F_{34}^{new}$. Increasing only $F_{12}^{\max}$ will not change this relationship since $F_{34}^{\max} \le F_{12}^{new} = F_{34}^{new}$ must still hold. A similar argument holds for increasing only $F_{34}^{\max}$.

If we increase the stability limit of both lines by the same amount, to $F^{max,new}$, then the flows along lines S_{12} and S_{34} can simultaneously increase while maintaining the relationship $F_{34}^{\max,new} = F_{12}^{\max,new} \le F_{12}^{new} = F_{34}^{new}$. A corollary to this result is that if the total cost (capital cost plus congestion cost) of the Wheatstone bridge exceeds the cost of upgrading the boundary links to the point where a failure on one link would not violate reliability criteria, then the Wheatstone bridge provides no net social benefit and should not be built.

Result 3: In the symmetric unbalanced Wheatstone network of Figure 5, the Lagrange multipliers on the congested lines are not necessarily unique. However, the sum of the multipliers on the two congested lines is unique. Further, the non-uniqueness is solely a function of the network topology, and is independent of the actual number of binding constraints.

Proof of Result 3.3: Another way of stating Result 3 is that in the symmetric unbalanced Wheatstone network of Figure 5, the DC line-flow constraints are not linearly independent. Thus, proving the result amounts to demonstrating that the constraint set violates the constraint qualification condition of the Kuhn-Tucker theorem [15]. We will show this using the linearized DC optimal power flow.

Define $\mathbf{H} = \mathbf{A'B^{diag}A}$, and also define $\mathbf{c}$ to be a vector of generator marginal costs. In the linearized DC optimal power flow, $\mathbf{c}$ contains constants (i.e., all generators have constant marginal costs), and the power flow problem can be written as the following linear program:

$$\min \mathbf{c'P} \tag{18}$$

such that:

$$\begin{aligned} &\mathbf{P} = \mathbf{B\theta} \\ &\mathbf{F} = \mathbf{H\theta} \\ &\mathbf{F} \le \mathbf{F^{max}}. \end{aligned} \tag{19}$$

Rewriting to include the equality constraints, the optimal power flow problem is:

$$\min \mathbf{c'B\theta} \tag{18'}$$

such that:

$$\mathbf{H\theta} \le \mathbf{F^{max}}. \tag{19'}$$

The constraint qualification condition says that at the optimal solution $\boldsymbol{\theta}^*$, the matrix derivative of the constraint set must have rank equal to the number of binding constraints. That is, if we define $g(\boldsymbol{\theta}) = \mathbf{H\theta} - \mathbf{F^{max}}$, and define $Dg(\boldsymbol{\theta})$ as the matrix whose (i, j)th entry is $\frac{\partial g_i}{\partial \theta_j}(\theta)$, then the constraint qualification condition is that the rank of $Dg(\boldsymbol{\theta}^*)$ be equal to the number of constraints that hold with equality.

In the case of the DC optimal power flow problem, we see that $Dg(\boldsymbol{\theta}^*)$ is a linear function of $\boldsymbol{\theta}^*$ and is independent of $\boldsymbol{\theta}^*$. Specifically,

$$Dg(\boldsymbol{\theta}^*) = \mathbf{H'}.$$

Thus, the constraint qualification condition for the DC power flow problem is that rank($\mathbf{H}$) be equal to the number of binding constraints.

The matrix $\mathbf{H}$ in the Wheatstone network is given by:

$$\mathbf{H'} = \begin{bmatrix} B_{12} & B_{13} & & & \\ -B_{12} & & B_{24} & & B_{23} \\ & -B_{13} & & B_{34} & -B_{23} \\ & & -B_{24} & -B_{34} & \end{bmatrix}.$$

Since we assume that the Wheatstone network is symmetric, it is easy to see that any column of $\mathbf{H}$ is a linear combination of the other columns. Thus, rank($\mathbf{H}$) = 3, meaning that only three of the five network constraints are linearly independent.[8]

Due to the symmetry and parameters of the problem, we can see that, at most, two constraints can be binding at the optimal solution. Since the combined capacity of links S_{12} and S_{13} is greater than the production capability of the generator at bus 1, these two lines cannot be simultaneously congested. Thus, the number of active constraints is less than the rank of $\mathbf{H}$', and the constraint qualification condition for the Kuhn-Tucker theorem is not satisfied. The dual variables thus may not be unique.

In this special case, it is easy to see that the constraint qualification condition (for the inequality constraints) is violated. In larger and more complex networks, it may be more difficult; verifying the constraint qualification condition would require checking all possible combinations of binding constraints. In real systems, this combinatorial problem could get prohibitively large, particularly for calculations requiring fast solutions (such as calculating nodal prices).

Another way to verify the constraint qualification condition is to consider the dual of the DC optimal power flow problem. If we let $\boldsymbol{\mu}$ be the vector of dual variables associated with the network line flow constraints in equation (19'), the dual problem can be written:

$$\max - \boldsymbol{\mu}'\mathbf{F^{max}} \tag{20}$$

such that

$$\mathbf{H'}\boldsymbol{\mu} + \mathbf{Bc} = \mathbf{0}. \tag{21}$$

In the dual problem the matrix derivative of the dual constraints, evaluated at the optimal (dual) solution $\boldsymbol{\mu}^*$, is $\mathbf{H}$. Since the rank of $\mathbf{H}$' is equal to the rank of $\mathbf{H}$, we again find that the rank of the derivative of the constraints is 3. In order to satisfy the constraint qualification condition for equality constraints, the rank of the derivative of the constraint matrix must be equal to the number of constraints.[9] In the dual formulation of the DC optimal power flow problem, there are four constraints. Thus, the dual problem does not satisfy the constraint qualification condition.

[8] Ordinarily, we would expect the rank of $\mathbf{H}$ to be equal to $n - 1$, to account for the reference bus. The result that $\mathbf{H}$ is not of full rank is independent of the choice of reference bus. .

[9] This is really just the same condition as in the primal formulation, except that in the dual problem all constraints are assumed to hold with equality.

The rank of **H** provides a simple test for uniqueness of the nodal prices. The nodal prices are unique if the rank of **H** is equal to the number of buses in the network. Any system containing a symmetric Wheatstone sub-network will fail this rank test, since in that sub-network there will be linearly dependent line constraints. The Wheatstone network need not be unbalanced for the network to fail the rank test. Thus, the mere existence of a symmetric Wheatstone network is enough to result in a degenerate solution to the DC optimal power flow problem.

Result 4: Suppose that $F_l^{\max} = F_l$ for some line l in a symmetric unbalanced Wheatstone network in which $\delta_l^{old} > 0$. Increasing B_l for any l while simultaneously increasing $F_l^{\max}$ will increase the power flow on that line, even if the Wheatstone network is unbalanced.

Proof of Result 4: We are most interested in those situations in which line l is not the Wheatstone bridge, but the result will hold either way. The proof is a direct application of the formulae [12], using equation (10'):

$$F_l^{new} = \left(F_l^{old} - \Delta B_l \delta_l^{old}\right)\left(1 - b_l^{-1}\mathbf{A'_l}\,\mathbf{B^{-1}A_l}\right). \qquad (10')$$

Calculate the sensitivity:

$$\begin{aligned}\frac{\partial F_l^{new}}{\partial \Delta B_l} &= \frac{\partial}{\partial \Delta B_l}\left(\Delta B_l \delta_l^{old} b_l^{-1}\mathbf{A'_l}\,\mathbf{B^{-1}A_l}\right)\\ &= \frac{\partial}{\partial \Delta B_l}\left(\frac{\Delta B_l \delta_l^{old}}{\left(\Delta B_l \mathbf{A'_l}\,\mathbf{B^{-1}A_l}\right)^{-1} + 1}\right)\\ &= \frac{\delta_l^{old}\left(\mathbf{A'_l}\,\mathbf{B^{-1}A_l}\right)^2}{\left(\Delta B_l + \mathbf{A'_l}\,\mathbf{B^{-1}A_l}\right)^2}.\end{aligned} \qquad (22)$$

To show that (22) is greater than zero, we note that $\delta_l^{old} >$ 0 and $\Delta B_l > 0$ by assumption. For the power flow to have a solution, **B** must be positive definite, implying that $\mathbf{B^{-1}}$ is also positive definite and $\mathbf{A'_l B^{-1} A_l} > 0$.

Result 5: In a symmetric unbalanced Wheatstone network with fixed susceptances on the boundary links, the stability limits on the boundary links required to avoid congestion are strictly increasing in the susceptance of the Wheatstone bridge. Further, there is an upper bound on the boundary-link flow F_{12}^{crit} once the Wheatstone bridge is added.

Proof of Result 5: The first part of the claim, that the $F_{12}^{\max}$ required to keep the Wheatstone network from becoming congested is strictly increasing in ΔB_{23}, follows from (19). To prove the second part of the claim, we examine F_{12}^{new} in the limit as ΔB_{23} becomes arbitrarily large:

$$\begin{aligned}F_{12}^{crit} &= \lim_{\Delta B_{23}\to\infty} F_{12}^{new} = \lim_{\Delta B_{23}\to\infty} F_{12}^{old} + \frac{\mathbf{A'}_{12}\,\mathbf{B^{-1}A}_{23} B_{12}\delta_{23}^{old}}{\Delta B_{23}^{-1} + \mathbf{A}_{23}'\mathbf{B^{-1}A}_{23}}\\ &= F_{12}^{old} + \frac{\mathbf{A'}_{12}\,\mathbf{B^{-1}A}_{23} B_{12}\delta_{23}^{old}}{\mathbf{A}_{23}'\mathbf{B^{-1}A}_{23}}.\end{aligned}$$

VII. Discussion

Result 1 establishes that the conditions under which the Braess Paradox will occur in a power network are more strict than for other types of networks. Thus, the theorem of [8], that an embedded Wheatstone sub-network is a necessary and sufficient condition for the network to exhibit the Braess Paradox, does not apply in power networks. Results 2 through 5 have more interesting implications for pricing, grid management, and investment in the electric transmission network.

Result 2 mirrors the results of [8] and [5], [10] for internet routing networks. It says that in a symmetric unbalanced Wheatstone network, congestion will occur on two of the four boundary lines (if any congestion occurs at all), and that network upgrades amounting to a capacity expansion on only one of those lines will not alter the dispatch. Both congested lines must be upgraded before the dispatch will be altered and the marginal and total system costs will be lowered. In other words, relieving congestion in Wheatstone configurations requires more than simply upgrading the most congested line. Congestion may occur on two of the boundary links in the Wheatstone network, but the active system constraint is either in those two links together, or in the Wheatstone bridge. Both interpretations are technically correct, but the policy implications are different. If the two boundary links represent the active system constraint, then either reducing demand or expanding capacity on both links would be optimal policies. If the Wheatstone bridge is viewed as the active system constraint, then the optimal policy would be to remove the bridge entirely, or (if the bridge was viewed as beneficial for reliability reasons) equip the bridge with fast relays or phase-angle regulation devices that would permit power to flow over the bridge only during contingencies. The preferable policy is largely a matter of network parameters and the state of technology.

Result 3 illustrates how nodal prices in the DC optimal power flow formulation may not always send clear signals to system operators and planners. Note that this phenomenon is general in the sense that it depends only on the network topology and not on the level of demand. The two congested lines in the Wheatstone network will, indeed, sport non-negative shadow prices (see Figure 5, for example). While

[16] note that nodal price differences do not always have the physical interpretation of being congestion costs, Results 2 and 3 taken together would seem to say that shadow prices in power networks do not necessarily represent the equilibrium value of capacity expansion in the network. Result 3 is particularly important in the context of electric industry restructuring, where nodal prices and shadow prices are supposed to guide operations and investment decisions. In the symmetric unbalanced Wheatstone network, the nodal prices and shadow prices are not representative of investments that would be profitable or socially beneficial (see also the example in [17], Ch. 3 and 4).

Result 3 also suggests that using the DC power flow approximation for the purpose of calculating nodal prices (as well as the value of transmission congestion contracts) may not be appropriate for all systems. One possible remedy is to replace the DC power flow model with a full AC power flow. The nonlinearities in the AC power flow model imply that the flow constraints on the transmission lines are more likely to be independent. As a side benefit, it would also allow for the optimization of real and reactive power dispatch and would account for marginal losses. The tradeoff is that the AC model is computationally more expensive.[10]

Another possible remedy would be a two-stage calculation of the nodal prices. The first stage would run the DC optimal power flow, as in (18) and (19). The second-stage optimization problem would choose from among the set of shadow prices satisfying the first-stage problem according to some decision rule. This two-stage method has been proposed in the telecommunications literature, where linear independence of the network flow constraints is rarely satisfied [18].

Result 4 shows that increasing the susceptance, rather than the stability limit, of congested lines in the unbalanced Wheatstone network will have the desired effect of relieving some congestion. With respect to the current issue of investment in the grid, this suggests that strategically adding susceptance in concert with capacity should be considered as part of an optimal policy. In market settings, where policymakers have emphasized the role of non-utility parties in grid expansion, Result 4 also suggests that any market-based transmission investment should be compensated with a portfolio of rights related to the physical and electrical properties of new lines, and not just for capacity (megawatts) as is currently the case.[11]

Result 5 shows how congestion in Wheatstone networks can be prevented altogether. It provides an upper bound for the new flows on the boundary links following the construction of the Wheatstone bridge (for a fixed level of demand in the system). In the planning stage, the stability limit should be set above F^{crit} to avoid the problem of congestion in the Wheatstone network. This introduces yet another aspect of the cost-benefit calculus of the Wheatstone bridge. If the cost of attaining F^{crit} on the boundary links exceeds the total social cost of the Wheatstone bridge, then the boundary links should be strengthened and the bridge should not be built; reliability criteria can be met more cheaply with a smaller number of higher-capacity transmission links.

VIII. Economic Welfare Analysis

Aside from causing congestion in the network, Braess's Paradox has implications for the economic welfare of the system, as well as distributional implications. These implications are particularly important in systems that have undergone restructuring and where power is supplied (and congestion is managed) using market mechanisms. The power market represented in the Wheatstone network of Figure 5 is shown here as Figure 6. Suppose that T_0 is the amount of power transferred across the network (from bus 1 to bus 4) for a given level of demand in the absence of any transmission constraints. Once the Wheatstone bridge is built, the transfer capability decreases to T^* (thus, $T_0 - T^*$ megawatts must be generated at the load bus).

Figure 3.7 illustrates two distinct welfare concepts. First, congestion in the network results in the generator at bus 4 collecting revenue that would not have been generated in the absence of congestion. This amount, equal to area A + B in Figure 6, is known as congestion rent (or merchandizing surplus) and can be calculated as $(\pi_4 - \pi_1)F_{12}^{\max}$. The area C + D represents extra congestion costs borne by the system, equal to the higher generation cost from having to use the generator at bus 4. Area C represents social wealth that is lost by consumers in the form of higher congestion costs, while area D represents the loss borne by producers. The general expression for the congestion cost is:

$$\text{CongCost} = \int_0^{T_0 - T^*} MC(P_{G4})dP_{G4} - \int_{T^*}^{T_0} MC(P_{G1})dP_{G1}. \quad (23)$$

Assume that there are no congestion contracts in the system; that is, assume that the merchandizing surplus amounts to a transfer of social wealth to the generator located at bus 4. Some of this wealth is transferred from consumers and some is transferred from the remaining producers in the system (i.e., the generator at bus 1).

[10] The solution to the AC power flow also suffers from the potential problem of non-uniqueness.

[11] Of course, as Wu et. al. (1996) point out, it is possible to cause congestion by raising the susceptance of a line, so any such payments would need to be structured carefully.

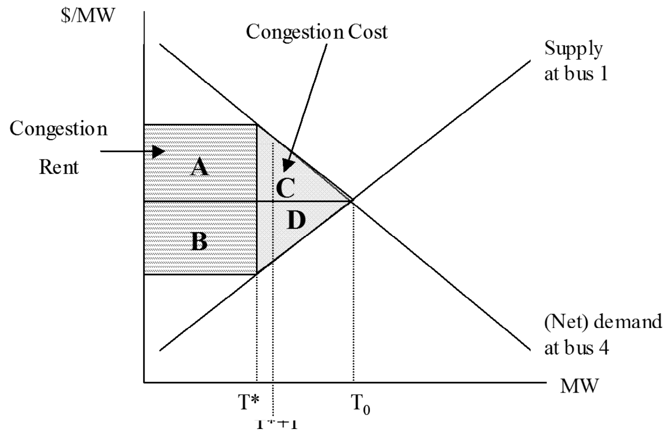

Figure 6: Distribution of congestion rent, congestion cost, and social surplus in the four-bus Wheatstone network.

The share of the social surplus transferred from the generator at bus 1 to the generator at bus 4 is given by:

$$\text{Producer surplus transferred to } G_4 = (\pi_0 - \pi_1)F_{12}^{\max}, \quad (24)$$

where π_0 is the marginal cost of serving the load without congestion (this is also equal to the nodal price prevailing at all four buses in the system). Thus, the share of the social surplus transferred from consumers to the generator at bus 4 is given by the remainder:

$$\text{Consumer surplus transferred to } G_4 = (\pi_4 - \pi_0)F_{12}^{\max}. \quad (25)$$

Given (24) and (25), we can calculate the total loss in producer surplus (ΔPS) and consumer surplus (ΔCS) as:

$$\Delta PS = (F_{12}^{old} - F_{12}^{\max})\pi_0 - \int_{F_{12}^{old}}^{F_{12}^{\max}} MC(P_{G1})dP_{G1} \quad (26a)$$

$$\Delta CS = (\text{CongCost - }\Delta PS) + (\pi_4 - \pi_0)F_{12}^{\max}. \quad (26b)$$

Equation (26a) represents producer surplus lost by the generator at bus 1. An amount $(\pi_0 - \pi_1)F_{12}^{\max}$ is simply transferred to the generator at bus 4 (and thus is still social wealth captured by producers), while the remainder of equation (26a) represents the social loss associated with having to use the generator at bus 4 to serve $F_{12}^{old} - F_{12}^{\max}$ megawatts of demand. The first term in the right-hand side of equation (26b) represents the consumers' share of total social wealth lost to congestion costs, while the second represents the transfer from consumers to the generator at bus 4 in the form of congestion rent.

Equations (26) can be applied to the system in Figure 5 to illustrate the social losses associated with a congested Wheatstone network. First, we note that with the Wheatstone bridge, the hourly prices for a demand of 100 MW are π_1 = \$11.96/MW and π_4 = \$51.67/MW, per Figure 5. In the absence of the Wheatstone bridge, the prevailing price at all four nodes would be π_0 = \$12.11/MW. We note also that T_0 = 100 MW, and T^* = 91.67 MW (these quantities must be determined empirically). As noted in Section 3, the hourly congestion cost is equal to \$323.27.

First, we calculate the loss in producer surplus due to the addition of the Wheatstone bridge, using equation (3.34a):

$$\Delta PS = (100 - 91.67)\times 12.11 - (C_{G1}(100) - C_{G1}(91.67)) = \$0.71$$

per hour.

From this, we evaluate equation (3.34b) to calculate the loss accrued by consumers:

$$\Delta CS = (323.27 - 0.71) + (51.67 - 12.11)\times 91.67 = \$3{,}949.03.$$

Thus, we see that nearly all of the social losses are borne by consumers. Note that more than 90% of the loss in consumer surplus is reflected in the congestion rent transferred to the generator at bus 4.

IX. Conclusions

Let us briefly return to the three network characteristics arising from the study of Braess' Paradox in networks other than power systems, as mentioned in Section 2:

1. Braess's Paradox occurs only in Wheatstone networks, and these networks are guaranteed to exhibit the Paradox over a certain range of flows;
2. When the network is upgraded, such upgrades should be made system-wide and should not focus on correcting local congestion;
3. "Sources" should not be located far from "sinks," at least not topologically.

This paper has largely addressed the first two points, although easy arguments can be made that the third is applicable to power systems just as it is to other systems. The first point, that the existence of Braess's Paradox and the Wheatstone network structure are equivalent, simply does not hold in power networks. The dependency actually fails to hold both ways. A network exhibiting Braess's Paradox is neither a necessary nor a sufficient condition for that network to have an embedded Wheatstone structure. Nor is the presence of congestion a necessary or sufficient condition for the network to have an embedded Wheatstone sub-network. The most general form of Braess's Paradox, that adding capacity can constrain a network, has been shown to hold for a simple two-bus parallel network. The conditions for a

Wheatstone exhibiting Braess's Paradox are much more stringent in power systems than they appear to be in other networks. The line limits and susceptance of the Wheatstone bridge must be within certain limits for the addition of the bridge to constrain the system. Transmission and resource planners might keep this condition in mind to help determine optimal line limits for new and existing lines.

If a Wheatstone network is constrained by the addition of the bridge, increasing the capacity of one congested line will not remove the constraint. All congested lines must receive capacity upgrades, or the bridge must be disconnected. Thus, the second point (that system upgrades should not be made locally) seems to hold true in power systems. Local upgrades will at best do nothing and at worst shift the problem somewhere else. Further, focusing attention to upgrading the megawatt capacity of a line may, in the Wheatstone network, be misguided. Upgrading the line's susceptance can also have a beneficial effect, depending on the relative upgrade cost.

The discussion in this paper has mentioned several times that the primary motivation for installing the Wheatstone bridge is that it may provide a reliability benefit. This reliability, however, comes at the cost of increased congestion. The amount of congestion actually caused is representative of the system's willingness-to-pay for flow control devices (relays, FACTS, phase-shifting transformers, and so on). Based on some simple simulations using the four-bus Wheatstone test network, the amount by which the reliability benefit will exceed the congestion cost over time is increasing in the variability of demand. Even with more variable demand levels, the probability of a line outage must be relatively large for the Wheatstone network to yield any net benefit over time.

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