Some Implications of Braess' Paradox in Electric Power Systems

Seth Blumsack*
Department of Engineering and Public Policy
Carnegie Mellon University
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Abstract: Braess' Paradox describes a situation in which constructing a Wheatstone bridge causes or worsens congestion in a network, thus increasing the cost to users. While this behavior has been extensively studied in other network industries, its implications for power systems have not. The steady-state conditions under which Braess' Paradox holds in a simple symmetric unbalanced Wheatstone network are derived. While these conditions are more stringent than in other types of networks, Wheatstone structures are quite common in actual power networks and can sometimes provide reliability benefits to the system. The price paid for this reliability benefit is increased congestion throughout the network; eliminating congestion in a Wheatstone network also eliminates the reliability benefit of the meshed network structure. Thus, awareness of these network structures is critical for the planning process, and can also yield useful "rules of thumb" for network planners.

*Contact information: Department of Engineering and Public Policy, 129 Baker Hall, Carnegie Mellon University, Pittsburgh PA 15213. Tel (412) 268-2670, Email blumsack@cmu.edu, WWW *http://www.andrew.cmu.edu/~sblumsac*.

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I. Notation

NL = Number of lines in the network
NB = Number of buses in the network
B_{ij} = Susceptance of the link connecting buses i and j
X_{ij} = Reactance of the link connecting buses i and j
θ_i = Phase angle at the ith bus
P_i = Real power injection at the ith bus
δ_{ij} = Phase angle difference between buses i and j
F_{ij} = Real power flow between buses i and j
π_i = Nodal price at bus i
μ_{ij} = Shadow price of transmission between buses i and j
$\mathbf{B}$ = ($NB \times NB$) positive definite system susceptance matrix
$\mathbf{A}$ = ($NB \times NL$) system incidence matrix
$\mathbf{P}$ = ($NB \times 1$) vector of bus injections
$\mathbf{F}$ = ($NL \times 1$) vector of line flows
$\boldsymbol{\theta}$ = ($NB \times 1$) vector of bus angles
$\boldsymbol{\delta}$ = ($NL \times 1$) vector of bus angle differences

II. Introduction

The U.S. blackout of 2003 focused policymakers' attention on the state of the North American transmission interconnection, with even skeptics showing concern over a multi-decade lull in transmission investment (Kirby and Hirst 2001). System reliability has become a primary concern, with the Energy Policy Act of 2005 specifying the creation of a U.S. Nationwide Reliability Organization that may have the power to set national reliability standards to replace the current voluntary inudstry standards.

Under industry restructuring, the transmission system is being asked to fulfull two roles. The first is to deliver power reliably to customers, and the second is to support a growing number of market transactions. The current transmission grid may find these two obligations conflicting. Many market transactions involve buyers and sellers separated by large geographic or topological distances. The resulting pattern of network loadings is very different from the regulated era, in which vertically-integrated utilities largely relied on self-scheduling to fill demand.[1]

One policy response is to build more transmission lines, in much the same way that transportation officials order new highways built to ease traffic congestion. In certain portions of the North American grid, expansion is a wise course of action, particularly locations where large DC links are feasible and required. In other portions, however, simply building up capacity may increase reliability, but at the cost of increased congestion. The unbalanced Wheatstone network provides a good framework to illustrate these tradeoffs, since adding certain links for reliability reasons causes congestion and

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[1] "Wheeling" transactions were commonplace prior to industry restructuring, but were not as numerous and sometimes involved long-term bilateral contracts. The overbuilding of transmission capacity by utilities decades prior to restructuring also likely dulled the impact of bilateral market transactions.

increases the dispatch cost. This paper uses the DC load-flow model to describe the conditions under which this seemingly paradoxical behavior occurs in a simple test system.

III. Wheatstone Networks and Braess' Paradox

The Wheatstone network describes a graph consisting of four nodes, with four corresponding edges on the boundary creating a diamond or circular shape. A fifth edge connects two of the nodes across the interior of the network, thus splitting the network into two triangular (or semicircular) subsystems. This fifth edge is aptly named the "Wheatstone bridge." Although the network is named for Charles Wheatstone, who was the first to publish the network topology in 1843, the network design was apparently the work of Samuel Christie some ten years earlier (Ekelöf 2001).

The original motivation for the Wheatstone network was the precise measurement of resistances, as shown in Figure 1. In the network, resistances R_1, R_2, and R_3 are known to very high precision, and R_2 is adjustable. The problem is to measure R_x with similar precision. The voltage V across the bridge is equal to:

$$(1) \quad V = \frac{R_2}{R_1 + R_2} V_s + \frac{R_x}{R_3 + R_x} V_s$$

where V_s is the voltage source. Assuming that $V_s \neq 0$, then the voltage drop across the bridge will be zero at that value of R_x where $R_2 / R_1 = R_x / R_3$. If this condition is satisfied, then the Wheatstone network is said to be *balanced*. If this condition is not satisfied, then there will be a voltage drop across the bridge and the network is said to be *unbalanced*.

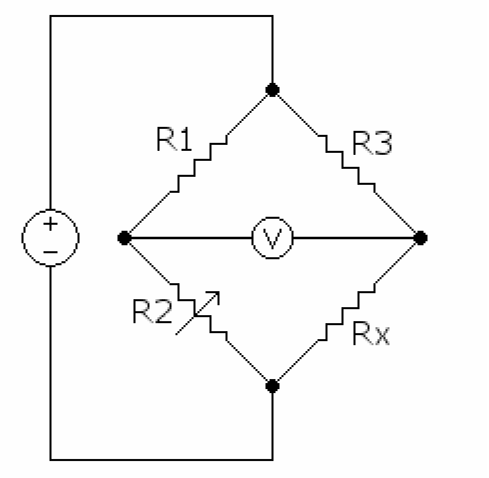

Figure 1: Wheatstone circuit example

As a fairly general topology, Wheatstone networks have arisen as structures of interest in other network situations such as traffic, pipes, and computer networks. Much of the attention paid to Wheatstone structures has centered around the network's seemingly paradoxical behavior. Under certain conditions, connecting a Wheatstone bridge to a formerly parallel network (or, in the context of the circuit in Figure 1, adjusting the boundary resistances so that the network is unbalanced) can actually increase the total user cost of the network. First studied by Braess (1968) in the context of traffic networks, this behavior has come to be known as Braess' Paradox.

The exact meaning of the "user cost" of the network has various interpretations depending on the network of interest. In Braess' original example, and in Arnott and Small (1994), the user cost of highways is the time it takes motorists to reach their final destination. An increase in the user cost, therefore, corresponds to wasted time and irritation sitting in larger traffic jams. Costs incurred through internet routing networks, as in Calvert and Keady (1993) and Korilis, Lazar, and Orda (1999), arise through increased latency and possibly lost information (Bean, Kelly, and Taylor 1997). Even in circuits, "user cost" can be interpreted as the voltage drop across the circuit as a whole. Cohen and Horowitz (1991) describe an example in which the addition of a Wheatstone bridge lowers the voltage drop across the network (assuming the network is unbalanced to begin with); thus the "cost" incurred by the Wheatstone bridge is reduced voltage over the circuit as a whole. Braess' Paradox suggests that user costs may increase for reasons independent of the amount of traffic on the network. The network itself, and not its users, may be the ultimate problem, and managing flows or disconnecting certain network links may actually serve to decrease congestion costs for all users.[2]

The question of whether Braess' Paradox is unique to the Wheatstone network has been studied by Milchtaich (2005). Using a result from Duffin (1965) that every network topology can be decomposed into purely series-parallel subnetworks and Wheatstone subnetworks, Milchtaich concludes that (apart from uninteresting situations such as simple bottlenecks) the paradoxical behavior cannot occur outside the Wheatstone structure. Thus, observation of the paradox serves as proof of an embedded Wheatstone subnetwork. Milchtaich (2005), Calvert and Keady (1993), and Korilis, Lazar and Orda (1997, 1999) offer the following technical and policy implications of Braess' Paradox:

1. Braess' Paradox occurs in any network that is not purely series-parallel;
2. Local network upgrades (that is, upgrading only congested links) will not resolve Braess' Paradox. Upgrades must be made throughout the system in order to reduce the user cost of the network.;
3. System upgrades should focus on connecting "sources" as close as possible to "sinks."

Underlying the policy recommendations is the assumption that flow networks all behave similarly, at least on the surface. While there are good analogies between the behavior in electric power networks and other networks, the analogies are ultimately flawed.

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[2] Viewing network traffic as a routing game, Braess' Paradox does not seem all that paradoxical. Each user choosing a network path to minimize their private costs easily lends itself to coordination failures such as the Prisoner's Dilemma. All users would benefit through coordination and cooperation, but no individual user has the incentive to initiate (or perhaps even sustain) this coordination.

Kirchoff's Laws do not hold in other networks.[3] In traffic and some internet systems, routing is determined by user preference rather than by physical laws (e.g., current flows follow Ohm's Law), although installation of FACTS devices could change this for those lines outfitted with devices. Congestion costs in systems with nodal pricing are discontinuous, while in other networks the cost of additional traffic can be described as a continuous function of current traffic. Despite these differences, power networks do exhibit some of the behavior described in other networks; in particular, Braess' Paradox can hold in simple systems or subsets of more complex systems. The aim of this paper is to describe the conditions under which Braess' Paradox does hold in electric power systems (Section V), and to elaborate on some implications for management and investment (Section VII).

IV. A Simple Wheatstone Test System

The four-bus test system used in this discussion is shown in Figure 2. There is one generator located at bus 1, an additional generator at bus 4, and one load at bus 4. Buses 2 and 3 are merely tie-points; power is neither injected at nor withdrawn from these two buses. From the analogy to Figure 1, the Wheatstone bridge is the link connecting buses 2 and 3. The test system is assumed to be symmetric, in the sense that $B_{12} = B_{34}$ and $B_{13} = B_{24}$. The susceptance of the Wheatstone bridge is given by B_{23} and will be a variable of interest in the discussion that follows. The symmetry assumption implies, among other things, that in the DC load flow, $F_{12} = F_{34}$ and $F_{13} = F_{24}$.[4]

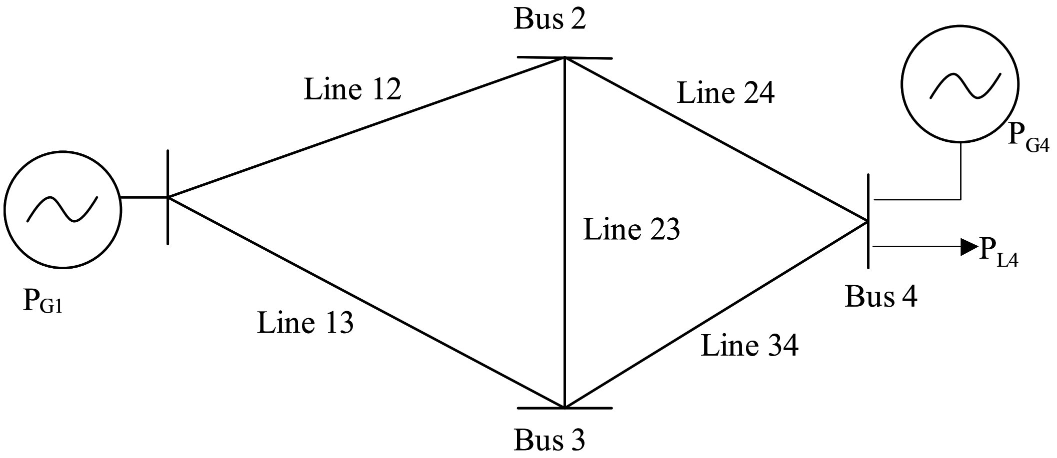

Figure 2: The symmetric Wheatstone network

The following definitions will help solidify concepts:

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[3] In the case of laminar flow, a version of Kirchoff's Law does hold in piping networks. However, real flows through pipes are almost always turbulent, rather than laminar.

[4] In the DC load flow, the current magnitude is identical to the admittance (since the voltage magnitudes are all set to 1 per-unit). The symmetry of the admittance matrix implies that the two cut sets in the system (buses 1, 2, and 3, and buses 2, 3, and 4) are also symmetric, and Kirchoff's Current Law must hold for each cut set.

Definition 1: A four-node network is said to be a Wheatstone network if its topology is the same as that in Figure 2.

Definition 2: A four-node network is said to be a symmetric Wheatstone network if it is a Wheatstone network, and if the susceptance conditions $B_{12} = B_{34}$ and $B_{13} = B_{24}$ hold.

Definition 3: A four-node network is said to be a symmetric unbalanced Wheatstone network if it is a symmetric Wheatstone network, and the magnitude of the flow across link (2,3) is nonzero.

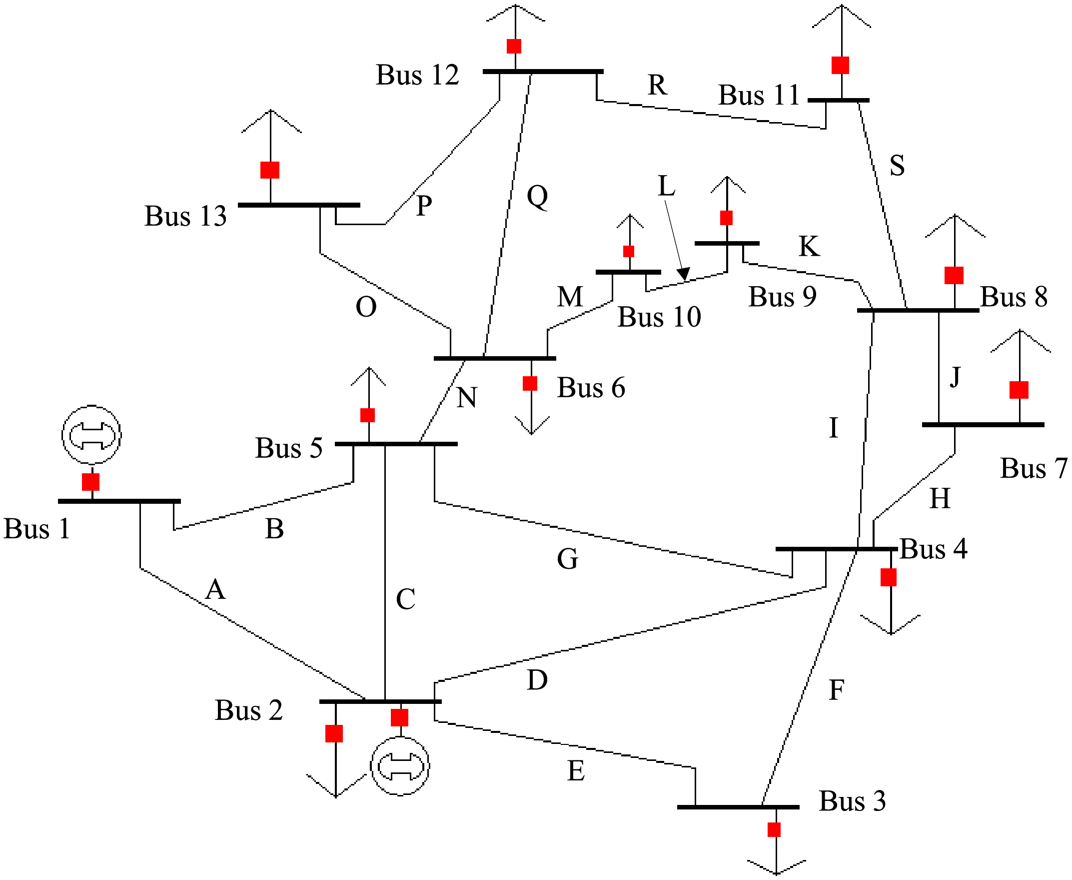

Figure 3: Thirteen-bus system based on the IEEE 14-bus system. There are at least six Wheatstone subnetworks in the system. Examples include those formed by lines A, B, C, G, and D; lines C, G, D, E, and F; and lines O, P, Q, (R+S), and (M+L+K).

Although the Wheatstone network shown in Figure 2 is simplistic, the Wheatstone structure is actually quite common in actual systems. Figure 3 shows a slightly modified version of the IEEE 14-bus test network.[5] The test network is based on an actual portion of AEP's high-voltage system. There are at least six embedded Wheatstone networks in

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[5] It has been modified by removing the synchronous condensers in the system. This reduces the network to 13 buses.

the 13-bus system of Figure 3. An interesting issue is how to decompose a large complex network into its component Wheatstone subnetworks and those subnetworks that are purely series-parallel, and how to treat adjacent and overlapping Wheatstone networks.[6]

V. Conditions for Braess' Paradox to Hold

Of particular interest here is how the addition of the Wheatstone bridge affects the flows on the boundary lines relative to the "base case" with no Wheatstone bridge. Here we are implicitly assuming that the generator injections, load withdrawals, and line susceptances are such that there is no congestion in the system prior to the addition of the bridge. If we take lines (1,2) and (2,4), and combine them in series to form a line with equivalent susceptance B_a, and we combine lines (1,3) and (3,4) in a similar fashion to construct an equivalent line B with equivalent susceptance B_b, the network is free of congestion if and only if:

$$(2) \quad \frac{B_k}{(B_a + B_b)} P_{G1} < F_k^{\max}, \text{ for } k = \{a, b\}$$[7]

To derive an explicit expression for the new network flows following the addition of the Wheatstone bridge, we will use the method derived in Ejebe and Wollenberg (1979) and Irisarri, Levner, and Sasson (1979), which compares steady-state line flows in the network before and after the network modification. Such modifications are represented as changes in the susceptance matrix **B**. Although the Ejebe-Wollenberg method was originally designed to model the effects of contingencies (so that the susceptance change in a given line, ΔB_k, is simply equal to $-B_k$) it is easily adaptable to the construction of a new line.

Since we are using the DC load flow approximation, the admittance matrix consists solely of susceptances:

$$(3) \quad B_{ij} = \begin{cases} -\dfrac{1}{X_{ij}} & i \neq j \\ \displaystyle\sum_{i \neq j} \dfrac{1}{X_{ij}} & i = j \end{cases}.$$

We start with the DC model: [8]

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[6] Duffin (1965) has shown that any non-radial network can be decomposed into series-parallel and embedded-Wheatstone subnetworks; Milchtaich (2005) also proves the same result. In the limit, where the mesh network consists essentially of everything connected to everything else, isolating particular Wheatstone structures may be difficult. Particularly when the network exhibits Braess' Paradox, the hardest question will be to pinpoint which Wheatstone is "causing" the paradoxical behavior. These issues are the subject of papers-in-progress.

[7] A similar condition also holds in AC networks, but uses the complex admittance instead of the susceptance.

[8] We could also start with the distribution-factor representation of the DC model, $\mathbf{F} = \mathbf{A}'\mathbf{B}^{\mathbf{diag}}\mathbf{A}\boldsymbol{\theta}$, where $\mathbf{B}^{\mathbf{diag}}$ is a $(NL \times NL)$ diagonal matrix of line susceptances. However, starting with the injection equations will allow us to write the new flows in the form $\mathbf{F}^{new} = \mathbf{F}^{old}$ + {adjustment factor}.

(4) $\mathbf{P} = \mathbf{B}\boldsymbol{\theta}$

Note that equation (4) represents the system prior to the addition of the Wheatstone bridge. After the Wheatstone bridge is connected, the load flow equations become:

(4') $\mathbf{P} = (\mathbf{B} + \mathbf{A'}\Delta\mathbf{B}^{\mathbf{diag}}\mathbf{A})\boldsymbol{\theta}^{new}$,

where $\Delta\mathbf{B}^{\mathbf{diag}}$ is a diagonal matrix of changes to the line susceptances. $\Delta\mathbf{B}^{\mathbf{diag}}$ has dimensionality ($NL \times NL$). Solving equation (4') for the vector of phase angles yields:

(5) $\boldsymbol{\theta}^{new} = (\mathbf{B} + \mathbf{A'}\Delta\mathbf{B}^{\mathbf{diag}}\mathbf{A})^{-1}\mathbf{P}$.

Using the Sherman-Morrison-Woodbury matrix inversion lemma and substituting equation (4), we get:

(6) $\boldsymbol{\theta}^{new} = (\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\mathbf{A'}\mathbf{B}^{-1})\mathbf{B}\boldsymbol{\theta}^{old}$.

Distributing terms,

(7) $\boldsymbol{\theta}^{new} = \boldsymbol{\theta}^{old} - \mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}$,

where δ is the ($NL \times 1$) vector of phase angle differences.

Following the network modification, the DC flow equations can be written

(8) $\mathbf{F}^{new} = (\mathbf{A'}(\mathbf{B}^{\mathbf{diag}} + \Delta\mathbf{B}^{\mathbf{diag}})\mathbf{A})\boldsymbol{\theta}^{new}$,

Inserting (7) into (8) and distributing terms yields:

(9)
$$\begin{aligned}\mathbf{F}^{new} &= \mathbf{A'}(\mathbf{B}^{\mathbf{diag}})\mathbf{A}\boldsymbol{\theta}^{old} - \mathbf{A'}(\mathbf{B}^{\mathbf{diag}})\mathbf{A}\mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old} + \mathbf{A'}(\Delta\mathbf{B}^{\mathbf{diag}})\mathbf{A}\boldsymbol{\theta}^{old} \\ &\quad - \mathbf{A'}(\Delta\mathbf{B}^{\mathbf{diag}})\mathbf{A}\mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old} \\ &= \mathbf{F}^{old} + \mathbf{A'}(\Delta\mathbf{B}^{\mathbf{diag}})\boldsymbol{\delta}^{old} - (\mathbf{A'}(\mathbf{B}^{\mathbf{diag}} + \Delta\mathbf{B}^{\mathbf{diag}})\mathbf{A})\mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}\end{aligned}$$

The adjustment is $\mathbf{A'}(\Delta\mathbf{B}^{\mathbf{diag}})\boldsymbol{\delta}^{old} - (\mathbf{A'}(\mathbf{B}^{\mathbf{diag}} + \Delta\mathbf{B}^{\mathbf{diag}})\mathbf{A})\mathbf{B}^{-1}\mathbf{A}(\Delta\mathbf{B}^{\mathbf{diag}^{-1}} + \mathbf{A'}\mathbf{B}^{-1}\mathbf{A})^{-1}\boldsymbol{\delta}^{old}$.

In the special case where the susceptance of only one line changes (as is the case with the Wheatstone bridge example), we can replace the $\Delta\mathbf{B}^{\mathbf{diag}}$ matrix with a scalar ΔB_k (where k indexes the line whose susceptance has been altered), and we can replace the incidence matrix with its kth column, denoted $\mathbf{A_k}$. In this case, equation (6) is modified to read:

$$(6') \quad \boldsymbol{\theta}^{new} = (\mathbf{B} + \Delta B_k \mathbf{A_k} \mathbf{A_k}')^{-1}\, \mathbf{P},$$

and equation (8) becomes:

$$(8') \quad \boldsymbol{\theta}^{new} = (\mathbf{I} - (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{B}^{-1}\mathbf{A_k}'\mathbf{A_k})\boldsymbol{\theta}^{old}.$$

The term $\Delta B^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k}$ can get pulled out because it is a scalar. Recognizing that $\mathbf{A_k}\boldsymbol{\theta} = \delta_k$, the phase angle difference along line *k*, we get:

$$(10) \quad \boldsymbol{\theta}^{new} = \boldsymbol{\theta}^{old} - (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{B}^{-1}\mathbf{A_k}'\delta_k.$$

Returning to the DC load flow equations, the flow across the *l*th line following the network modification is:

$$(11) \quad F_l^{new} = B_l \delta_l^{new}.$$

In the case where $l = k$, equation (16) can be modified to read $F_k^{new} = (B_k + \Delta B_k)\delta_k^{new}$, although the emphasis here will be on lines other than *k* (since the object of interest is calculating the effect of the Wheatstone bridge on flows on the other lines). Rewriting equation (16) as:

$$(12) \quad F_l^{new} = B_l \mathbf{A'_l}\, \boldsymbol{\theta}^{new}$$

and substituting equation (15), we get:

$$\begin{aligned}(13) \quad F_l^{new} &= B_l \mathbf{A'_l} \left[\boldsymbol{\theta}^{old} - (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{B}^{-1}\mathbf{A_k}\delta_k^{old}\right] \\ &= F_l^{old} + (\Delta B_k^{-1} + \mathbf{A_k}'\mathbf{B}^{-1}\mathbf{A_k})^{-1}\mathbf{A'_l}\,\mathbf{B}^{-1}\mathbf{A_k} B_l \delta_k^{old} \\ &= F_l^{old} + b_k^{-1}\mathbf{A'_l}\,\mathbf{B}^{-1}\mathbf{A_k} B_l \delta_k^{old}.\end{aligned}$$

In the special case where $l = k$, equation (13) becomes:

$$(13') \quad F_l^{new} = \left(F_l^{old} - \Delta B_l \delta_l^{old}\right)\!\left(1 - b_l^{-1}\mathbf{A'_l}\,\mathbf{B}^{-1}\mathbf{A_l}\right).$$

Of particular interest here are the conditions under which any of the boundary links will become congested with the addition of the Wheatstone bridge (congestion occurs when the generator at bus 4 either does not exist or is not turned on). Without loss of generality, assume that $B_{12} > B_{13}$. Thus, once the bridge is added, more power will flow over link (1,2) than over link (2,3).[9] So we are really interested in the conditions under

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[9] The symmetry assumption implies that equal amounts of power will flow over both paths in the absence of the Wheatstone bridge.

which link (1,2) will become congested. The symmetry assumption implies that a similar condition will hold for link (3,4) to become congested.

Link (1,2) becomes congested if $F_{12}^{new} \geq F_{12}^{\max}$. Thus, an equivalent condition is:

$$(14)\quad F_{12}^{old} + b_{23}^{-1}\mathbf{A'}_{12}\,\mathbf{B}^{-1}\mathbf{A}_{23}B_{12}\delta_{23}^{old} \geq F_{12}^{\max}$$

$$\Rightarrow \Delta B_{23}^{-1} \geq \frac{\mathbf{A'}_{12}\,\mathbf{B}^{-1}\mathbf{A}_{23}B_{12}\delta_{23}^{old}}{F_{12}^{\max} - F_{12}^{old}} - \mathbf{A}_{23}\mathbf{'}\mathbf{B}^{-1}\mathbf{A}_{23}.$$

This "feasible region" for the susceptance of the Wheatstone bridge is shown in Figure 4 for the configuration where $B_{12} = B_{34} = 0.06$ p.u., $B_{13} = B_{24} = 0.03$ p.u., $F_{12}^{\max} = 55$ MW, and $P_{G1} = P_{L4} = 100$ MW. Thus, from the DC power flow on this network we get $F_{12}^{old} = 50$ MW and $\delta_{23}^{old} = 1.5$ degrees.

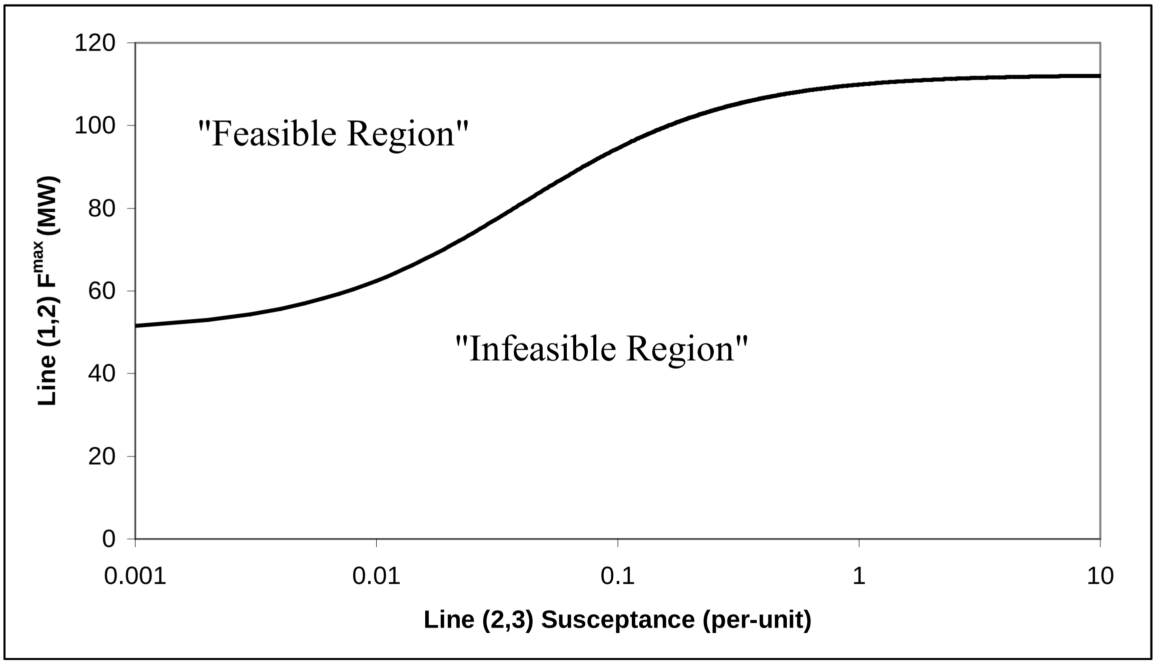


Figure 4: Whether the Wheatstone bridge causes congestion on line (1,2) (and also congestion on line (3,4) in the case of a symmetric Wheatstone network) depends on the susceptance of the Wheatstone bridge and the thermal limit of line (1,2). The "feasible region" above the line indicates susceptance-thermal limit combinations which will not result in congestion on the network. The "infeasible region" below the line represents susceptance-thermal limit combinations for which the network will become congested. Note that the x-axis has a logarithmic scale.

The point of this exercise is to show that unlike other networks such as internet communications (Milchtaich 2005), the existence of a Wheatstone configuration is not in itself sufficient for the network to exhibit Braess' Paradox. Equation (14) thus provides two "rules of thumb" for transmission planning. First, it shows conditions under which parallel networks can become more interconnected without causing congestion in the modified system. Second, it provides a condition on the line limit $F_{12}^{\max}$ under which a conversion of a parallel network to a Wheatstone network would be socially beneficial.

Why would the Wheatstone bridge ever be installed in a parallel system? One obvious answer is that it may provide reliability benefits. In the parameterization of the network represented in Figure 4, suppose that the load at bus 4 represents a customer with a high demand for reliability, that link (2,4) had an abnormally high outage rate, and that the generator at bus 4 did not exist. In this case, if the remainder of the links had sufficiently small thermal limits, the network would not meet (*N* – *1*) reliability criteria. With the addition of the Wheatstone bridge, the reliability criteria might be satisfied, but at the cost of a certain amount of congestion during those times in which link (2,4) was operating normally.[10]

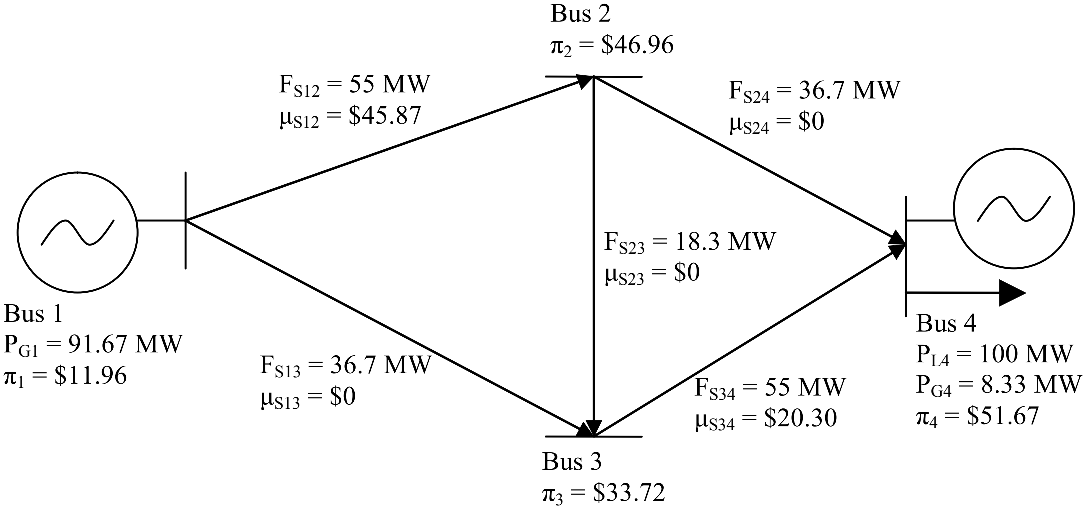
VI. DC Optimal Power Flow on the Wheatstone Network

Assume that the cost curves for the two generators in the symmetric unbalanced Wheatstone network are quadratic with the following parameterization:

(15) $C(P_{G1}) = 200 + 10.3P_{G1} + 0.008P_{G1}^2$

(16) $C(P_{G4}) = 300 + 50P_{G4} + 0.1P_{G4}^2$.

Also assume that every line in the network has a thermal limit of 55 MW. Prior to the addition of the Wheatstone bridge, the DC optimal power flow results show that 50 MW flows on each line towards bus 4; thus there is no congestion in the system. The nodal prices are all equal to \$12.11/MWh, and the total system cost is \$1,620 per hour.[11]



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[10] Ideally, controllers would be installed on the system to prevent power from flowing over the Wheatstone bridge except during contingencies on link (2,4). The congestion cost thus represents the value of such a controller to the system.

[11] The optimal power flow calculations were performed with the aid of Matpower, a free collection of Matlab files for power flow analysis, available at http://www.pserc.cornell.edu/matpower.

Figure 5: The addition of the Wheatstone bridge connecting buses 2 and 3 causes congestion along links (1,2) and (3,4). The total system cost rises from $1,620 per hour without the Wheatstone bridge to $1,945 per hour with the bridge.

Following the addition of the Wheatstone bridge, lines (1,2) and (3,4) become congested, as shown in Figure 5. The total system cost rises to $1,945 per hour as the economic dispatch is forced to run the expensive generator located at bus 4. Among other things, this implies that the value of reliability to the load is at least $325 per hour that link (2,4) remains functional.

VII. Implications of Braess' Paradox

Equations (13) and (14) from Section V have a number of implications for grid management and investment. Some of these implications mirror results described in Section III for other types of networks, while some appear to be unique to electric power networks.

Result 1: A symmetric Wheatstone network is balanced (that is, $F_{23} = 0$) if and only if $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Before proving the claim, we note that under the DC power flow approximation, $P_{23} = 0$ is equivalent to $\theta_2 = \theta_3$, so another way of stating the claim is that $\theta_2 = \theta_3$ if and only if $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Proof of Result 1: The first part of the proof is to show that $\theta_2 = \theta_3 \Rightarrow \frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Suppose that $\theta_2 = \theta_3$, and thus $F_{23} = 0$. Because all the power is flowing towards Bus 4, and since there are no losses, this condition is equivalent to stating that $F_{12} = F_{24}$ and $F_{13} = F_{34}$. From the DC load flow equations, we see that

$$F_{12} = F_{24} \Rightarrow \frac{1}{X_{12}}(\theta_1 - \theta_2) = \frac{1}{X_{24}}(\theta_2 - \theta_4)$$

$$\Rightarrow \frac{X_{24}}{X_{12}} = \frac{(\theta_2 - \theta_4)}{(\theta_1 - \theta_2)}.$$

and

$$F_{13} = F_{34} \Rightarrow \frac{1}{X_{13}}\left(\theta_1 - \theta_3\right) = \frac{1}{X_{34}}\left(\theta_3 - \theta_4\right)$$

$$\Rightarrow \frac{X_{34}}{X_{13}} = \frac{\left(\theta_3 - \theta_4\right)}{\left(\theta_1 - \theta_3\right)}.$$

Since $\theta_2 = \theta_3$, we see that $\frac{X_{24}}{X_{12}} = \frac{\left(\theta_2 - \theta_4\right)}{\left(\theta_1 - \theta_2\right)}$ and $\frac{X_{34}}{X_{13}} = \frac{\left(\theta_2 - \theta_4\right)}{\left(\theta_1 - \theta_2\right)}$; thus, $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

The second part of the proof is to show that $\theta_2 = \theta_3 \Leftarrow \frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$.

Suppose that $\frac{X_{13}}{X_{12}} = \frac{X_{34}}{X_{24}}$. From the DC load flow equations, we see that

$$\frac{X_{24}}{X_{12}} = \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)}$$

and

$$\frac{X_{34}}{X_{13}} = \frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)}.$$

Since $\frac{X_{24}}{X_{12}} = \frac{X_{34}}{X_{13}}$, it must be true that $\frac{X_{24}}{X_{12}} \div \frac{X_{34}}{X_{13}} = 1$, and thus it must also be true that:

$$(17) \quad \frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)} \div \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)} = 1.$$

By the symmetry of the network, we have $F_{12} = F_{34}$ and $F_{13} = F_{24}$. Thus,

$$(18) \quad \frac{F_{13}(\theta_3 - \theta_4)}{F_{34}(\theta_1 - \theta_3)} \div \frac{F_{12}(\theta_2 - \theta_4)}{F_{24}(\theta_1 - \theta_2)} = \frac{F_{13}^2}{F_{34}^2}\frac{(\theta_3 - \theta_4)(\theta_1 - \theta_2)}{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}.$$

For (18) to hold, it must be true that $\theta_2 = \theta_3$ and thus, $F_{13} = F_{34}$.

Result 2: In a symmetric unbalanced Wheatstone network, suppose that links (1,2) and (3,4) are congested following the construction of the Wheatstone bridge, as in Figure 5. The congestion will be relieved, and the total system cost will decline, only to the extent that upgrades are performed on both lines.

Proof of Result 2: Using equations (12) and (14), for the case in which the Wheatstone bridge causes congestion:

$$(19) \quad F_{12}^{\max} - F_{12}^{old} \leq F_{12}^{new} - F_{12}^{old} = B_{23}^{-1}\mathbf{A'}_{12}\,\mathbf{B}^{-1}\mathbf{A}_{23}B_{12}\delta_{23}^{old}$$

and

$$(20) \quad F_{34}^{\max} - F_{34}^{old} \leq F_{34}^{new} - F_{34}^{old} = B_{23}^{-1}\mathbf{A'}_{34}\,\mathbf{B}^{-1}\mathbf{A}_{23}B_{34}\delta_{23}^{old}\ .$$

Since $B_{12} = B_{34}$, we get that $F_{34}^{\max} = F_{12}^{\max} \leq F_{12}^{new} = F_{34}^{new}$. Increasing only $F_{12}^{\max}$ will not change this relationship since $F_{34}^{\max} \leq F_{12}^{new} = F_{34}^{new}$ must still hold. A similar argument holds for increasing only $F_{34}^{\max}$.

If we increase the thermal limit of both lines by the same amount, to $F^{\max,new}$, then the flows along lines (1,2) and (3,4) can simultaneously increase while maintaining the relationship $F_{34}^{\max,new} = F_{12}^{\max,new} \leq F_{12}^{new} = F_{34}^{new}$. A corollary to this result is that if the total cost (capital cost plus congestion cost) of the Wheatstone bridge exceeds the cost of upgrading the boundary links to the point where a failure on one link would not violate reliability criteria, then the Wheatstone bridge provides no net social benefit and should not be built.

Result 3: In the symmetric unbalanced Wheatstone network of Figure 5, the lagrange multipliers on the congested lines are not unique. However, the sum of the multipliers on the two congested lines is unique.

Proof of Result 3: The first part of the result, that the multipliers on the two congested lines are not unique, can be shown directly via the linearized DC optimal power flow.

Define $\mathbf{H} = \mathbf{A'B^{diag}A}$, and also define **c** to be a vector of generator marginal costs. In the linearized DC optimal power flow, **c** contains constants (i.e., all generators have constant marginal costs), and the power flow problem can be written as the following linear program:

$$(21) \quad \min \mathbf{c'P}$$

such that:

$$(22a) \quad \mathbf{P} = \mathbf{B\theta}$$

$$(22b) \quad \mathbf{F} = \mathbf{H\theta}$$

$$(22c) \quad \mathbf{F} \leq \mathbf{F^{max}}.$$

Rewriting to include the equality constraints, the optimal power flow problem is:

(21') $\min \mathbf{c'B\theta}$

such that:

(22') $\mathbf{H\theta} \leq \mathbf{F}^{\mathbf{max}}$.

Let $\boldsymbol{\mu}$ be the vector of dual variables associated with the network line flow constraints in equation (22'). The dual program is thus:

(23) $\max - \boldsymbol{\mu}' \mathbf{F}^{\mathbf{max}}$

such that

(24a) $\mathbf{H}'\boldsymbol{\mu} + \mathbf{Bc} = \mathbf{0}$

(24b) $\boldsymbol{\mu} \geq \mathbf{0}$.

Note that $\mathbf{c}$ = (c_1, 0, 0, c_4). Without loss of generality, we may assume that congestion occurs on the lines connecting buses 1 and 2, and buses 3 and 4. Thus, the optimal value of $\boldsymbol{\mu}$ is of the form $\boldsymbol{\mu}^*$ = (μ_{12}, 0, 0, μ_{34}, 0), and the constraint set in equations (24a) becomes, at the optimum:

$$\begin{bmatrix} b_{12} & b_{13} & & & \\ -b_{12} & & b_{24} & & b_{23} \\ & -b_{13} & & b_{34} & -b_{23} \\ & & -b_{24} & -b_{34} & \end{bmatrix} \begin{bmatrix} \mu_{12} \\ 0 \\ 0 \\ \mu_{34} \\ 0 \end{bmatrix} + \mathbf{B} \begin{bmatrix} c_1 \\ 0 \\ 0 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As a function of the dual variables μ_{12} and μ_{34}, the constraint set can thus be written:

(25a) $B_{12}\mu_{12} + c_1 B_{11} + c_4 B_{14} = 0$

(25b) $-B_{12}\mu_{12} + c_1 B_{21} + c_4 B_{24} = 0$

(25c) $B_{34}\mu_{34} + c_1 B_{31} + c_4 B_{34} = 0$

(25d) $-B_{34}\mu_{34} + c_1 b_{41} + c_4 B_{44} = 0.$

Meanwhile, the dual objective function is:

(26) $v^* = -\mu_{12} F_{12}^{\max} - \mu_{34} F_{34}^{\max}$

$$\Rightarrow \mu_{12} = -\mu_{34} - v^* / F_{12}^{\max},$$

where we have used that $F_{12}^{\max} = F_{34}^{\max}$, and v^* indicates, again, that we are considering the dual objective function evaluated at its optimum.

Noting that $B_{14} = B_{41} = 0$, and defining $B_{12} = B_{34} = B'$, and $B_{13} = B_{24} = B''$, we can equate (25a) and (25d) to yield:

$$(27) \quad B'\mu_{12} - c_1(B'+B'') = -B'\mu_{34} - c_4(B'+B'')$$
$$\Rightarrow \mu_{12} = -\mu_{34} - \frac{1}{B'}\left(c_1(B'+B'') - c_4(B'+B'')\right).$$

Comparing (26) and (27), we see that both equations are of the form $\mu_{12} = -\mu_{34} -$ constant. Thus, the dual objective function is parallel to the dual constraint set, and the set of dual variables $\{\mu_{12}, \mu_{34}\}$ which solves the dual program in (23) is not unique.

The second part of the result, that the sum of the shadow prices on the two congested lines is constant, follows from equation (27):

$$(28) \quad \mu_{12} + \mu_{34} = -\frac{1}{B'}\left(c_1(B'+B'') - c_4(B'+B'')\right).$$

Equation (28) can be further simplified by noting that the ratio B'/B'' is a constant, so we can write B'' = αB' for some constant α. Substituting into (28) yields:

$$(28') \quad \mu_{12} + \mu_{34} = -\frac{1}{B'}\left(c_1(B'+\alpha B') - c_4(B'+\alpha B')\right)$$
$$\Rightarrow \mu_{12} + \mu_{34} = (1+\alpha)(c_4 - c_1).$$

Result 4: Suppose that $F_l^{\max} = F_l$ for some line *l* in a symmetric unbalanced Wheatstone network in which $\delta_l^{old} > 0$. Increasing B_l for any *l* while simultaneously increasing $F_l^{\max}$ will increase the power flow on that line, even if the Wheatstone network is unbalanced.

Proof of Result 4: We are most interested in those situations in which line *l* is not the Wheatstone bridge, but the result will hold either way. The proof is a direct application of the formula of Ejebe and Wollenberg (1979), using equation (13'):

$$(13') \quad F_l^{new} = \left(F_l^{old} - \Delta B_l \delta_l^{old}\right)\left(1 - b_l^{-1}\mathbf{A'_l}\,\mathbf{B}^{-1}\mathbf{A_l}\right).$$

Calculate the sensitivity:

$$\frac{\partial F_l^{new}}{\partial \Delta B_l} = \frac{\partial}{\partial \Delta B_l}\left(\Delta B_l \delta_l^{old} b_l^{-1} \mathbf{A'}_\mathbf{l}\, \mathbf{B}^{-1} \mathbf{A}_\mathbf{l}\right)$$

$$(29) \qquad = \frac{\partial}{\partial \Delta B_l}\left(\frac{\Delta B_l \delta_l^{old}}{\left(\Delta B_l \mathbf{A'}_\mathbf{l}\, \mathbf{B}^{-1} \mathbf{A}_\mathbf{l}\right)^{-1} + 1}\right)$$

$$= \frac{\delta_l^{old}\left(\mathbf{A'}_\mathbf{l}\, \mathbf{B}^{-1} \mathbf{A}_\mathbf{l}\right)^2}{\left(\Delta B_l + \mathbf{A'}_\mathbf{l}\, \mathbf{B}^{-1} \mathbf{A}_\mathbf{l}\right)^2}.$$

To show that (29) is greater than zero, we note that $\delta_l^{old} > 0$ and $\Delta B_l > 0$ by assumption. For the power flow to have a solution, **B** must be positive definite, implying that $\mathbf{B}^{-1}$ is also positive definite and $\mathbf{A'}_\mathbf{l}\, \mathbf{B}^{-1} \mathbf{A}_\mathbf{l} > 0$.

Result 5: In a symmetric unbalanced Wheatstone network with fixed susceptances on the boundary links, the thermal limits on the boundary links required to avoid congestion are strictly increasing in the susceptance of the Wheatstone bridge. Further, there is an upper bound on the boundary-link flow F_{12}^{crit} once the Wheatstone bridge is added.

Proof of Result 5: The first part of the claim, that the $F_{12}^{\max}$ required to keep the Wheatstone network from becoming congested is strictly increasing in ΔB_{23}, follows from equation (22). To prove the second part of the claim, we examine F_{12}^{new} in the limit as ΔB_{23} becomes arbitrarily large:

$$(30) \quad F_{12}^{crit} = \lim_{\Delta B_{23} \to \infty} F_{12}^{new} = \lim_{\Delta B_{23} \to \infty} F_{12}^{old} + \frac{\mathbf{A'}_{12}\, \mathbf{B}^{-1} \mathbf{A}_{23} B_{12} \delta_{23}^{old}}{\Delta B_{23}^{-1} + \mathbf{A}_{23}' \mathbf{B}^{-1} \mathbf{A}_{23}}$$

$$= F_{12}^{old} + \frac{\mathbf{A'}_{12}\, \mathbf{B}^{-1} \mathbf{A}_{23} B_{12} \delta_{23}^{old}}{\mathbf{A}_{23}' \mathbf{B}^{-1} \mathbf{A}_{23}}.$$

VIII. Discussion

Results 2 – 5 have the most interesting implications for pricing, grid management, and investment in the electric transmission network.

Result 2 mirrors the results of Milchtaich (2005) and Korilis et. al. (1997) for internet routing networks. It says that in a symmetric unbalanced Wheatstone network, congestion will occur on two of the four boundary lines (if any congestion occurs at all), and that network upgrades amounting to a capacity expansion on only one of those lines will not alter the dispatch. It requires upgrading the capacity of both lines to affect the economic dispatch and to lower the marginal and total system costs. In other words, congestion in Wheatstone configurations is a distinct concept from a constraint. The two are not interchangeable. Congestion may occur on two of the boundary links in the Wheatstone network, but the system constraint is either in those two links together, or in the Wheatstone bridge. Both interpretations are technically correct, but the policy implications are different. If the constraint is believed to be in the two boundary links, then either reducing demand or expanding capacity on both links would be optimal policies. If the Wheatstone bridge is viewed as the constraint, then the optimal policy would be to remove the bridge entirely, or (if the bridge was viewed as beneficial for reliability reasons) equip the bridge with fast relays or phase-angle regulation devices that would not permit power to flow over the bridge during normal operations, but would allow power to flow over the bridge in contingencies. Which is the preferable policy is largely a matter of network parameters and the state of technology.

Result 3 demonstrates the "knife-edge" property of ill-conditioned linear programs. The two congested lines in the Wheatstone network will, indeed, sport nonnegative shadow prices (see Figure 5, for example). Thus, Result 3 is essentially identical to Result 2, and expands on the now well-known proposition that nodal prices in electric power networks are not analagous to nodal prices in other transportation networks. While Wu et. al. (1996) note that nodal price differences do not represent congestion costs, Results 2 and 3 taken together would seem to say that shadow prices in power networks do not necessarily represent the equilibrium value of capacity expansion in the network.[12] Result 3 is particularly important in the context of electric-industry restructuring, where nodal prices and shadow prices are supposed to guide operations and investment decisions. In the symmetric unbalanced Wheatstone network, the nodal prices and shadow prices are not representative of investments that would be profitable or socially beneficial (Blumsack 2006).

Result 3 would also seem to imply that superposition does not always hold in the case of multiple constraints. In other words, in a system with multiple congested lines, analyzing the effect of isolated network upgrades may yield misleading results.

Result 4 says that increasing the susceptance, rather than the thermal limit, of congested lines in the unbalanced Wheatstone network will have the desired effect of relieving some congestion. In other words, even small changes in the network parameters can remove the "knife-edge property" shown in Result 3. With respect to the current issue of investment in the grid, this suggests that strategically adding susceptance should be considered as part of an optimal policy along with adding capacity. In market settings, where policymakers have emphasized the role of nonutility parties in grid expansion, Result 4 also suggests that investors in the grid should be compensated for a portfolio of capacity (megawatts) and susceptance, and not just for capacity as is currently the case.[13]

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[12] It is also interesting to note that when both congested lines in the Wheatstone network are upgraded, the total system benefit is less than the sum of the shadow prices on the individual congested lines. Thus, there is a deadweight loss in social surplus to upgrading both lines. Whether this is a general result or not has not yet been explored, though it is consistent with the "revenue adequacy theorem" of Hogan (1992).

[13] Of course, as Wu et. al. (1996) point out, it is possible to cause congestion by raising the susceptance of a line, so any such payments would need to be structured carefully. Gribik, Shirmohammadi, Graves, and Kritikson (2005) suggest a type of admittance payment involving transfers from holders of capacity rights. The structure of the admittance payments does not remedy the possible incentive effects discussed by Wu et. al. and Blumsack (2006), and also does not account for social welfare that may be created or destroyed by altering the system admittance matrix.

Result 5 says that congestion in Wheatstone networks can be prevented altogether. It provides an upper bound for the new flows on the boundary links following the construction of the Wheatstone bridge (for a fixed level of demand in the system). In the planning stage, the thermal limit should be set above F^{crit} to avoid the problem of congestion in the Wheatstone network. This introduces yet another aspect of the cost-beneft calculus of the Wheatstone bridge. If the cost of attaining F^{crit} on the boundary links exceeds the total social cost of the Wheatstone bridge, then the boundary links should be strengthened and the bridge should not be built; reliability criteria can be met more cheaply with a smaller number of higher-capacity transmission links.

IX. Conclusion

Let us briefly return to the three network characteristics arising from the study of Braess' Paradox in networks other than power systems, as mentioned in Section III:

- Braess' Paradox occurs only in Wheatstone networks, and these networks are guaranteed to exhibit the Paradox over a certain range of flows;
- When the network is upgraded, such upgrades should be made systemwide and should not focus on correcting local congestion;
- "Sources" should not be located far from "sinks," at least not topologically.

This paper has largely addressed the first two points, although easy arguments can be made that the third is also applicable to power systems just as it is to other systems. The first point, that the existence of Braess' Paradox and the Wheatstone network structure are equivalent, simply does not hold in power networks. The dependency actually fails to hold both ways. A network exhibiting Braess' Paradox is neither a necessary nor a sufficient condition for that network to have an embedded Wheatstone structure. Nor is the presence of congestion a necessary or sufficient condition for the network to have an embedded Wheatstone subnetwork. The most general form of Braess' Paradox, that adding capacity can constrain a network, has been shown to hold for a simple two-bus parallel network. The conditions for a Wheatstone exhibiting Braess' Paradox are much more stringent in power systems than they appear to be in other networks. The line limits and susceptance of the Wheatstone bridge must be within certain limits for the addition of the bridge to constrain the system. Transmission and resource planners might keep this condition in mind to help determine optimal line limits for new and even existing lines.

If a Wheatstone network is constrained by the addition of the bridge, increasing the capacity of one congested line will not remove the constraint. All congested lines must receive capacity upgrades, or the bridge must be disconnected. Thus, the second point (that system upgrades should not be made locally) seems to hold true in power systems. Local upgrades will at best do nothing and at worst shift the problem somewhere else. Further, focusing attention to upgrading the megawatt capacity of a line may, in the Wheatstone network, be misguided. Upgrading the line's susceptance can also have a beneficial effect, depending on the relative upgrade cost.

This paper has mentioned several times that the primary motivation for installing the Wheatstone bridge is that it may provide a reliability benefit. This reliability, however, comes at the cost of increased congestion. The amount of congestion actually caused is representative of the system's willingness-to-pay for flow control devices (relays, FACTS, and so on). This sort of cost-benefit calculus, and a discussion of finding Wheatstone structures within larger systems will be the subject of future papers.

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