6. THE BOMBIERI-VINOGRADOV THEOREM

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1. The Main Theorem

The Bombieri-A. I. Vinogradov Theorem is concerned with the distribution of primes into arithmetic progressions. By the way, the other Vinogradov, I. M., will also make an appearance, albeit somewhat fleeting, in this story.

Let

$$\Lambda(n) = \begin{cases} \log p & \text{when } n = p^k \text{ for some } p \text{ and } k \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

the von Mangoldt function, and define

$$\psi(x;q,a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \Lambda(n) \tag{2}$$

which essentially counts the primes not exceeding x in the residue class a moduluo p with weight $\log p$. The higher powers of primes contribute, hopefully, a relatively small amount to the total, and anyway

$$\vartheta(x;q,a) = \psi(x;q,a) + O(x^{\frac{1}{2}})$$

where

$$\vartheta(x;q,a) = \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \log p. \tag{3}$$

All the main theorems stated here can be restated with $\psi(x;q;a)$ replaced by $\vartheta(x;1,a)$ or

$$\pi(x;q;a) = \sum_{\substack{p \leq x \\ p \equiv \pmod q}} 1.$$

Note that

$$\pi(x;q,a) = \frac{\vartheta(x;q,a)}{\log x} + \int_2^x \frac{\vartheta(u;q,a)}{u \log^2 u} du \tag{4}$$

The main reason for preferring Λ is that it arises naturally as the coefficient in the Dirichlet series expansion of the logarithmic derivative of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

viz.

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

when $\Re s>1$.

The best general estimate we have for an individual pair q, a, which is uniform in q, is the

Siegel [1935]–Walfisz [1936] Theorem. *Suppose that $A>0$ is a fixed real number. When $(a,q)=1$ and $q\le(\log x)^A$ we have*

$$\psi(x;q,a)=\frac{x}{\phi(q)}+O_A\Big(\exp\big(-c_1\sqrt{\log x}\big)\Big)$$

where c_1 is an absolute positive constant.

Let χ denote a Dirichlet character modulo q and put

$$\psi(x;\chi)=\sum_{n\le x}\chi(n)\Lambda(n). \tag{5}$$

Then, by orthogonality

$$\psi(x;q,a)=\frac{1}{\phi(q)}\sum_{\chi\pmod q}\overline{\chi}(a)\psi(x;\chi), \tag{6}$$

and clearly

$$\psi(x;\chi)=\sum_{a=1}^{q}\chi(a)\psi(x;q,a). \tag{7}$$

The proof of the above also establishes the

Siegel–Walfisz Theorem variant. *Suppose that $A>0$ is a fixed real number. When $q\le(\log x)^A$ and χ is a Dirichlet character modulo q we have*

$$\psi(x;\chi)-\delta(\chi)x\ll_A x\exp\big(-c_1\sqrt{\log x}\big)$$

where c_1 is an absolute positive constant and $\delta(\chi)$ is 1 or 0 according as χ is principal or non-principal.

Good references for these two results are Davenport [2000] or Estermann [1952] or Montgomery and Vaughan [2006].

When $\Re s>1$ we define

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

This has an analytic continuation to $\mathbb{C}$, and is entire except when χ is principal, in which case it is analytic except at 1 where it has a simple pole with residue

$$\frac{\phi(q)}{q}.$$

Indeed,

$$L(s,\chi_0)=\zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right).$$

The Generalised Riemann Hypothesis (GRH) is the statement that $L(s,\chi)\neq 0$ when $\Re s>\frac{1}{2}$. If GRH holds for $L(s,\chi)$, then we know (Titchmarsh [1930]) that

$$\psi(x;\chi)-\delta(\chi)x\ll x^{\frac{1}{2}}(\log qx)^2,$$

and so GRH for all χ modulo q implies that

$$\psi(x;q,a)=\frac{x}{\phi(q)}+O\big(x^{\frac{1}{2}}(\log x)^2\big)$$

uniformly for all q. We can compare this with

Theorem 6.1 Bombieri's version of the Bombieri [1965]-A.I. Vinogradov [1965,1966] theorem. *For any fixed positive number A,*

$$\sum_{q\leq Q}\max_{(a,q)=1}\sup_{y\leq x}\left|\psi(y;q,a)-\frac{y}{\phi(q)}\right|\ll_A x(\log x)^{-A}+x^{1/2}Q(\log xQ)^4.$$

Bombieri had a somewhat inflated logarithmic factor compared with the above, but in applications that is usually of no significance. Vinogradov had an x^{ε}. We see that the above is practically as good, when we average over q, as having GRH for all χ to all moduli $q\leq x^{1/2}(\log x)^{4-A}$. Consequently this theorem has many applications. Also, apart from the log power there is no known way in general of improving the crucial term $x^{1/2}Q(\log xQ)^4$ even if one assumes GRH. Something can be done if one fixes a for all q, replaces y by x or does not take absolute values, but such results are of limited applicability.

By the way, the crude estimate $(x/q+1)\log x$ for each term in the sum gives the trivial bound

$$x(\log xQ)^2+Q\log x$$

which is better than the theorem when $Q>x^{\frac{1}{2}}$, so we can suppose in any proof that

$$Q\leq x^{\frac{1}{2}}.$$

All proofs of the above start off the same way. One observes that, by (6),

$$\left|\psi(y;q,a)-\frac{y}{\phi(q)}\right|\le\frac{1}{\phi(q)}\sum_{\chi\pmod q}|\psi(y;\chi)-\delta(\chi)y|$$

and so it suffices to bound

$$\sum_{q\le Q}\frac{1}{\phi(q)}\sum_{\chi\pmod q}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \tag{8}$$

This already throws away some likely cancellation in the summation over χ, cancellation which almost certainly any improvements on Bombieri–Vinogradov will have to make some use of. When χ is induced by the primitive character χ^*, so that the conductor q^* divides q we have

$$\psi(y;\chi)=\psi(y;\chi^*)+O\left(\sum_{p|q,p\nmid q^*}(\log p)\sum_{k\le(\log y)/\log p}1\right).$$

The error term here is $\ll(\log q)\log y$ and so (8) is

$$=\sum_{q\le Q}\frac{1}{\phi(q)}\sum_{q^*|q}\sum_{\chi^*\pmod{q^*}}^{*}\sup_{y\le x}|\psi(y;\chi^*)-\delta(\chi^*)y|+O\left(Q(\log Q)(\log x)\right)$$

where $\sum^*$ indicates that the sum is restricted to primitive characters. The error term here is more than acceptable, and on interchanging the order of summation and replacing q by q^*r, the main term becomes

$$\sum_{q^*\le Q}\sum_{r\le Q/q^*}\frac{1}{\phi(q^*r)}\sum_{\chi\pmod{q^*}}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y|. \tag{9}$$

Now

$$\frac{1}{\phi(q^*r)}\le\frac{1}{\phi(q^*)\phi(r)}$$

and

$$\sum_{q\le Q}\frac{1}{\phi(q)}\ll\log 2Q.$$

[To see this write $1/\phi(q)=\frac{1}{q}\sum_{r|q}\frac{\mu(r)}{\phi(r)}$, and put $q=rm$. Then the sum is $\sum_{r\le Q}\mu(r)r^{-2}\sum_{m\le Q/r}\frac{1}{m}$.] Hence, on replacing q^* by q (9) is

$$\ll\sum_{q\le Q}\frac{\log 2Q}{\phi(q)}\sum_{\chi\pmod q}^{*}\sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y|.$$

Let $R = (\log x)^{6+A}$. Then, by the variant Siegel–Walfisz theorem we have

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\ \sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A (\log x)Rx\exp\left(-c_2\sqrt{\log x}\right)$$

where c_2 is a positive constant. We can suppose that $x > x_0(A)$. Then we distinguish two cases. If $y \le \sqrt{x}$, then we get the conclusion at once. If $\sqrt{x} \le y \le x$, then the conditions of the Siegel-Walfsiz theorem are satisfied, possibly with a slightly large value of A. Hence

$$\sum_{q\le R}\frac{\log 2Q}{\phi(q)}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\ \sup_{y\le x}|\psi(y;\chi)-\delta(\chi)y| \ll_A x(\log x)^{-A},$$

which is acceptable. Everything so far is classical and could have been done in 1935.

By definitiion $\delta(\chi)=0$ for primitive characters with conductor $q>1$. Thus it remains (!) to deal with the sum

$$(\log 2Q)\sum_{R<q\le Q}\frac{1}{\phi(q)}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\ \sup_{y\le x}|\psi(y;\chi)|\,. \tag{10}$$

The essential extra ingredient is the following

Theorem 6.2, Basic Mean Value Theorem. *Let*

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\ \sup_{y\le x}|\psi(y;\chi)|$$

where $\sum^*$ *indicates that the sum is over primitive characters modulo* q*, and suppose that* $Q\ge 1$*,* $x\ge 2$*. Then*

$$T(x,Q)\ll\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log xQ)^3.$$

We remark in passing that by working harder it is possible to replace the middle term by $x^{4/5}Q$.

The desired conclusion now follows from the above by partial summation. To see this, let

$$f(q)=\frac{1}{\phi(q)}\ \sideset{}{^*}\sum_{\chi\ (\mathrm{mod}\ q)}\ \sup_{y\le x}|\psi(y;\chi)|.$$

Then the sum in question is

$$\begin{aligned}(\log 2Q)\sum_{R<q\le Q} f(q) &= (\log 2Q)\sum_{R<q\le Q} f(q)q\left(\frac{1}{Q}+\int_q^Q \frac{dt}{t^2}\right)\\ &= (\log 2Q)Q^{-1}\sum_{R<q\le Q} qf(q) + (\log 2Q)\int_R^Q t^{-2}\sum_{R<q\le t} qf(q)dt\\ &\le (\log 2Q)Q^{-1}T(x,Q) + \int_R^Q t^{-2}T(x,t)dt.\end{aligned}$$

By the Basic Mean Value Theorem this is

$$\begin{aligned}&\ll Q^{-1}\left(x+x^{5/6}Q+x^{1/2}Q^2\right)(\log x)^4 + \int_R^Q t^{-2}\left(x+x^{5/6}t+x^{1/2}t^2\right)(\log x)^4 dt\\ &\ll \left(xR^{-1}+x^{5/6}\log(2Q/R)+x^{1/2}Q\right)(\log x)^4.\end{aligned}$$

We recall our choice $R=(\log x)^{6+A}$ to conclude that

$$(\log 2Q)\sum_{R<r\le Q} f(r) \ll x(\log x)^{-A} + x^{1/2}Q(\log x)^4$$

as required.

2. Dealing with the von Mangoldt function

The general philosophy is that we have good information about various kinds of bilinear forms, at least on average. Thus we want to convert our sums involving $\Lambda(n)$ into double sums. One, possibly naive, way of doing this is *via* the formula

$$\Lambda(n) = \sum_{lm=n} \mu(l)\log m$$

so that, for example,

$$\sum_{n\le x}\Lambda(n)f(n) = \sum_{l\le x}\sum_{m\le x/n}\mu(l)(\log m)f(lm)$$

and we would think of $\mu(l)$ and $\log m$ as being values of the variables in the bilinear form and f(lm) as being the coefficient of the bilinear form. The first person to successfully attack such a problem was I. M. Vinogradov [1937] in his proof that every sufficiently large odd number is the sum of three primes. He needed to bound

$$\sum_{p\le x} e(p\alpha)$$

when α does not have a good rational approximation with a relatively small denominator. His first step is not dissimilar to that mentioned above in the case of $\Lambda(n)$. It was while examining Vinogradov's methods that Vaughan [1977] found a way of dealing with

$$\sum_{n\leq x}\Lambda(n)e(n\alpha)$$

which was intrinsically more direct, and focussed towards the available information on bilinear forms.

In considering bilinear forms

$$\sum_m\sum_n a_m b_n c_{mn}$$

which might arise one has to have some idea of which ones can be sensibly dealt with. Here we should think of the c_{mn} as oscillating and potentially giving some cancellation. Typical examples are additive or multiplicative characters.

It is useful to divide bilinear forms into two categories.

Type I. In these one of the variables is smooth, ideally always 1, such as

$$\sum_m\sum_n a_m c_{mn}$$

and it is possible to perform the summation over n with effect. Usually the only constraint is that the sum over m should not be too long, i.e. ideally we want to ensure that the m are restricted to a fairly short interval.

Type II. In these we are not lucky enough to find that one of the variables is congenial. One needs to use quite general bounds, such as those provided by the large sieve. To illustrate this let us look at the bound provided by Lemma 3. For sake of argument, lets suppose that $MN \asymp x$, and

$$\sum_m |a_m|^2 \ll M, \quad \sum_n |b_n|^2 \ll N.$$

Then Lemma 3 gives the bound

$$\ll \sqrt{(M+Q^2)(N+Q^2)MN} \ll x + xQM^{-\frac{1}{2}} + xQN^{-\frac{1}{2}} + x^{\frac{1}{2}}Q^2$$

and this is a good bound (cf BMVT) provided that M and N are both large (or equivalently M is large but not too close to x). In effect we are saying that the rectangular coefficient matrix (c_{mn}) should not be too "thin".

It turns out that there is a way of dealing with the von Mangoldt function which gives rise solely to "good" bilinear forms of types I and II.

Lemma 6.3. *Suppose that $u > 0$, $v > 0$, $y \geq 2$ and $f : \mathbb{N} \to \mathbb{C}$. Then*

$$\sum_n \Lambda(n) f(n) = S_1 - S_2 - S_3 + S_4$$

where

$$S_1 = \sum_{m \leq u} \mu(m) \sum_{n \leq y/m} (\log n) f(mn),$$

$$S_2 = \sum_{m \leq uv} c_m \sum_{n \leq y/m} f(mn) \text{ where } c_m = \sum_{\substack{k \leq u, l \leq v \\ kl = m}} \Lambda(k)\mu(l),$$

$$S_3 = \sum_{m > u} \sum_{\substack{n > v \\ mn \leq y}} \left(\sum_{\substack{k \mid m \\ k > u}} \Lambda(k) \right) \mu(n) f(mn),$$

$$S_4 = \sum_{n \leq v} \Lambda(n) f(n).$$

One can see that if u and v are allowed to grow, but not too fast, then S_1 and S_2 will be good bilinear forms of type I and S_3 will be a good bilinear form of type II. Presumably the number of terms in S_4 will be relatively small so it can be bounded trivially.

Proof. Consider the identity

$$-\frac{\zeta'}{\zeta}(s) = G(s)(-\zeta'(s)) - F(s)G(s)\zeta(s) - (-\zeta'(s) - F(s)\zeta(s)) \left(G(s) - \frac{1}{\zeta(s)} \right) + F(s)$$

where

$$F(s) = \sum_{n \leq u} \Lambda(n) n^{-s}, \quad G(s) = \sum_{n \leq v} \mu(n) n^{-s},$$

and write this as

$$D_1(s) - D_2(s) - D_3(s) + D_4(s)$$

Each of the $D_j(s)$ can be written as a Dirichlet series. Let $\Lambda_j(n)$ be the coefficient of n^{-s} in $D_j(n)$. Then, by the identity theorem for Dirichlet series,

$$\Lambda(n) = \Lambda_1(n) - \Lambda_2(n) - \Lambda_3(n) + \Lambda_4(n).$$

By inspection of each of the Dirichlet series $D_j(s)$ we can see that each S_j satisfies

$$S_j = \sum_n \Lambda_j(n) f(n).$$

3. Proof of the Basic Mean Value Theorem

We now return to the proof of the theorem, that is, we bound

$$T(x,Q)=\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \pmod q}\sup_{y\le x}|\psi(y;\chi)|$$

It is useful to deal with some special situations first. If $Q^2>x$, then using Lemma 5.6 directly with $M=1$, $a_1=1$, $N=\lfloor x\rfloor$, $b_n=\Lambda(n)$ gives the bound

$$\ll\left(Q^2(x+Q^2)\sum_{n\le x}\Lambda(n)^2\log x\right)^{\frac12}\ll x^{\frac12}Q^2\log Qx.$$

Thus we can suppose that $Q^2\le x$. Let

$$u=v=\min\left(Q^2,x^{1/3},xQ^{-2}\right)$$

Then in the same way from Lemma 5.6, when the supremum is restricted to $y\le u^2$, we get

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi}\sup_{y\le u^2}|\psi(y;\chi)|\ll\left(Q^2(u^2+Q^2)\sum_{n\le u^2}\Lambda(n)^2\log x\right)^{\frac12}$$
$$\ll(Qx^{2/3}+Q^2x^{1/3})\log x \tag{15}$$

which is good enough. Thus it suffices to bound

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi}\sup_{u^2<y\le x}|\psi(y;\chi)|.$$

In view of Lemma 6.3 with $f(n)=\chi(n)$ when $n\le y$ and $f(n)=0$ otherwise it then suffices to bound

$$T_j=\sum_{q\le Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi}\sup_{u^2<y\le x}|S_j(\chi)|$$

for $j=1,2,3,4$. The case $j=4$ is easy since

$$S_4(\chi)=\sum_{n\le u}\chi(n)\Lambda(n)=\psi(u;\chi)$$

and $u\le u^2$, and so we can appeal to (15).

The expression T_1 is also fairly easy, since log is smooth and

$$S_1(\chi) = \sum_{m\le u} \mu(m)\chi(m) \sum_{n\le y/m} \chi(n) \int_1^n \frac{dt}{t}$$
$$= \int_1^y \sum_{m\le \min(u,y/t)} \mu(m)\chi(m) \sum_{t<n\le y/m} \chi(n)\frac{dt}{t}$$

and so when $q > 1$ the Pólya-Vinogradov inequality gives the bound

$$\ll \int_1^y uq^{1/2}\log q \frac{dt}{t} \ll uq^{1/2}(\log q)(\log y).$$

This together with the trivial bound $x(\log x)^2$ for the term $q = 1$ gives

$$T_1 \ll (x + uQ^{5/2})(\log xQ)^2 \ll (x + x^{1/2}Q^2)(\log xQ)^2$$

on examining the different cases $Q \le x^{1/6}$, $x^{1/6} < Q \le x^{1/3}$ and $Q > x^{1/3}$, i.e. $u = Q^2$, $u = x^{1/3}$, $u = xQ^{-2}$ respectively.

The expression T_3 is more complicated to deal with. We want $MN \asymp x$ but both m and n have to range over more than $x^{\frac12}$ values. We keep control of the overall number of pairs by splitting up the range for m dyadically. Let

$$\mathcal{M} = \left\{2^k\lfloor u\rfloor : k = 0, 1, ...; 2^k\lfloor u\rfloor \le x/u\right\}$$

so that

$$\mathrm{card}\mathcal{M} \ll \log x.$$

Then

$$S_3(\chi) \ll \sum_{M\in\mathcal{M}} |S_3(\chi; M)|$$

where

$$S_3(\chi; M) = \sum_{M<m\le 2M} \sum_{\substack{u<n\le x/M\\ mn\le y}} \left(\sum_{\substack{k|m\\ k>u}} \Lambda(k)\right) \mu(n)\chi(mn).$$

Note that here the upper limit x/M is never smaller than y/m and will only come into play after we have used Lemma 5.6 to remove the condition $mn \le y$. It is also useful to note that

$$\left|\sum_{\substack{k|m\\ k>u}} \Lambda(k)\right| \le \sum_{k|m} \Lambda(k) = \log m.$$

It follows now that

$$T_3 \le \sum_{M\in\mathcal{M}} T_3(M)$$

where

$$T_3(M) = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2<y\le x} |S_3(\chi; M)| \,.$$

By Lemma 5.6,

$$T_3(M) \ll (\log x)\sqrt{(M+Q^2)(xM^{-1}+Q^2) \sum_{m\le 2M} (\log m)^2 \sum_{n\le x/M} \mu(n)^2}.$$

Since

$$\sum_{m\le z} (\log m)^2 \ll z(\log 2z)^2, \quad \sum_{n\le z} \mu(n)^2 \ll z$$

we have

$$T_3(M) \ll (\log x)^2 \left(x + xM^{-1/2}Q + x^{1/2}M^{1/2}Q + x^{1/2}Q^2\right).$$

The estimation of T_3 is completed by summing over the elements of $\mathcal{M}$. Thus

$$T_3 \ll (\log x)^3 \left(x + xu^{-1/2}Q + x^{1/2}Q^2\right).$$

Again, separate inspection of the ranges $Q \le x^{1/6}$ $(u = Q^2)$, $x^{1/6} < Q \le x^{1/3}$ $(u = x^{1/3})$, $Q > x^{1/3}$ $(u = xQ^{-2})$, establishes that

$$T_3 \ll (\log x)^3 \left(x + x^{5/6}Q + x^{1/2}Q^2\right).$$

The final sum to consider is T_2 and for this we use a hybrid method. We have

$$S_2(\chi) = \sum_{m\le u^2} \sum_{n\le y/m} c_m \chi(mn)$$

We now split this sum into two parts, so that

$$S_2(\chi) = S_2'(\chi) + S_2''(\chi)$$

where $S_2'(\chi)$ contains the terms with $m \le u$ and $S_2''(\chi)$ the terms with $u < m \le u^2$. The sum

$$T_2' = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2<y\le x} |S_2'(\chi)|$$

is then treated *via* a direct use of the Pólya-Vinogradov inequality and in a concomitant manner to T_1 and the sum

$$T_2'' = \sum_{q\le Q} \frac{q}{\phi(q)} \sum_{\chi}{}^{*} \sup_{u^2<y\le x} |S_2''(\chi)|$$

is treated in the same way as T_3. Note that

$$|c_m| = \left| \sum_{\substack{k\le u,l\le u\\ kl=m}} \mu(k)\Lambda(l) \right| \le \sum_{l|m} \Lambda(l) = \log m$$

and this completes the proof of Bombieri's theorem.

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