Throughout we suppose that for each prime $p$, $0 \leq \omega(p) < p$ and that $F(Q) = \sum_{q \leq Q} \mu(q)^2 \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)}$.

1. (Vaughan 1973). (a) By considering $F(Q) = \sum_{h=1}^{\infty} \left( \frac{\omega(1)}{1} \right)^h$, or otherwise, prove that

$$F(Q) = \sum_{s(r) \leq Q} \prod_{p \mid r} \left( \frac{\omega(p)}{p} \right)^h = \sum_{m} \sum_{s(r) \leq Q} \prod_{p \mid m} \left( \frac{\omega(p)}{p} \right)^h$$

where $s(r)$ is the squarefree kernel of $r$ and $\Omega(m)$ is the total number of prime factors of $m$.

(b) Suppose that $s(r) = m \prod_{p \mid r} \frac{\omega(p)}{p - \omega(p)}$.

(c) Suppose that $\sum_{p \leq X} \frac{\omega(p)}{p} \geq f(X)$. Prove that $F(Q) \geq \max_{m} \exp (m \log (m^{-1} f(1/m)))$.

2. ("Rankin's Trick" but Rankin said that it was shown to him by Ingham). Suppose that $Q \geq 1$, and $P \in \mathbb{N}$. Let

$$F(Q, P) = \sum_{q \leq Q, \mu(q)} \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)}.$$ 

(a) Prove that, with the obvious abuse of notation, $F(\infty, P) = \prod_{p \mid P} \frac{p}{p - \omega(p)}$.

(b) Suppose that $\theta \geq 0$. Prove that $F(\infty, P) - F(Q, P) \leq Q^{-\theta} \sum_{q \leq P} q^{\theta} \mu(q)^2 \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)}$ and

$$\frac{F(Q, P)}{F(\infty, P)} \geq 1 - Q^{-\theta} \prod_{p \mid P} (1 + (p^\theta - 1) \omega(p)/p).$$

(c) Suppose that $f$ is a strictly increasing and continuous real function on $[1, \infty)$ and suppose that for $X \geq X_0$

$$\log X \leq ef(X) \quad \text{and} \quad \sum_{p \leq X} \omega(p)p^{-1} \log p \leq f(X). \quad (1)$$

Show that $\sum_{p \leq X} (p^\theta - 1)p^{-1} \omega(p) \leq (X^\theta - 1)(\log X)^{-1}f(X)$ and that $-\theta \log Q + (X^\theta - 1)(\log X)^{-1}f(X)$ is minimised as a function of $\theta$ by the choice

$$\theta = \frac{1}{\log X} \frac{\log \log Q}{f(X)}.$$ 

(d) Suppose that $X \leq Q$ and $P = \prod_{p \leq X} p$. Deduce that

$$F(Q, P) \geq 1 - \exp \left( -\frac{\log Q}{\log X} \frac{\log \log Q}{f(X)} - f(X)(\log X)^{-1} \right)$$

and show that if $f$ satisfies

$$f(X)(\log X)^{-1} \rightarrow \infty \quad \text{as} \quad X \rightarrow \infty, \quad \text{and} \quad X = g(e^{-1} \log Q) \quad (2)$$

where $g$ is the inverse function of $f$, and $Q > Q_0$, then $\frac{1}{2}F(\infty, P) < F(Q, P) \leq F(\infty, P)$.

(e) Prove that $F(\infty, P) \geq \exp \left( \sum_{p \leq X} \frac{\omega(p)}{p} \right)$ and that if there is a positive constant $C$ such that

$$\sum_{p \leq X} \omega(p)p^{-1} \log p \geq Cf(X), \quad (3)$$

then $\sum_{p \leq X} \omega(p)p^{-1} \geq C f(X)(\log X)^{-1}$.

(f) Deduce that if (1), (2), (3) hold, then for $Q > Q_1$, $F(Q) \geq F(Q, P) \gg \exp \left( \frac{C \log Q}{e \log (g(e^{-1} \log Q))} \right)$.

Surprisingly the fact that $X$ is much smaller than $Q$ does not lose too much. By using the Rankin trick in the form $F(Q) \leq Q^\theta \sum_{q=1}^{\infty} q^{-\theta} \mu(q)^2 \prod_{p \mid q} \frac{\omega(p)}{p - \omega(p)}$ combined with a condition of the kind (1) it can be shown that, for $Q > Q_2$, $F(Q) \leq \exp \left( C' \log Q (\log g(e^{-1} \log Q)) \right)$.

Generally both methods lead to the same sort of conclusion. Thus when $\lambda > 1$, in Q1 if $f(X) = C(\log X)^{\lambda-1}$ or if $f(X) = C(\log X)^{\lambda}$ in Q2, then one gets $F(Q) > \exp(C'(\log Q)^{1-1/\lambda})$ and if $f(x) = C(\log \log X)^{\lambda}$ in Q1 or $F(X) = (\log X)(\log \log X)^{\lambda}$ in Q2, then $F(Q) > \exp(C'(\log \log Q)^{\lambda})$. 

Due Tuesday 19th February