Due Tuesday 30th October

As in class define
\[ f(\alpha) = \sum_{p \leq n} (\log p)e(\alpha p) \]
and let \( P = (\log n)^B \) and let \( \mathcal{M} \) be the union of the intervals \( \mathcal{M}(q, a) = \left[ \frac{a}{q} - \frac{P}{n}, \frac{a}{q} + \frac{P}{n} \right] \) with \( 1 \leq a \leq q \leq P \) and \( (a, q) = 1 \). Let \( \mathcal{U} = (P/n, 1 + P/n] \) and \( m = \mathcal{U} \setminus \mathcal{M} \).

1. Show that
\[
\int_0^1 f(\alpha)^2f(-2\alpha)d\alpha = \sum_{p_1, p_2, p_3 \leq n \atop p_1 + p_2 = 2p_3} (\log p_1)(\log p_2)(\log p_3).
\]
Note that the equation \( p_1 + p_2 = 2p_3 \) counts, with logarithmic weight, the number of three term arithmetic progressions in the set of primes \( \leq n \). Let \( R(n) \) denote this expression.

2. Show that
\[
\int_0^1 |f(\alpha)f(2\alpha)|d\alpha \ll n(\log n)
\]
and that
\[
\int_m |f(\alpha)^2f(2\alpha)|d\alpha \ll n^2(\log n)^{9/2-B/2}.
\]

3. Prove that
\[
\int_{\mathcal{M}} f(\alpha)^2f(-2\alpha)d\alpha = \mathcal{G}(P)I(P/n) + O(n^2(\log n)^{-B})
\]
where
\[
\mathcal{G}(Q) = \sum_{q \leq Q} \frac{\mu(q)^2\mu(q')}{\Phi(q)\Phi(q')} \quad \text{with} \quad q' = q/(q, 2),
\]
\[
I(\delta) = \int_{-\delta}^{\delta} v(\beta)^2v(-2\beta)d\beta \quad \text{and} \quad v(\beta) = \sum_{m=1}^{n} e(\beta m).
\]

4. Prove that
\[
\int_{P/n \leq |\beta| \leq 1/2} |v(\beta)|^2d\beta \ll n/P
\]
and hence deduce that
\[
R(n) = CI(1/2) + O(n^2(\log n)^{9/2-B/2})
\]
where \( C = 2\prod_{p \geq 2}(1 - (p - 1)^{-2}) \).

5. Prove that \( R(n) = \frac{1}{2}Cn^2 + O(n^2(\log n)^{9/2-B/2}) \).