MATH 571, FALL 2018, PROBLEMS 5

Due Tuesday 2nd October

1. Let \( f(n) \) be an arithmetic function such that \( f(1) = 1 \). Show that \( f \) is multiplicative if and only if \( f(m)f(n) = f((m, n))f([m, n]) \) for all pairs of positive integers \( m, n \).

2. (Hooley (1972), Montgomery & Vaughan (1979)) By lower and upper bound sifting functions we mean functions \( \lambda^\pm : \mathbb{N} \to \mathbb{R} \) with the properties

\[
\sum_{m|n} \lambda^-_m \leq \sum_{m|n} \mu(m) \leq \sum_{m|n} \lambda^+_m
\]

respectively.

(i) Let \( \lambda^+_d \) be an upper bound sifting function such that \( \lambda^+_d = 0 \) for all \( d > z \). Show that for any \( q \),

\[
0 \leq \frac{\varphi(q)}{q} \sum_{d|q, (d,q)=1} \frac{\lambda^+_d}{d} \leq \sum_d \frac{\lambda^+_d}{d}
\]

(Hint: Multiply both sides by \( P/\varphi(P) = \sum 1/m \) where \( m \) runs over all integers composed of the primes dividing \( P \), and \( P = \prod_{p \leq z} p \)).

(ii) Let \( \Lambda_d \) be real with \( \Lambda_d = 0 \) for \( d > z \). Show that for any \( q \),

\[
0 \leq \frac{\varphi(q)}{q} \sum_{d,e} \frac{\Lambda_d \Lambda_e}{[d,e]} \leq \sum_{d,e} \frac{\Lambda_d \Lambda_e}{[d,e]}
\]

(iii) Let \( \lambda^-_d \) be a lower bound sifting function such that \( \lambda^-_d = 0 \) for \( d > z \). Show that for any \( q \),

\[
\frac{\varphi(q)}{q} \sum_{d|q, (d,q)=1} \frac{\lambda^-_d}{d} \geq \sum_d \frac{\lambda^-_d}{d}
\]