Math 571 Analytic Number Theory I, Fall 2018, Problems 4
Due Tuesday 25th September 2018

These questions are interconnected.

1. Let \( \pi_2(X) \) denote the number of prime numbers \( p \leq X \) for which \( p + 2 \) is also prime and let \( D = \sqrt{X} \) and \( P = \prod_{p \leq D} p \). Define \( b_n \) to be 0 unless \( (n(n + 2), P) = 1 \) in which case take \( b_n = 1 \), and define \( Z = \sum_{n \leq X} b_n \). Prove that \( \pi_2(X) \leq Z + D \).

2. Prove that if \( \omega \) is the multiplicative function with \( \omega(2) = 1 \), \( \omega(p) = 2 \) when \( 2 < p \leq D \) and \( \omega(p^k) = 0 \) otherwise, then

\[
\pi_2(N) \ll \frac{X}{S(D)} + R
\]

where \( S(D) = \sum_{q \leq D} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p - \omega(p)} \) and \( R = \sum_{q \leq D} \sum_{r \leq D} \mu(q)^2 \mu(r)^2 \omega([q, r]) \).

3. Prove that \( S(D) = T(D) + T(D/2) \) where \( T(Q) = \sum_{q \leq Q} \mu(q)^2 \prod_{p|q} \frac{2}{p - 2} \).

4. Prove that if \( p > 2 \), then \( \frac{2}{p - 2} = \sum_{k=1}^{\infty} \frac{2^k}{p^k} \) and that if \( g \) is the multiplicative function with \( g(p^k) = 2^k \), then \( T(Q) \geq \sum_{\substack{q \leq Q \ \text{odd}}} \frac{g(q)}{q} \).

5. Prove that \( g(q) \geq d(q) \) and that \( T(Q) \geq \sum_{\substack{q \leq Q \ \text{odd}}} \frac{d(q)}{q} \).

6. Prove that if \( Q \geq 2 \), then \( \sum_{\substack{q \leq Q \ \text{odd}}} \frac{d(q)}{q} \gg (\log Q)^2 \) and hence that if \( X \geq 2 \), then

\[
\pi_2(X) \ll \frac{X}{(\log X)^2}.
\]

Note: By working a bit harder it can be shown that \( S(\sqrt{X}) \sim \frac{\log^2 X}{C} \) where \( C \) is the twin prime constant, i.e. Selberg’s sieve applied in this way gives an upper bound 8 times larger than the conjectured asymptotic formula.

7. (Brun 1919) Let \( \mathcal{P}_2 \) denote the set of primes \( p \) for which \( p + 2 \) is also prime. Prove that \( \sum_{p \in \mathcal{P}_2} \frac{1}{p} \) converges.