Throughout this course we will use $e(\alpha)$ to denote $e^{2\pi i \alpha}$.

1. (i) Let $a, b, h \in \mathbb{R}$. Show that

$$\int_{a}^{b} e(\alpha h) d\alpha = 0$$

iff $h(b - a) \in \mathbb{Z}$.

(ii) Let $\mathcal{R} = [a, b] \times [c, d]$. Prove that

$$\int_{\mathcal{R}} e(\alpha_1 + \alpha_2) d\alpha_1 d\alpha_2 = 0$$

iff at least one of $b - a$, $d - c$ is an integer.

(iii) Prove that if $\mathcal{R} = [a, b] \times [c, d]$ can be partitioned into $s$ subrectangles $\mathcal{R}_j = [a_j, b_j] \times [c_j, d_j]$ ($j = 1, \ldots, s$) such that for every $j$ at least one of $b_j - a_j$ and $d_j - c_j$ is an integer, then $\mathcal{R}$ has the same property.

2. (i) Prove that

$$\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \begin{cases} 1 & \text{when } q|n, \\ 0 & \text{when } q \nmid n. \end{cases}$$

(ii) Let $a_n, M+1, \ldots, M+N$ be $N$ complex numbers and write

$$S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha)$$

and $Z(q; a) = \sum_{n=M+1}^{M+N} a_n \pmod{q}$.

Prove that

$$\frac{1}{q} \sum_{a=1}^{q} |S(a/q)|^2 = \sum_{a=1}^{q} |Z(q; a)|^2.$$

(iii) Let $Z = Z(1; 1)$. Prove that

$$\frac{1}{q} \sum_{a=1}^{q-1} |S(a/q)|^2 = \sum_{a=1}^{q} |Z(q; a) - Z/q|^2.$$