Math 568, Analytic Number Theory I, Spring 2018, Problems 10

Due Tuesday 27th March

This homework is a continuation of the previous one. Here we investigate the zero-free region for \(L\)-functions formed from quadratic characters. We suppose throughout that \(\chi\) is real but non-principal and that \(\rho_0 = \beta_0 + i\gamma_0\) is a zero of \(L(s, \chi)\). Now we have an additional problem in that \(\chi^2 = \chi_0\), so that \(L(s, \chi^2)\) has a pole at \(s = 1\). This homework is an example of how a simple idea which works in an original situation can be adapted and amended to deal with more awkward ones.

1. Suppose that \(|\gamma_0| \geq 6(1 - \beta_0)\). Since we know that \(L(1, \chi) \neq 0\) we may suppose that \(\gamma_0 \neq 0\).
   (i) Prove that there is a positive constant \(C\) such that if \(1 < \sigma < 2\), then
   \[-\Re\frac{L'}{L}(\sigma + 2i\gamma_0, \chi^2) \leq \frac{\sigma - 1}{(\sigma - 1)^2 + 4\gamma_0^2} + C\log(4 + |\gamma_0|)\].
   (ii) Prove that there is a positive constant \(C\) such that if \(\sigma > 1\), then
   \[0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta_0} + \frac{\sigma - 1}{(\sigma - 1)^2 + 4|\gamma_0|^2} + C\log(4 + |\gamma_0|)\]
   and deduce that \(\beta_0 \neq 1\) and that \(0 \leq \frac{3}{\sigma - 1} - \frac{4}{\sigma - \beta_0} + \frac{\sigma - 1}{(\sigma - 1)^2 + 144(1 - \beta_0)^2} + C\log(4 + |\gamma_0|)\).
   (iii) Prove that there is a positive constant \(C\) such that \(\beta_0 \leq 1 - c/\log(q(4 + |\gamma_0|))\).

2. Suppose that \(0 < |\gamma_0| \leq 6(1 - \beta_0)\). Note that then \(\beta_0 \neq 1\).
   (i) Prove that \(L(\beta_0 - i\gamma_0, \chi) = 0\).
   (ii) Prove that if \(\sigma > 1\), then \(-\frac{L'}{L}(\sigma, \chi_0) - \frac{L'}{L}(\sigma, \chi) > 0\).
   (iii) Prove that if \(\sigma > 1\), then
   \[\frac{1}{\sigma - \beta_0 - i\gamma_0} + \frac{1}{\sigma - \beta_0 + i\gamma_0} = \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0^2} \geq \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 36(1 - \beta_0)^2}\].
   (iv) Prove that there is a positive constant \(C\) such that if \(\sigma > 1\), then
   \[0 \leq \frac{1}{\sigma - 1} - \frac{2(\sigma - \beta_0)}{(\sigma - \beta_0)^2 + 36(1 - \beta_0)^2} + C\log(4 + |\gamma_0|)\).
   (v) Prove that there is a positive constant \(c\) such that \(\beta_0 \leq 1 - c/\log(q(4 + |\gamma_0|))\).

3. Suppose the \(L(s, \chi)\) has two real zeros \(\beta_0\) and \(\beta_1\) with \(\beta_0 \leq \beta_1 \leq 1\). Note that from the proof of Dirichlet’s theorem we have \(\beta_1 < 1\).
   (i) Prove that there is a positive constant \(C\) such that if \(1 < \sigma < 2\), then
   \[-\Re\frac{L'}{L}(\sigma, \chi) \leq -\frac{1}{\sigma - \beta_0} - \frac{1}{\sigma - \beta_1} + C\log 4q \leq -\frac{2}{\sigma - \beta_0} + C\log 4q\].
   (ii) Prove that there is a constant \(C > 0\) such that if \(\sigma > 1\), then \((2)\) is useful
   \[0 \leq \frac{1}{\sigma - 1} - \frac{2}{\sigma - \beta_0} + C\log(4q)\).
   (iii) Prove that there is a positive constant \(c\) such that \(\beta_0 \leq 1 - c/\log(4q)\).

To summarise. The above shows that there is a region \(\{s : \sigma \geq 1 - c/(\log(q(4 + |\ell|)))\}\) in which \(L(s, \chi)\) has at most one zero, and if such a zero exists, then it is real and \(\chi\) is real but non-principal. Such a zero is known as a Siegel zero. No such zero has ever been found.