The object of this and the next homework is to imitate as far as possible for Dirichlet $L$–functions the zero–free region we obtained for $\zeta(s)$. As usual $\tau = 2 + |t|$.

1. Let $\chi_0$ denote the principal character modulo $q$.
   (i) Prove that the zeros of $L(s; \chi_0)$ with $\sigma > 0$ coincide with those of $\zeta(s)$.
   (ii) Prove that if $\sigma > 0$, $s \neq 1$ and $\zeta(s) \neq 0$, then
   $$\frac{L'(s; \chi_0)}{L(s; \chi_0)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p \nmid q} \frac{\log p}{p^s - 1}.$$  
   (iii) Prove that if $5/6 \leq \sigma \leq 2$, then
   $$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log \tau)$$
   where the sum is over all zeros $\rho$ of $\zeta(s)$ for which $|\rho - (3/2 + it)| \leq 5/6$. Hint; apply the argument of Lemma 6.4 to $\zeta(s)(s-1)$.
   (iv) Prove that if $5/6 \leq \sigma \leq 2$, then
   $$-\frac{L'(s; \chi_0)}{L(s; \chi_0)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log q \tau)$$
   where the sum is over all zeros $\rho$ of $L(s, \chi_0)$ for which $|\rho - (3/2 + it)| \leq 5/6$.

2. In this question the results of Homework 9.3 will be useful. Suppose that $\chi$ is a non–principal character modulo $q$. Prove that if $5/6 \leq \sigma \leq 2$ then
   $$-\frac{L'(s; \chi)}{L(s; \chi)} = -\sum_{\rho} \frac{1}{s-\rho} + O(\log q \tau)$$
   where the sum is over all zeros $\rho$ of $L(s, \chi)$ for which $|\rho - (3/2 + it)| \leq 5/6$. Note that as $L(s, \chi)$ has no pole at 1 a uniform treatment can be given.

3. Prove that if $\sigma > 1$, then
   $$\Re\left(-3\frac{L'}{L}(\sigma; \chi_0) - 4\frac{L'}{L}(\sigma + it; \chi) - \frac{L'}{L}(\sigma + 2it; \chi^2)\right) \geq 0.$$  

4. Prove that there is a constant $c > 0$ such that if $\chi$ is a complex character modulo $q$, i.e $\chi^2 \neq \chi_0$, then the region $R_q = \{s : \sigma > 1 - c/\log q \tau\}$ contains no zero of $L(s, \chi)$. 