1. (a) Let \( d_n = \text{lcm}[1, 2, \ldots, n] \). Show that \( d_n = e^{\psi(n)} \).

(b) Let \( P \in \mathbb{Z}[x] \), deg \( P \leq n \). Put \( I = I(P) = \int_0^1 P(x) \, dx \). Show that \( Id_{n+1} \in \mathbb{Z} \), and hence that \( d_{n+1} \geq 1/|I| \) if \( I \neq 0 \).

(c) Show that there is a polynomial \( P \) as above so that \( Id_{n+1} = 1 \).

(d) Verify that \( \max_{0 \leq x \leq 1} |x^2(1-x)^2(2x-1)| = 5^{-5/2} \).

(e) For \( P(x) = (x^2(1 - x)^2(2x - 1))^{2n} \), verify that \( 0 < I < 5^{-5n} \).

(f) Show that \( \psi(10n + 1) \geq (\frac{1}{2} \log 5) \cdot 10n \).

2. (i) Prove that
\[
\int_1^x \frac{\psi(u)}{u^2} \, du = \log x + O(1).
\]

(ii) Prove that \( \limsup_{x \to \infty} \frac{\psi(x)}{x} \geq 1 \) and \( \liminf_{x \to \infty} \frac{\psi(x)}{x} \leq 1 \).

(iii) Prove that if there is a constant \( c \) such that \( \psi(x) \sim cx \) as \( x \to \infty \), then \( c = 1 \).

(iv) Prove that if there is a constant \( c \) such that \( \pi(x) \sim c \frac{x}{\log x} \) as \( x \to \infty \), then \( c = 1 \).

3. Let \( \omega(n) \) denote the number of different prime factors of \( n \). Suppose that \( n \geq 3 \). Prove that
\[
\omega(n) \leq \frac{\log n}{\log \log n} \left( 1 + O\left( \frac{1}{\log \log n} \right) \right).
\]