9. MODULAR FORMS

9.1. Introduction A modular form is an analytic function which satisfies a certain simple relationship under the action of Möbius transformations together with some other simple properties, to be defined. The importance of modular forms is that they underpin a lot of interesting number theoretic structures.

9.2. Properties of Möbius transformations. Let

\[ f(z) = \frac{az + b}{cz + d}; a, b, c, d \in \mathbb{C}, ad \neq bc. \]  

The assumption \( ad \neq bc \) is to ensure that \( f \) is not a constant and is well defined (\( c \) and \( d \) cannot both be 0). This defines \( f(z) \) for all \( z \) in the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) except for \( z = -d/c \) and \( z = \infty \). We extend the definition to \( \hat{\mathbb{C}} \) by taking

\[ F(-d/c) = \infty, \quad f(\infty) = a/c \]

with the usual convention that \( w/0 = \infty \) when \( w \neq 0 \), and vice versa. Clearly \( f \) is analytic on \( \hat{\mathbb{C}} \) except for a simple pole at \( -d/c \) and maps \( \hat{\mathbb{C}} \) onto \( \hat{\mathbb{C}} \). Moreover given \( w \in \hat{\mathbb{C}} \) the point

\[ z = \frac{dw - b}{cw + a} \]

has the property that \( f(z) = w \). Thus

\[ g(z) = \frac{dw - b}{cw + a} \]

is the inverse of \( f \) and \( f \) is a bijection from \( \hat{\mathbb{C}} \) to itself. We have

\[ \frac{f(w) - f(z)}{w - z} = \frac{ad - bc}{(cw + d)(cz + d)} \]  

and letting \( w \to z \) gives

\[ f'(z) = \frac{ad - bc}{(cz + d)^2}. \]

This is non-zero. Thus \( f \) is conformal except possibly at \( z = -d/c \).

Consider the equation

\[ A\overline{z} + Bz + \overline{B}z + C = 0 \]

with \( a, b, c, d \in \mathbb{C}, ad \neq bc \) and \( f(z) = \frac{az + b}{cz + d} \).
where $A$ and $C$ are real. The points on any circle satisfy such an equation with $A \neq 0$ 
$(A|z + B/A|^2 = |B|^2/A - C)$ and the points on any line satisfy such an equation with $A = 0$. 
Suppose that
\[
z = \frac{aw + b}{cw + d}
\]
Then on substituting in the above equations, clearing the denominators $cw + d$. $c\overline{w} + d$ and 
collecting together coefficients of $w$, $w^2$, and $w$ gives
\[
A'w\overline{w} + B'w + B'\overline{w} + C' = 0.
\]
Hence every Möbius transformation maps circles and lines into circles and lines.

Since for any $D \in \mathbb{C}\\{0\}$ we have
\[
\frac{az + b}{cz + d} = \frac{(a/D)z + b/D}{(c/D)z + d/D}
\]
and
\[
\frac{a}{D} \cdot \frac{d}{D} - \frac{b}{D} \cdot \frac{c}{D} = \frac{ad - bc}{D^2}
\]
we can suppose that
\[
ad - bc = 1.
\]
We can associate with
\[
f(z) = \frac{az + b}{cz + d}
\]
the $2 \times 2$ matrix
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Then det $A = 1$. If $f$ and $g$ are Möbius transformations with associated matrices $A$ and $B$, 
then $(f \circ g)(z) = f(g(z))$ has associated matrix $AB$. The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 
corresponds to $f(z) = z$ and the inverse matrix
\[
A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{(note det } A^{-1} = da - bc = 1)
\]
is associated with $f^{-1}(z)$.

9.3. **The modular group.** The set of all Möbius transforms form a group under 
composition, and this is associated with $\text{SL}_2(\mathbb{C})$. We will mostly be concerned with 
the subgroup $\text{SL}_2(\mathbb{Z})$. When $a$, $b$, $c$, $d$ are real one has
\[
\Im f(z) = \Im \frac{(az + b)(cz + d)}{|cz + d|^2} = \Im \frac{z + bc(z + \overline{z})}{|cz + d|^2} = \Im \frac{z + bc(z + \overline{z})}{|cz + d|^2},
\]
so
\[ \Im f(z) = \frac{\Im z}{|cz + d|^2}. \]

Thus \( f \) maps the upper half-plane
\[ \mathbb{H} = \{ z : \Im z > 0 \} \]
bijectively to \( \mathbb{H} \).

Another important remark is that
\[ \frac{az + b}{cz + d} = \frac{(-a)z + (-b)}{(-c)z + (-d)}. \]

In other words,
\[ A \quad \text{and} \quad A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]
give identical maps. Thus it is normal to restrict one’s attention to
\[ \text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\{ \pm I \} \]
and
\[ \text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{ \pm I \}. \]

Since \( \text{PSL}_2(\mathbb{Z}) \) is a handful to write one tends to use a shorthand. Serre uses \( G \) and Apostol and many others use \( \Gamma \), and we will follow the herd. This group is called the modular group.

**Theorem 9.1.** The modular group \( \Gamma \) is generated by
\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]
i.e. every \( A \in \Gamma \) can be expressed in the form
\[ A = T^{n_1} S^{n_2} \ldots S T^{n_k} \]
where the \( n_j \in \mathbb{Z} \).

**Remark.** The matrices \( S \) and \( T \) correspond to \( z \rightarrow -1/z \) and \( z \rightarrow z + 1 \) respectively.

**Proof.** Since we are working modulo \( \pm I \) we need only consider the
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
with \( c \geq 0 \). We argue by induction on \( c \). If \( c = 0 \), then \( ad = 1 \) so \( a = d = \pm 1 \) and

\[
A = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} = T^{\pm b}.
\]

If \( c = 1 \), then \( ad - bc = 1 \), so \( b = ad - 1 \) and

\[
A = \begin{pmatrix} a & ad - 1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = T^a ST^d.
\]

Now suppose that \( c > 1 \) and assume the conclusion for all \( A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \) with \( 0 \leq c' < c \). Since \( ad - bc = 1 \) we have \((d,c) = 1\). Hence \( d = cq + r \) where \( 0 < r < c \). Then

\[
AT^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - aq \\ c & r \end{pmatrix}
\]

and

\[
AT^{-q} S = \begin{pmatrix} a & b - aq \\ c & r \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b - aq & -a \\ r & -c \end{pmatrix}.
\]

The only other observation we need is that \( S^2 = -I \equiv I \).

### 9.3 Fundamental Domains

We are interested in the behaviour of the modular group acting on points in \( \mathbb{H} \).

**Definition 9.1.** Let \( G \) be a subgroup of \( \Gamma \). Two points \( z, w \in \mathbb{H} \) are equivalent under \( G \) when \( z = Aw \) for some \( A \) in \( G \). This equivalence relation partitions \( \mathbb{H} \) into equivalence classes called orbits (of \( G \)), i.e. for a given \( z \in \mathbb{H} \) an orbit is the set of all \( Az \) with \( A \in G \).

**Definition 9.2.** Let \( G \) be a subgroup of \( \Gamma \). Any simply connected subset \( D_G \) of \( \mathbb{H} \) is called a fundamental domain (or region) of \( G \) when it satisfies the following.

(i) No two distinct points of \( D_G \) are in the same orbit of \( G \).

(ii) Every orbit of \( G \) contains a point of \( D_G \).

When \( G = \Gamma \) we simplify the notation by writing \( D \) for \( D_\Gamma \).

**Theorem 9.2.** Let

\[
D = \{ z : \text{either } |z| > 1, -\frac{1}{2} \leq \Re z < \frac{1}{2} \text{ and } \Im z > 0, \text{ or } |z| = 1, -\frac{1}{2} \leq \Re z \leq 0 \text{ and } \Im z > 0 \}.
\]

Then \( D \) is a fundamental domain for \( \Gamma \).

**Proof.** Suppose that \( z \in \mathbb{H} \). Let \( N \) denote the number of integers \( c \) and \( d \) such that \( |cz + d| \leq 1 \). Since \( \Im z > 0 \) we have \( |c| \Im z = |\Im(cz + d)| \leq |cz + d| \leq 1 \), so that \( |c| \leq 1/3z \)
and $|d| = |cz + d - cz| \leq |cz + d| + |cz| \leq 1 + |z|/3z$. Thus $N \leq (1 + 2/3z)(3 + 2|z|/3z)$. Thus for all but $N$ choices of $c$ and $d$ we have $|cz + d| > 1$ and so

$$\Im(Az) = \frac{3z}{|cz + d|^2} < \Im z.$$ 

Thus there is an $A \in \Gamma$ for which $\Im(Az)$ is maximal. Now choose $n \in \mathbb{Z}$ so that $-\frac{1}{2} \leq \Re Az + n < \frac{1}{2}$. In other words $-\frac{1}{2} \leq \Re T^n Az < \frac{1}{2}$. Then $\Im T^n Az = \Im Az$ is also maximal. If $|T^n Az| < 1$, then $|ST^n Az| = | -1/(T^n Az)| > 1$ so that $\Im(ST^n Az) = \Im(T^n Az)|T^n Az|^2 > \Im(T^n Az) = \Im(Az)$ which would contradict the maximality of $\Im(Az)$. Hence $|T^n Az| \geq 1$. If $|T^n Az| > 1$ or $|T^n Az| = 1$ and $-\frac{1}{2} \leq \Re T^n Az \leq 0$, then $T^n Az \in \mathbb{D}$. If $|T^n Az| = 1$ and $0 < \Re T^n Az < \frac{1}{2}$, then $ST^n Az \in \mathbb{D}$.

We complete the proof by showing that if $z, w \in \mathbb{D}$, $A \in \Gamma$, $z = Aw$, then $z = w$. As usual we associate $A$ with the element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\text{SL}_2 \mathbb{Z}$. If $c = 0$, then $ad = 1$, $a = d = \pm 1$ and $w = Az = z \pm b$. Hence $b = 0$ and $w = z$. Now suppose that $c \neq 0$. We have (3). Since $A^{-1}w = z$ we also have

$$\Im z = \Im(A^{-1}w) = \frac{3z}{|cw + a|^2}. \tag{4}$$

Moreover

$$|cz + d|^2 = c^2|z|^2 + 2cd\Re z + d^2 \geq c^2|z|^2 - |cd| + d^2 \geq c^2 - |cd| + d^2.$$

Since $c \neq 0$ and $u^2 - u + 1$ has no real roots we have

$$|cz + d|^2 \geq c^2|z|^2 - |cd| + d^2 \geq 1. \tag{5}$$

Likewise

$$|cw + a|^2 \geq c^2|w|^2 - |ca| + a^2 \geq 1. \tag{6}$$

Note that equality could only occur in these last two inequalities if $|z| = |w| = 1$. By (2) and (5), $\Im w \leq \Im z$ and by (4) and (6), $\Im z \leq \Im w$, so $\Im z = \Im w$. But then we have equality in (5) and (6), so $|z| = |w|$, and hence $|Rz| = |Rw|$. But on that part of $\mathbb{D}$ with $|z| = 1$ we have $Rz \leq 0$ hence $Rz = Rw$.

**Exercises 9.1.**

$\Gamma$ denotes the modular group and $S, T$ are its generators, $S(z) = -1/z$, $T(z) = z + 1$. Given a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with real coefficients, $d = d_Q = b^2 - 4ac$ is called the discriminant of $Q$. 
1. (i) Find all elements \( A \) of \( \Gamma \) which commute with \( S \).
(ii) Find all elements \( A \) of \( \Gamma \) which commute with \( T \).
(iii) Find the smallest \( n > 0 \) such that \((ST)^n = I\).
(iv) Determine all \( A \) in \( \Gamma \) which leave \( i \) fixed.
(v) Determine all \( A \) in \( \Gamma \) which leave \( \rho = e(1/3) \) fixed.

2. Prove that if \( A \in \Gamma \), and \((x,y)^T = A(x',y')^T\), then the quadratic form \( Q' \) defined by \( Q'(x',y') = Q(x,y) \) satisfies \( d_{Q'} = d_Q \). Two forms related in this way are called equivalent. This relation separates all forms into equivalence classes. The forms in the same class have the same discriminant and the ranges \( Q(\mathbb{Z}^2) \) coincide.

In the remaining exercises it will be supposed that the quadratic forms have positive coefficients of \( x^2 \) and \( y^2 \) and negative discriminant. The associated polynomial \( Q(z,1) \) has two complex roots. The one in \( \mathbb{H} \) is called the representative of \( Q \).

3. (i) If \( d \) is fixed, prove that there is a bijection between the set of forms with discriminant \( d \) and the members of \( \mathbb{H} \).
(ii) Prove that two quadratic forms with discriminant \( d \) are equivalent iff their representatives are equivalent under \( \Gamma \).

A reduced form is one whose representative lies in the fundamental domain \( \mathbb{D} \), the set of \( z \) such that either \( |z| > 1 \) and \(-1/2 \leq \Re z < 1/2 \) or \( |z| = 1 \) and \(-1/2 \leq \Re z \leq 0 \). Thus two reduced forms are equivalent iff they are identical, and moreover each equivalence class contains exactly one reduced form.

4. Prove that \( Q(x,y) = ax^2 + bxy + cy^2 \) is reduced iff either \(-a < b \leq a < c \) or \( 0 \leq b \leq a = c \).

In questions 5,6 it is assumed that the quadratic forms have integer coefficients.

5. Prove that the number of reduced forms with a given discriminant \( d < 0 \) is finite. The number of such classes is called the class number and is denoted by \( h(d) \).

6. When \( d = -3,-4,-7,-8,-11,-15,-19,-20,-23 \) determine all reduced forms with discriminant \( d \), and the corresponding class number \( h(d) \).

7. (i) Prove that if \( p \equiv 1 \pmod{3} \), then \( \left( \frac{-3}{p} \right)_L = 1 \).
(ii) Let \( \mathcal{M} = \{ n \in \mathbb{N} : p|n \implies p \equiv 1 \pmod{3} \} \). Prove that if \( n \in \mathcal{M} \), then \( x^2 + 3 \equiv 0 \pmod{4n} \) is soluble in \( x \).
(iii) Let \( n \in \mathcal{M} \). Prove that there are \( a, B \in \mathbb{Z} \) with \( a > 0 \) such that \( B^2 + 12 = 4an \). Let \( b = B - 2a, c = (b^2 + 12)/4a \). Prove that \( b^2 - 4ac = -12 \) and \( a + b + c = n \).
(iv) Let \( h(d) \) be defined as in homework 11. Prove that \( h(-12) = 2 \).
(v) Prove that if \( n \in \mathcal{M} \), then \( x^2 + 3y^2 = n \) is soluble in integers \( x \) and \( y \).

8. (i) Prove that if \( p \equiv 1, 4 \pmod{7} \), then \( \left( \frac{-2}{p} \right)_L = 1 \).
(ii) Let \( \mathcal{N} = \{ n \in \mathbb{N} : p|n \implies p \equiv 1, 4 \pmod{7} \} \). Prove that if \( n \in \mathcal{N} \), then \( x^2 + 7 \equiv 0 \pmod{4n} \) is soluble in \( x \).
(iii) Let $n \in \mathbb{N}$. Prove that there are $a, B \in \mathbb{Z}$ with $a > 0$ such that $B^2 + 7 = 4an$. Let $b = B - 2a$, $c = (b^2 + 7)/4a$. Prove that $b^2 - 4ac = -7$ and $a + b + c = n$.

(iv) Recall from homework 11 that $h(-7) = 1$. Prove that if $n \in \mathbb{N}$, then $x^2 + xy + 2y^2 = n$ is soluble in integers $x$ and $y$.

(v) Let $n \in \mathbb{N}$. Prove that $x^2 + 7y^2 = 4n$ is soluble in integers $x$, $y$. Moreover prove that $x$ and $y$ are both even, and thus $x^2 + 7y^2 = n$ is also soluble in integers $x$, $y$.

### 9.4. Modular functions.

**Definition 9.3.** Let $k \in \mathbb{Z}$. Then $f$ is weakly modular of weight $2k$ when $f$ is meromorphic on $\mathbb{H}$ and satisfies

$$f(z) = (cz + d)^{-2k} f\left(\frac{az + b}{cz + d}\right)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$.

**Theorem 9.3.** Let $f$ be meromorphic on $\mathbb{H}$. Then $f$ is weakly modular of weight $2k$ where $k \in \mathbb{Z}$ if and only if

$$f(z + 1) = z + 1,$$

$$f(-1/z) = z^{2k} f(z)$$

for all $z \in \mathbb{H}$.

**Proof.** If $f$ is weakly modular of weight $2k$, then at once it must satisfy the above relations. Suppose conversely that it satisfies them. Then we can apply Theorem 9.1 to obtain $f(Az)$ where $A$ is any member of $\text{SL}_2(\mathbb{Z})$. We need to show that the correct factor $(cz + d)^{-2k}$ arises. It suffices to show that if $A = S$ or $T$, so that $a = 1$, $b = 1$, $c = 0$, $d = 1$ or $a = 0$, $b = 1$, $c = -1$, $d = 0$, and

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z}),$$

then, for example inductively on the number of terms in Theorem 9.1, either

$$((c\alpha + d\gamma)z + c\beta + d\delta)^{-2k} f(ABz) = (\gamma z + \delta)^{-2k} f(Bz) = f(z)$$

or

$$((c\alpha + d\gamma)z + c\beta + d\delta)^{-2k} f(ABz) = (\alpha z + \beta)^{-2k} f\left(-\frac{1}{Bz}\right) = (\alpha z + \beta)^{-2k} f\left(\frac{-\gamma z - \delta}{\alpha z + \beta}\right) = f(z).$$

The first of the above relationships tells us that $f$ is periodic with period 1. Thus we can write $f$ as a function of

$$q = e^{2\pi i z}.$$
More precisely we could put $|q| = e^{-2\pi |z|}$, $\arg q = 2\pi (\Re z - |z|)$. Then $z \in \mathbb{Z}$ and $z$ satisfying, say, $-\frac{1}{2} < \Im z < \frac{1}{2}$ is equivalent to $0 < |q| < 1$. In other words, regardless of the branch of the logarithm,

$$f(z) = f \left( \frac{\log q}{2\pi i} \right) = \tilde{f}(q)$$

where $\tilde{f}$ is meromorphic on the punctured disc $A = \{ q : 0 < |q| < 1 \}$. If we can extend $\tilde{f}$ to being meromorphic (or analytic) at $0$, then we can say that $f$ is meromorphic (or analytic) at $\infty$. More precisely this would mean that $\tilde{f}$ has a Laurent expansion about $0$,

$$\tilde{f}(q) = \sum_{n=-N}^{\infty} a_n q^n.$$  

**Definition 9.4.** A weakly modular function is called a modular function when it is meromorphic at $\infty$, and if it is analytic there we write $f(\infty) = \tilde{f}(0)$. A modular function which is analytic on $\mathbb{H} = \mathbb{H} \cup \{ \infty \}$ is called a modular form. If such a function is $0$ at $\infty$, then it is called a cusp form.

Thus a modular form of weight $2k$ is given by a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

which converges for all $q \in D = \{ q : |q| < 1 \}$ and satisfies

$$f(-1/z) = z^{2k} f(z).$$

It is a cusp form when $a_0 = 0$. The expansion (7) is called the Fourier expansion of $f$.

**9.5. Lattice functions and modular forms.** A lattice $\Lambda$ can be thought of in various ways. One is that it is a discrete subgroup of a finite dimensional vector space $V$ over $\mathbb{R}$ and there is an $\mathbb{R}$-basis $(e_1, \ldots, e_n)$ of $V$ which is a $\mathbb{Z}$-basis of $\Lambda$. Thus when $V = \mathbb{C}$ we could suppose that there are $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$ such that $\Im(\omega_1/\omega_2) > 0$ and

$$\Lambda(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2,$$

i.e.

$$\Lambda(\omega_1, \omega_2) = \{ m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z} \}.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \in \text{SL}_2(\mathbb{Z}).$$

Then

$$\omega'_1 = a\omega_1 + b\omega_2,$$
$$\omega'_2 = c\omega_1 + d\omega_2.$$
is another basis of $\Lambda(\omega_1, \omega_2)$. Since
\[
\frac{\omega'_1}{\omega'_2} = \frac{a\omega_1/\omega_2 + b}{c\omega_1/\omega_2 + d}
\] (8)
it follows from (2) that $\Im(\omega'_1/\omega'_2) > 0$ also. Let
\[
\mathcal{M} = \{ (\omega_1,\omega_2) \in (\mathbb{C} \setminus \{0\})^2 : \Im(\omega_1/\omega_2) > 0 \}.
\]

**Theorem 9.4.** Two elements of $\mathcal{M}$ define the same lattice if and only if they are congruent modulo $\text{SL}_2(\mathbb{Z})$.

**Proof.** In view of the discussion above it suffices to show that if $(\omega_1, \omega_2)$ and $(\omega'_1, \omega'_2)$ define the same lattice, then (8) holds with $\det A = 1$. In fact it suffices to show that (8) holds with $\det A = \pm 1$ for then the positive sign follows from (2) and the facts that $\Im(\omega_1/\omega_2) > 0$ and $\Im(\omega'_1/\omega'_2) > 0$.

We have $\omega' = A\omega$ and $\omega = A'\omega'$ where $w$ denotes the column vector $(w_1, w_2)^T$ and $A, A' \in \text{GL}_2(\mathbb{Z})$. Then $\omega = A'\omega' = A' A \omega$ and since $\omega_1$ and $\omega_2$ are linearly independent over $\mathbb{Z}$ we have $A' A = I$. Thus $\det A' \det A = 1$. But $\det A', \det A \in \mathbb{Z}$. Hence $\det A = \pm 1$.

Let $\mathcal{R}$ denote the set of lattices $\Lambda(\omega_1, \omega_2)$ with $(\omega_1, \omega_2) \in \mathcal{M}$ and suppose that $F$ satisfies
\[
F : \mathcal{R} \to \mathbb{C}.
\]
Let $k \in \mathbb{Z}$. Then $F$ is of weight $2k$ when
\[
F(\lambda \Lambda) = \lambda^{-2k} F(\Lambda)
\]
for every $\Lambda \in \mathcal{R}$ and every $\lambda \in \mathbb{C}$. Now $\Lambda$ is invariant under the action of $\text{SL}_2 \mathbb{Z}$. Moreover
\[
\lambda \Lambda(\omega_1, \omega_2) = (\lambda \omega_1, \lambda \omega_2).
\]
Thus
\[
\omega_2^{-2k} F(\Lambda(\omega_1, \omega_2)) = F(\omega_2^{-1} \Lambda(\omega_1, \omega_2)) = F(\Lambda(\omega_1/\omega_2, 1)).
\]
Thus there is a function $f$ on $\mathbb{H}$ such that
\[
F(\Lambda(\omega_1, \omega_2)) = \omega_2^{-2k} f(\omega_1/\omega_2).
\]
Since $F$ is invariant under $\text{SL}_2 \mathbb{Z}$,
\[
f(z) = (cz + d)^{-2k} f(Az) \quad \text{for all } A \in \text{SL}_2(\mathbb{Z}), \ z \in \mathbb{H}.
\]
On the other hand given such a function $f$ we can reverse the process and obtain a lattice function of weight $2k$. Thus lattice functions are a fruitful way of creating and identifying modular forms. Perhaps the easiest way is by considering Eisenstein series
\[
G_k(\Lambda) = \sum_{\omega \in \Lambda(\omega_1, \omega_2) \setminus \{0\}} \frac{1}{\omega^{2k}} = \sum_{m,n \neq 0,0} \frac{1}{(m \omega_1 + n \omega_2)^{2k}}.
\]
The corresponding function on $\mathbb{H}$ is

$$G_k(z) = \sum_{m,n \neq 0,0} \frac{1}{(mz + n)^{2k}}.$$  \hfill (9)

By the way the above construction would fail if the exponent $2k$ were to be replaced by an odd exponent, for then the function would be identically 0.

Before proceeding further we need to discuss convergence. The following Lemma provides a basis for sufficiency.

**Lemma.** Suppose that $\sigma > 2$, $0 < v_1 < v_2$ and $0 < u$, and $\mathcal{H}$ denotes the closed rectangle \{ $z \in \mathbb{C} : -u \leq \Re z \leq u, v_1 \leq \Im z \leq v_2$ \}. Then

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \sup_{z \in \mathcal{H}} \frac{1}{|mz + n|^{\sigma}}$$

converges.

**Proof.** For each pair $(m,n)$ which we sum over,

$$|mz + n|^2 = (m\Re z + n)^2 + (m\Im z)^2 \geq v_1^2 m^2$$

and

$$|mz + n|^2 = |z|^2 |m + nz^{-1}|^2 \geq |z|^2 (n \Im(z^{-1}))^2 = n^2 |z|^{-2} (\Im z)^2 \geq \frac{v_1^2 n^2}{u^2 + v_2^2}.$$ 

Thus $|mz + n|^{-1} \ll (\max(m,n))^{-1}$ uniformly for $z \in \mathcal{H}$, and so for any real $R > 1$

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \mid |mz + n| \leq R} \sup_{z \in \mathcal{H}} \frac{1}{|mz + n|^{\sigma}} \ll \sum_{n \in \mathbb{Z} \setminus \{0\} \mid n \ll R} |n|^{-\sigma} + \sum_{m,n} |n|^{-\sigma} \ll \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-1}}$$

**Theorem 9.5.** Let $k \in \mathbb{N}$, $k > 1$. Then the Eisenstein series $G_k(z)$ given by (9) is a modular form of weight $2k$ and $G_k$ has the Fourier expansion

$$G_k(z) = 2\zeta(2k) + \frac{2^{2k+1} \pi^{2k} (-1)^k}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi inz}.$$ 

**Proof.** By the Lemma $G_k$ is uniformly and absolutely convergent on $\mathcal{H}$, and each term of the series is analytic on $\mathbb{H}$. Hence, by a theorem of Weierstrasse $G_k$ is analytic in $\mathcal{H}$, and hence at every point of $\mathcal{H}$.

We have $m(z+1) + n = mz + m + n = 0 \cdot z + 0$ if and only if $m = n = 0$. Thus

$$G_k(z+1) = G_k(z).$$
Obviously $m(-1/z) + n = (-1/z)((-n)z + m)$, so

$$G_k(-1/z) = z^{2k}G_k(z).$$

Thus, by Theorem 9.3, $G_k$ is weakly modular. We have to show that $G_k$ is analytic at $\infty$. We establish this by exhibiting a Fourier series for $G_k$ that is analytic at $q = 0$. We start from the partial fraction decomposition

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=\infty}^{n=-\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right)$$

which is valid for all $z \in \mathbb{C} \setminus \mathbb{Z}$ and converges locally uniformly and absolutely in that domain. For $z \in \mathbb{H}$ we have

$$\pi \cot \pi z = \pi i \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} = \pi i \frac{q + 1}{q - 1}.$$

Thus

$$\frac{1}{z} + \sum_{n=\infty}^{n=-\infty} \left( \frac{1}{z + n} - \frac{1}{n} \right) = -\pi i \left( 1 + 2 \sum_{r=1}^{\infty} q^r \right).$$

Differentiating both sides $l$ times gives

$$\frac{(-1)^l l!}{z^{l+1}} + \sum_{n=\infty}^{n=-\infty} \frac{(-1)^l l!}{(z + n)^{l+1}} = -(2\pi i)^{l+1} \sum_{r=1}^{\infty} r^le^{2\pi irz}.$$

Now for $m \in \mathbb{N}$, we have $z \in \mathbb{H}$ if and only if $mz \in \mathbb{H}$. Thus

$$\sum_{m=1}^{\infty} \sum_{n=\infty}^{n=-\infty} \frac{(-1)^l l!}{(mz + n)^{l+1}} = -(2\pi i)^{l+1} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^l e^{2\pi irmz} = -(2\pi i)^{l+1} \sum_{n=1}^{\infty} \sigma_l(n) e^{2\pi inz}$$

where

$$\sigma_l(n) = \sum_{d|n} d^l.$$

When $l$ is odd, say $l = 2k - 1 \geq 3$,

$$\sum_{m=1}^{\infty} \sum_{n=\infty}^{n=-\infty} \frac{(2k-1)!}{(mz + n)^{2k}} = (2\pi i)^{2k} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi inz}.$$

Moreover

$$\sum_{m=\infty}^{\infty} \sum_{n=\infty}^{n=-\infty} \left( \frac{1}{(mz + n)^{2k}} \right) = \sum_{m=1}^{\infty} \sum_{n=\infty}^{n=-\infty} \left( \frac{1}{(mz + n)^{2k}} \right).$$
Hence
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^{2k}} = \frac{2^{2k+1} \pi^{2k} (-1)^k}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{2\pi inz}.
\]

Adding in the terms with \(m = 0\) (and \(n \neq 0\)) gives an extra \(2\zeta(2k)\).

Recall that
\[
\zeta(2k) = (-1)^{k-1} 2^{2k-1} \pi^{2k} B_{2k} / (2k)!.
\]

where \(B_l\) is the \(l\)-th Bernoulli number, and

**Table 1**

<table>
<thead>
<tr>
<th>(k)</th>
<th>(B_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/1 = 1.00000 00000</td>
</tr>
<tr>
<td>1</td>
<td>-1/2 = -0.50000 00000</td>
</tr>
<tr>
<td>2</td>
<td>1/6 = 0.16666 66667</td>
</tr>
<tr>
<td>4</td>
<td>-1/30 = -0.03333 33333</td>
</tr>
<tr>
<td>6</td>
<td>1/42 = 0.02380 95238</td>
</tr>
<tr>
<td>8</td>
<td>-1/30 = -0.03333 33333</td>
</tr>
<tr>
<td>10</td>
<td>5/66 = 0.07575 75758</td>
</tr>
<tr>
<td>12</td>
<td>-691/2730 = -0.25311 35531</td>
</tr>
<tr>
<td>14</td>
<td>7/6 = 1.16666 66667</td>
</tr>
<tr>
<td>16</td>
<td>-3617/510 = -7.09215 68627</td>
</tr>
<tr>
<td>18</td>
<td>43867/798 = 54.97117 79449</td>
</tr>
<tr>
<td>20</td>
<td>-174611/330 = -529.12424 24242</td>
</tr>
</tbody>
</table>

Thus \(\zeta(4) = \frac{\pi^4}{90}\), \(\zeta(6) = \frac{\pi^6}{945}\). There are various standard notations. For example

\[g_2(z) = 60G_2(z), \ g_3(z) = 140G_3(z)\]

and then it follows that the Fourier expansion of

\[
\Delta(z) = g_2(z)^3 - 27g_2(z)^2
\]

has no constant term. Thus \(\Delta\) is a cusp form of weight 12. By multiplying out the series and collecting together like powers of \(q\) it follows that

\[
\Delta(z) = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}
\]

where the \(\tau(n)\) are integers with \(\tau(1) = 1, \ \tau(2) = -24\). This function was first studied by Ramanujan, and we will come back to it in Chapter 10.
Other standard notation is

\[ E_k(z) = G_k(z)/(2\zeta(2k)) \]

and then the Fourier expansion has constant term 1. Moreover, by (11),

\[
\frac{2^{2k+1} \pi^{2k} (-1)^k}{(2k-1)! 2\zeta(2k)} = \frac{2^{2k+1} \pi^{2k} (-1)^k (2k)!}{(2k-1)! (-1)^k 12^{2k} \pi^{2k} B_{2k}} = -\frac{4k}{B_{2k}}.
\]

Thus

\[ E_k(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi inz}. \]

It should be born in mind that some authors write \( G_{2k} \) and \( E_{2k} \) for \( G_k \) and \( E_k \) respectively.

9.6. Zeros and poles of modular functions. For a function \( f \), meromorphic on \( \mathbb{H} \) and not identically 0 we define, for each \( w \in \mathbb{H} \), \( v = v_w(f) \) so that \( f(z)(z-w)^{-v} \) is analytic and non-zero at \( w \). \( v_w(f) \) is called the order of \( f \) at \( w \). If \( v_w(f) \) is positive, then it is the order of the zero of \( f \) at \( w \). Likewise if \( v_w(f) \) is negative, then \( -v_w(f) \) is the order of the pole at \( w \). When \( f \) is a modular function of weight \( 2k \) and \( w \) and \( Aw \) are both finite, then the relationship

\[ f(z) = (cz + d)^{-2k} f(Az) \]

shows that \( v_w(f) = v_{Aw}(f) \). For points at \( \infty \) we define \( v_{\infty} \) to be the order (in \( q \)) of \( \tilde{f}(q) \).

**Theorem 9.6.** Let \( f \) be a modular function of weight \( 2k \), not identically 0. Then

\[ v_{\infty} + \frac{1}{2} v_i(f) + \frac{1}{3} v_{\rho}(f) + \sum_{w \in \mathbb{D}^*} v_w(f) = \frac{k}{6} \]

where \( \rho = e^{2\pi i/3} \) and \( \mathbb{D}^* = \mathbb{D} \setminus \{i, \rho\} \).

**Proof.** We consider

\[ \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz \]

where \( C \) is, with some provisos, the contour consisting of the horizontal line \( L \) from \( \frac{1}{2} + iY \) to \( -\frac{1}{2} + iY \) (where \( Y > 1 \)), the vertical line segment \( L_- \) from \( -\frac{1}{2} + iY \) to \( \rho \), the circular arc \( \hat{C} \) of radius 1, centred at 0 from \( \rho \) to \( -\hat{p} \) through \( i \) and the vertical line segment \( L_+ \) from \( -\hat{p} \) to \( \frac{1}{2} + iY \). The provisos are (i) that \( Y \) is chosen so that \( L \) avoids any singularity of the integrand, and (ii) if the integrand has a singularity on the remaining path, then the contour traverses a small detour consisting of a circular arc of small radius centred at the singularity and oriented so that singularities in \( \mathbb{D}^* \) are included in the interior and those not in \( \mathbb{D}^* \) are excluded from the interior. The integrand has singularities precisely at the
zeros and poles of $f$ and the residue at such points is the order of $f$ at that point. Thus, by Cauchy’s integral formula,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = \sum_{w \in \mathbb{D}^*(Y)} v_w(f)$$

where $\mathbb{D}^*(Y) = \{ z \in \mathbb{D}^* : \Im z \leq Y \}$.

Since $f(z) = f(z + 1)$ we have

$$\frac{f'(z)}{f(z)} = \frac{f'(z + 1)}{f(z + 1)}$$

and thus

$$\int_{L_-} \frac{f'(z)}{f(z)} \, dz = -\int_{L_+} \frac{f'(z)}{f(z)} \, dz$$

where any possible detours, except any which might occur at $\rho$ and $-\rho$, are included in the paths. In view of the relationship (13) such detours will match exactly. We also have

$$f(z) = z^{-2k} f(-1/z)$$

so

$$f'(z) = -2kz^{-2k-1} f(-1/z) - z^{-2k-2} f'(-1/z).$$

Thus

$$\frac{f'}{f}(z) = -\frac{2k}{z} - \frac{f'}{z^2 f}(-1/z).$$

Let $C_-$ be the subpath of $C$ from $\rho$ to $i$ and $C_+$ the subpath from $i$ to $-\rho$, with the pviso that we exclude any possible detours around $\rho$, $i$ and $-\rho$. Then

$$\int_{C_-} \frac{f'}{f} \, dz = \int_{C_-} \frac{2k}{z} - \frac{f'}{z^2 f}(-1/z) \, dz$$

and by the change of variable $w = -1/z$ this is

$$-2k \left( -\frac{2\pi i}{12} \right) - \int_{C_+} \frac{f'}{f} \, dw.$$

Thus

$$\int_{C} \frac{f'}{f} \, dz = \frac{2\pi i k}{6}.$$

There may be detours around $\rho$, $i$ and $-\rho$. If $f$ has a zero or pole at $\rho$, then by there will be one at $-\rho$ of the same order. Letting the radius of the detour at $\rho$ tend to 0 we pick up
the $i$ times the residue times minus the angle subtended by the paths $L_-$ and $C$ at $\rho$, which is $-2\pi/6$. Hence the contribution from $\rho$ and $-\bar{\rho}$ to the integral along the path is 

$$-2\pi iv_\rho(f)/3.$$ 

A detour around $i$ likewise will pick up $i$ times the residue times minus the angle subtended by the path $C$ at $i$, which is $-\pi$. Thus the contribution from $i$ to the integral along the path is 

$$-\pi iv_i(f).$$

It remains to deal with the contribution from $L$. To summarise so far

$$\int_L \frac{f'}{f}(z)dz + \frac{2\pi ik}{6} - 2\pi iv_\rho(f)/3 - \pi iv_i(f) = 2\pi i \sum_{w \in \mathbb{D}^*(Y)} v_w(f).$$

In the integral along $L$ we make the substitution $q = e^{2\pi iz}$. Then $L$ is transformed into the circle $C_0$ centred at 0 of radius $e^{-2\pi Y}$ and traversed in the clockwise direction. Moreover as $\tilde{f}(q) = f(z)$, we have $\frac{\tilde{f}'}{\tilde{f}}(q) \frac{dq}{dz} = \frac{f'(z)}{f(z)}$. Hence

$$\int_L \frac{f'}{f}(z)dz = \int_{C_0} \frac{f'}{f}(q) dq.$$ 

Since $\tilde{f}(q)$ is meromorphic at 0 there will be a punctured disc $\mathcal{A}$ centred at 0 on which $\tilde{f}$ is analytic. Thus if $Y$ is large enough $C_0 \subset \mathcal{A}$. Hence by Cauchy’s integral formula

$$\int_{C_0} \frac{f'}{f}(q) dq = -2\pi iv_\infty(f).$$

Moreover

$$\sum_{w \in \mathbb{D}^*(Y)} v_w(f) = \sum_{w \in \mathbb{D}^*} v_w(f).$$

This completes the proof of the theorem.

When $k \in \mathbb{Z}$, let $M_k$ denote the vector space over $\mathbb{C}$ of modular forms of weight $2k$, and let $M_k^0$ denote the subspace of cusp forms of weight $2k$. Let $f$ be a non-cusp member of $M_k$. If $g$ is another, then for some scalar $c$, $f - cg$ will be a cusp form. Thus every non-cusp member of $M_k$ is a linear combination of $f$ and a cusp form. Thus

$$\dim(M_k \setminus M_k^0) \leq 1.$$ (15)

Indeed a concomitant argument shows that if $M_k^j$ denotes the subspace of $f \in M_k$ in which $v_\infty(f) \geq j + 1$ in $q$, then

$$\dim(M_k^{j-1} \setminus M_k^j) \leq 1.$$ (16)
When $k \geq 2$, $G_k \in M_k$ but $G_k \not\in M_k^0$. Thus

$$M_k = G_k \oplus M_k^0 \quad (k \geq 2).$$

(17)

Let $f \in M_k$, so that $f$ is analytic on $\tilde{H}$. In Theorem 9.6 each $v_z(f)$ is non-negative. Hence $k \geq 0$. Thus $M_k$ is empty when $k < 0$. When $k = 1$ there is no solution to $l + \frac{1}{2}m + \frac{1}{2}n = \frac{k}{6}$ with $l, m, n$ non-negative. Hence

$$M_k = \emptyset \quad (k \geq 2).$$

When $k = 6$, we have seen that $\Delta$ is a cusp form of weight 12. Thus $v_{\infty}(\Delta) \geq 1$. Hence all other $v_z(\Delta)$ are 0. Thus $\Delta$ does not vanish on $\mathbb{H}$ and has a simple zero at $\infty$. Let $k$ be arbitrary and $f \in M_k^0$. Then $g = f/\Delta$ has weight $2k - 12$ and

$$v_z(g) = v_z(f) - v_z(\Delta) = \begin{cases} v_z(f) - 1 & (z = \infty), \\ v_z(f) & (z \neq \infty). \end{cases}$$

Thus $v_z(g) \geq 0$ and is analytic on $\tilde{H}$ and thus belongs to $M_{k-6}$. In fact the relationship

$$f \rightarrow f/\Delta$$

gives an isomorphism between the vector spaces $M_k^0$ and $M_{k-6}$. More generally this relationship gives an isomorphism between $M_{k+1}$ and $M_{k-6}$. We have seen that $M_k^0$ is empty when $k < 6$ or $k = 1$. Thus $M_k \leq 1$ when $1 \leq k \leq 5$ and $k = 7$. We have $1 \in M_0$. Hence

$$\dim M_k = 1.$$ 

Also, by (17), when $2 \leq k \leq 5$ or $k = 7$,

$$\dim M_k = 1.$$

**Theorem 9.7.** For convenience define $G_0(z) = 1$. Then

(i) $M_k$ is empty when $k < 0$ or $k = 1$.

(ii) when $k \geq 0$,

$$\dim M_k = \begin{cases} \lfloor k/6 \rfloor & k \equiv 1 \pmod{6}, \\ \lfloor k/6 \rfloor + 1 & k \not\equiv 1 \pmod{6}. \end{cases}$$

(iii) when $k \geq 0$ and $k \neq 1$,

$$M_k = \mathbb{C}G_k \oplus \mathbb{C}\Delta G_{k-6} \oplus \cdots \oplus \mathbb{C}\Delta^j G_{k-6j}$$

where

$$j = \begin{cases} \lfloor k/6 \rfloor - 1 & k \equiv 1 \pmod{6}, \\ \lfloor k/6 \rfloor & k \not\equiv 1 \pmod{6}. \end{cases}$$

Recall that $\Delta$ is a linear combination of $G_2$ and $G_3$. In fact it can be shown that every $G_k$ is polynomial in $G_2$ and $G_3$, and indeed that every $M_k$ is spanned by the monomials $G_2^u G_3^v$ where $u$ and $v$ run over the solutions to $2u + 3v = k$ with $u \geq 0, v \geq 0$.

It can also be shown that

$$G_2(\rho) = 0, \quad G_3(i) = 0,$$

either directly or by utilising Theorem 9.6.

The cusp form $\Delta$ has several remarkable properties. One of them is the product formula below.
**Theorem 9.8.** Let \( z \in \mathbb{H} \). Then

\[
\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad (q = e^{2\pi iz}).
\]

**Proof.** There is no very simple proof. We know that \( \Delta \in M_6^0 \), \( \dim M_6^0 = 1 \), and the coefficient of \( q \) in \( \Delta \) is \((2\pi)^{12}\). Thus it suffices to show that

\[
f(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}
\]

is of weight 12. Since it is immediate that it is periodic with period 1, it suffices to show that

\[
f(-1/z) = z^{12} f(z) \quad (z \in \mathbb{H}).
\]

Consider the function

\[
G_1(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n.
\]  

(18)

We will show that

\[
G_1(z) = \frac{2\pi i}{z} + \frac{1}{z^2} G_1(-1/z) \quad (z \in \mathbb{H}).
\]  

(19)

Then, by logarithmic differentiation,

\[
\frac{F'}{F}(z) = 2\pi i \left( 1 - \sum_{n=1}^{\infty} \frac{24q^n}{1 - q^n} \right)
= 2\pi i \left( 1 - 24 \sum_{m=1}^{\infty} \sigma(m) q^m \right)
= 2\pi i \cdot 3 \cdot \frac{\pi^2}{G_1(z)}
= 6i \pi G_1(z)
= 6i \pi \left( \frac{2\pi i}{z} + \frac{1}{z^2} G_1(-1/z) \right)
= -\frac{12}{z} + \frac{d}{dz} \log F(-1/z).
\]

Thus \( F \) satisfies

\[
z^{12} F(z) = C f(-1/z)
\]

for some \( C \in \mathbb{C} \). Since \( F(-1/i) = F(i) \) and \( i^{12} = 1 \) we have \( C = 1 \).
To complete the proof of the theorem it suffices to show that $G_1$, given by (18), satisfies (19). Following the proof of Theorem 9.5, with some care as the double series is no longer absolutely convergent, we have

$$G_1(z) = 2\zeta(2) + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2}.$$  

Then

$$G_1(-1/z) = 2\zeta(2) + z^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m+nz)^2} = 2\zeta(2) + z^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m+nz)^2}.$$  

Thus it suffices to show that $L(z)$ and $R(z)$ converge and

$$L(z) = -\frac{2\pi i}{z} + R(z) \quad (20)$$

where

$$L(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{n\neq 0} \frac{1}{(m+nz)^2}$$

and

$$R(z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{m\neq 0} \frac{1}{(m+nz)^2}.$$  

Note that the sums are different even though they are only interchanged. Let

$$S(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{(m,n)\neq (0,0),(1,0)} \frac{1}{(m-1+nz)(m+nz)}$$

and

$$T(z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{(m,n)\neq (0,0),(0,1)} \frac{1}{(m-1+nz)(m+nz)}.$$
We will show below that these series converge. Then the convergence of $L$ follows from the relationship

$$S(z) - L(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m-1 + nz)(m + nz)^2} + \sum_{m \neq 0,1} \frac{1}{m(m-1)}.$$ 

In the last sum the terms with $m > 1$ sum to 1 and those with $m < 0$ sum to $-1$. Hence the above becomes

$$S(z) - L(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m-1 + nz)(m + nz)^2}.$$ 

Similarly the convergence of $R$ follows from

$$T(z) - R(z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m-1 + nz)(m + nz)^2} + \sum_{m \neq 0,1} \frac{1}{m(m-1)}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m-1 + nz)(m + nz)^2}.$$ 

These series are absolutely convergent and hence can be interchanged. Thus they are identical. Therefore not only will the convergence of $L(z)$ and $R(z)$ follow from that of $S(z)$ and $T(z)$ but we will have

$$L(z) - R(z) = S(z) - T(z).$$

Thus to prove (20) it suffices to show that $S(z)$ and $T(z)$ converge and

$$S(z) - T(z) = -\frac{2\pi i}{z} \quad \text{(21)}$$

The sum over $m$ in $T$ when $n \neq 0$ is

$$\sum_{m=-\infty}^{\infty} \left( \frac{1}{m-1 + nz} - \frac{1}{m + nz} \right).$$

The part with $m \geq 0$ sums to $\frac{1}{1+ nz}$ and the part with $m \leq -1$ sums to $-\frac{1}{1+ nz}$ Hence when $n \neq 0$ the sum over $m$ in $T$ is 0. When $n = 0$ the sum over $m$ is

$$\sum_{m=2}^{\infty} \left( \frac{1}{m-1} - \frac{1}{m} \right) + \sum_{m=-\infty}^{-1} \left( \frac{1}{m-1} - \frac{1}{m} \right) = 1 + 1 = 2.$$
Hence $T(z)$ converges to 2.

The series $S(z)$ is more complicated. We will complete the proof of the theorem by showing that it converges to $2 - \frac{2\pi i}{z}$. We have

$$S(z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{1}{m-1+nz} - \frac{1}{m+nz} \right) + \sum_{m=-\infty}^{\infty} \left( \frac{1}{m-1} - \frac{1}{m} \right)$$

$$= 2 + \frac{1}{z} \sum_{m=-\infty}^{\infty} \left( U((m-1)/z) - U(m/z) \right)$$

where

$$U(w) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{w+n} - \frac{1}{n} \right).$$

For $m \neq 0$, by (10),

$$U(w) = \pi \cot \pi w - \frac{1}{w}.$$ 

Clearly

$$S(z) = 2 + \frac{1}{z} \left( \lim_{M \to -\infty} U(M/z) - \lim_{M \to \infty} U(M/z) \right)$$

and the convergence of $S(z)$ stands or falls on the existence of the limits above. Obviously

$$\lim_{M \to \pm\infty} U(M/z) = \lim_{M \to \pm\infty} \pi \cot \pi (M/z).$$

Now \(\Re 2\pi i M/z = \Re 2\pi i M (x - iy)/|z|^2 = 2\pi M y/|z|^2\) and as $M \to \infty$, $e^{-2\pi i M/z} \to 0$. Thus

$$\pi \cot \pi M/z = \pi i \frac{1 + e^{-2\pi i M/z}}{1 - e^{-2\pi i M/z}} \to \pi i.$$ 

On the other hand, as $M \to -\infty$

$$\pi \cot \pi M/z \to -\pi i.$$ 

This establishes the convergence of $S(z)$ and its evaluation, and completes the proof of the theorem.

**Exercises 9.2.**

1. Let $E_k(z) = G_k(z)/(2\zeta(2k))$, $q = e^{2\pi iz}$. Show that

$$E_2(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$
\[ E_3(z) = 1 - 540 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \]
\[ E_4(z) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \]
\[ E_5(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n, \]
\[ E_6(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n, \]

2. Prove that \( \sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n} \sigma_3(m) \sigma_3(n-m). \)

3. Prove that \( 11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m) \sigma_5(n-m). \)

4. Prove that \( 756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 691.252 \sum_{m=1}^{n-1} \sigma_5(m) \sigma_5(n-m). \) Deduce Ramanujan’s congruence \( \tau(n) \equiv \sigma_{11}(n) \pmod{691}. \)