1. (i) Prove that if \( p \equiv 1 \pmod{3} \), then \( \left( \frac{-3}{p} \right)_L = 1 \). By quad. recip. \( \left( \frac{-3}{p} \right)_L = (-1)^{\frac{p-1}{2}} \left( \frac{2}{p} \right)_L = 1 \).

   (ii) Let \( \mathcal{M} = \{ n \in \mathbb{N} : p|n \implies p \equiv 1 \pmod{3} \} \). Prove that if \( n \in \mathcal{M} \), then \( x^2 + 3 \equiv 0 \pmod{4n} \) is soluble in \( x \). By (i), \( x^2 + 3 \equiv 0 \pmod{p} \) is soluble when \( p \equiv 1 \pmod{3} \), and \( 1^2 + 3 \equiv 0 \pmod{4} \). Moreover a residue modulo \( p^k \) is a quadratic residue iff it is one modulo \( p \). Conclusion follows by Chinese remainder theorem.

   (iii) Let \( n \in \mathcal{M} \). Prove that there are \( a, B \in \mathbb{Z} \) with \( a > 0 \) such that \( B^2 + 12 = 4an \). Let \( b = B - 2a, c = (b^2 + 12)/4a \). Prove that \( b^2 - 4ac = -12 \) and \( a + b + c = n \). By (ii), \( x^2 + 3 \equiv 0 \pmod{m} \) is soluble. Hence there is a solution with \( x > n \). Let \( a = (x^2 + 3)/n \). Then \( B = 2x \) satisfies \( B^2 + 12 = 4an \). Moreover \( b^2 - 4ac = -12 \) and \( a + b + c = a + b + \frac{b^2 + 12}{4a} = a + B - 2a + B^2 - 4B + a^2 + 7 = B^2 + 7 \).

   (iv) Let \( h(d) \) be defined as in homework 11. Prove that \( h(-12) = 2 \). Consider \( b^2 - 4ac = -12 \) with \( -a < b \leq a < c \) or \( 0 \leq b \leq a = c \). In either case \( b \) is even and \( a^2 - 4a^2 \geq -12 \), so \( a^2 \leq 4 \), \( a = 1 \) or \( 2 \). When \( a = 1 \), since \( b \) is even, \( b = 0 \) and so \( c = 3 \) is the only solutions. When \( a = 2 \), \( 8 \nmid 12 \) so \( b \neq 0 \). Hence \( b = c = 2 \) is the only solution.

   (v) Prove that if \( n \in \mathcal{M} \), then \( x^2 + 3y^2 = n \) is soluble in integers \( x \) and \( y \). By (iii), when \( n \in \mathcal{M} \), \( n \) is represented by \( ax^2 + bxy + cy^2 \) where \( b^2 - 4ac = -12 \) and so is represented by at least one of the reduced forms. But \( n \) is odd, so it is represented by \( x^2 + 3y^2 \).

2. (i) Prove that if \( p \equiv 1, 4 \pmod{7} \), then \( \left( \frac{-2}{p} \right)_L = 1 \). By the law of quad. recip. \( \left( \frac{-2}{p} \right)_L = \left( \frac{7}{p} \right)_L = \left( \frac{7}{p} \right)_L = 1 \).

   (ii) Let \( \mathcal{N} = \{ n \in \mathbb{N} : p|n \implies p \equiv 1, 4 \pmod{7} \} \). Prove that if \( n \in \mathcal{N} \), then \( x^2 + 7 \equiv 0 \pmod{4n} \) is soluble in \( x \). \( 1^2 + 7 \equiv 0 \pmod{4} \). Moreover a residue modulo \( p^k \) is a quadratic residue iff it is one modulo \( p \). Hence by (i) and the Chinese remainder theorem \( x^2 + 7 \equiv 0 \pmod{4n} \)

   (iii) Let \( n \in \mathcal{N} \). Prove that there are \( a, B \in \mathbb{Z} \) with \( a > 0 \) such that \( B^2 + 7 = 4an \). Let \( b = B - 2a, c = (b^2 + 7)/4a \). Prove that \( b^2 - 4ac = -7 \) and \( a + b + c = n \). By (ii) there are \( B > n \) such that \( B^2 + 7 \equiv 0 \pmod{4n} \). Let \( a = (B^2 + 7)/4n \). Then \( b^2 - 4ac = -7 \) and \( a + b + c = a + b + \frac{b^2 + 7}{4a} = a + B - 2a + \frac{B^2 - 4B + a^2 + 7}{4a} = B^2 + 7 

   (iv) Recall from homework 11 that \( h(-7) = 1 \). Prove that if \( n \in \mathcal{N} \), then \( x^2 + xy + 2y^2 = n \) is soluble in integers \( x \) and \( y \). By (iii), \( n \) is represented by \( ax^2 + bxy + cy^2 \) with \( b^2 - 4ac = -7 \). Hence it is represented by the lone reduced form \( x^2 + y^2 + 2y^2 \) with discriminant \(-7 \).

   (v) Let \( n \in \mathcal{N} \). Prove that \( x^2 + 7y^2 = 4n \) is soluble in integers \( x, y \). Moreover prove that \( x \) and \( y \) are both even, and thus \( x^2 + 7y^2 = n \) is also soluble in integers \( x \). By (iv), \( x^2 + xy + 2y^2 = n \). Hence \( 4n = (2x + y)^2 + 7y^2 \), so \( 4n \) has a representation \( 4n = x^2 + 7y^2 \). Either \( x \) and \( y \) are both odd or both even. But if they are both odd, then \( x^2 + 7y^2 \equiv 1 + 7 \equiv 0 \pmod{8} \) and \( 8 \nmid n \).