Return by Tuesday 18th November

**Easier problems**

1. Show that for arbitrary real or complex numbers $c_1, \ldots, c_q$,

$$
\sum_{\chi} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \varphi(q) \sum_{n=1}^{q} |c_n|^2
$$

where the sum on the left hand side runs over all Dirichlet characters $\chi \pmod q$.

2. Show that for arbitrary real or complex numbers $c_\chi$,

$$
\sum_{n=1}^{q} \left| \sum_{\chi} c_\chi \chi(n) \right|^2 = \varphi(q) \sum_{\chi} |c_\chi|^2
$$

where sums over $\chi$ are extended over all Dirichlet characters $\pmod q$.

**Harder problem**

3. (Mertens (1895a,b)) Let $r(n) = \sum_{d|n} \chi(d)$.

(a) Show that if $\chi$ is a non-principal character $\pmod q$, then

$$
\sum_{n>x} \frac{\chi(n)}{\sqrt{n}} \ll_{\chi} \frac{1}{\sqrt{x}}.
$$

(b) Show that if $\chi$ is a non-principal character $\pmod q$, then

$$
\sum_{n\leq x} \frac{r(n)}{n^{1/2}} = 2x^{1/2}L(1, \chi) + O(1).
$$

(c) Recall that if $\chi$ is quadratic then $r(n) \geq 0$ for all $n$, and that $r(n^2) \geq 1$. Deduce that if $\chi$ is a quadratic character, then the left hand side above is $\gg \log x$.

(d) Conclude that if $\chi$ is a quadratic character, then $L(1, \chi) > 0$. 