MATH 567 INTRODUCTION TO NUMBER THEORY I, FALL TERM 2008, PROBLEMS 11

Return by Tuesday 11th November

1. (a) Prove that if \( x \geq 1 \), then

\[
\sum_{n \leq x} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1.
\]

(b) Prove that

\[-1 + 1/x \leq \sum_{n \leq x} \frac{\mu(n)}{n} \leq 1 + 1/x.\]

In fact we know that

\[\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0,\]

but this is equivalent to the prime number theorem in the sense that if follows from the prime number theorem and there is a relatively simple proof that it implies the prime number theorem.

2. (a) Let \( a_1, a_2, \ldots \) be non-zero integers, and define \( d_n = \text{lcm}[a_1, \ldots, a_n] \). Given \( n \), prove that there are integers \( b_1, b_2, \ldots, b_n \) such that

\[
\frac{1}{d_n} = b_1 + \cdots + b_n.
\]

(b) Let \( d_n = \text{lcm}[1, 2, \ldots, n] \). Prove that \( d_n = e^{\psi(n)} \).

(c) Let \( P \in \mathbb{Z}[x], \deg P \leq n \). Put \( I = I(P) = \int_0^1 P(x) \, dx \). Prove that \( I d_{n+1} \in \mathbb{Z} \), and hence that \( d_{n+1} \geq 1/|I| \) if \( I \neq 0 \).

(d) Prove that there is a polynomial \( P \) as above so that \( I d_{n+1} = 1 \).

(e) Prove that \( \max_{0 \leq x \leq 1} |x^2(1-x)^2(2x-1)| = 5^{-5/2} \).

(f) For \( P(x) = (x^2(1-x)^2(2x-1))^2 \), prove that \( 0 < I < 5^{-5n} \).

(g) Prove that \( \psi(10n + 1) \geq (\frac{1}{2} \log 5) \cdot 10n \).

3. Prove that all the characters modulo 8 are real.