Easier problems

1. Evaluate \( \left( \frac{313}{367} \right)_J, \left( \frac{367}{401} \right)_J, \left( \frac{401}{313} \right)_J \).

2. Show that the congruence \( x^6 - 11x^4 + 36x^2 - 36 \equiv 0 \pmod{p} \) is soluble for every prime \( p \). Hint: Factorise \( z^3 - 11z^2 + 36z - 36 \).

3. Suppose that \( a \in \mathbb{Z}\setminus\{0\} \), and there is a \( b \in \mathbb{Z} \) such that \( a = -b^2 \). Show that there is an odd positive integer \( m \) such that \( \left( \frac{a}{m} \right)_J = -1 \). Deduce that there is an odd prime \( p \) such that \( \left( \frac{a}{p} \right)_J = -1 \). Let \( m \) be a solution to \( m \equiv 5 \pmod{8}, m \equiv 1 \pmod{b} \).

4. Suppose that \( a \in \mathbb{Z}\setminus\{0\} \) and \( a = \pm 2^u b \) where \( u \in \mathbb{N} \) and \( b \in \mathbb{N} \) with both \( u \) and \( b \) odd. Show that there is an odd positive integer \( m \) such that \( \left( \frac{a}{m} \right)_J = -1 \). Deduce that there is an odd prime \( p \) such that \( \left( \frac{a}{p} \right)_J = -1 \). Hint: Let \( m \) be a solution to \( m \equiv 5 \pmod{8}, m \equiv 1 \pmod{b} \).

5. Suppose that \( a \in \mathbb{Z}\setminus\{0\} \) and \( a = \pm 2^u b q^t \) where \( u \) is a non-negative integer, \( b \in \mathbb{N} \) and \( t \in \mathbb{N} \) with both \( b \) and \( t \) odd, and \( q \) is an odd prime. Show that there is an odd positive integer \( m \) such that \( \left( \frac{a}{m} \right)_J = -1 \). Deduce that there is an odd prime \( p \) such that \( \left( \frac{a}{p} \right)_J = -1 \). Hint: Let \( m \) be a solution to \( m \equiv 1 \pmod{4b}, m \equiv n \pmod{q} \) where \( n \) is a quadratic non-residue modulo \( q \).

Harder problem

6. Show that an integer \( a \) is a perfect square if and only if it is a quadratic residue for every prime \( p \) not dividing \( a \). Questions 3, 4, 5, are relevant. This is a simple example of the “local-to-global” principle.