To be submitted by Tuesday, October 7th

**Easier problems**

1. Show that \( \left( \sum_{m|n} d(m) \right)^2 = \sum_{m|n} d(m)^3. \)

2. Show that if \( \sigma(n) \) is odd, then \( n \) is a square or twice a square.

3. Show that \( \sum_{l|(m,n)} \mu(l) \) is 1 when \( (m, n) = 1 \) and is 0 otherwise. Hence prove that \( \sum_{m=1:(m,n)=1} m = \frac{1}{2} n \phi(n) \) when \( n > 1. \)

4. Let \( \lambda(n) = (-1)^\Omega(n) \) (Liouville’s function). Show that \( \lambda(n) = \sum_{m|n} \mu(n/m^2). \)

5. Define \( f(n) \) to be \( (-1)^{n-1}/2 \) when \( n \) is odd, 0 when \( n \) is even. Show that \( f \) is totally multiplicative and is periodic with period 4.

**Harder problem**

6. Let \( k \in \mathbb{N}, z \in \mathbb{C}, e(\alpha) = \exp(2\pi i \alpha). \) Define \( \Phi_k(z) = \prod_{l|k} (z^l - 1)^{\mu(k/l)}. \) the \( k \)-th cyclotomic polynomial, i.e. the monic polynomial whose roots are the primitive \( k \)-th roots of unity.

   (i) Show that \( \prod_{l|k} \Phi_l(z) = z^k - 1 \) and \( \Phi_1(z) = z - 1. \)

   (ii) Deduce that \( \Phi_k(z) = \prod_{l|k} (z^l - 1)^{\mu(k/l)}. \)

   (iii) Show that if \( k > 1, \) then \( \Phi_k(z) = \prod_{l|k} (1 - z^l)^{\mu(k/l)} \) and \( \Phi_k(0) = 1. \)

   (iv) By considering the expansion \( (1 - z^l)^{-1} = 1 + z^l + z^{2l} + \cdots \) when \( |z| < 1 \) show that \( \Phi_k(z) \) has integer coefficients.

   (v) Let \( K \) be the largest squarefree divisor of \( k. \) Show that \( \Phi_k(z) = \Phi_K(z^k/K). \)

   (vi) Prove that \( \Phi_p(z) = 1 + z + \cdots + z^{p-1}. \)

   (vii) Show that if \( k \) is odd and \( k > 1, \) then \( \Phi_{2k}(z) = \Phi_k(-z^{2k-1}). \)

   (viii) Suppose that \( p \) and \( q \) are different primes. Show that, when \( |z| < 1, \Phi_{pq}(z) = (1-z)\sum_{n=0}^{\infty} b_n z^n \) where \( b_n \) is the number of choices of \( u, v \in \mathbb{Z} \) with \( 0 \leq u \leq q-1, \)

   (ix) Show that \( b_n = 0 \) or 1 and that the coefficients of \( \Phi_{pq}(z) \) are \( \pm 1 \) or 0.

   (x) Show that if \( k < 105, \) then the coefficients of \( \Phi_k(z) \) are \( \pm 1 \) or 0.

   (xi) Show that the coefficient of \( z^7 \) in \( \Phi_{105} \) is \(-2. \) It is known (Erdös 1948, Vaughan 1975) that sometimes the coefficients of \( \Phi_k(z) \) are as large as \( \exp \left( \frac{2 \log k}{ \log \log k} \right) \) and that “almost always” the largest coefficient is arbitrarily large (Meier 1995).

   So much for intuition ...!

   (xi) Prove that if \( k > 1, \) then \( \Phi_k(1) = e^{\Lambda(k)}. \)