9. MODULAR FORMS

9.1. Introduction A modular form is an analytic function which satisfies a certain simple relationship under the action of Möbius transformations together with some other simple properties, to be defined. The importance of modular forms is that they underpin a lot of interesting number theoretic structures.

9.2. Properties of Möbius transformations. Let

$$f(z) = \frac{az+b}{cz+d}; a, b, c, d \in \mathbb{C}, ad \neq bc. \tag{1}$$

The assumption $ad \neq bc$ is to ensure that f is not a constant and is well defined (c and d cannot both be 0). This defines $f(z)$ for all z in the extended complex plane $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ except for $z = -d/c$ and $z = \infty$. We extend the definition to $\tilde{\mathbb{C}}$ by taking

$$F(-d/c) = \infty, \quad f(\infty) = a/c$$

with the usual convention that $w/0 = \infty$ when $w \neq 0$, and *vice versa*. Clearly f is analytic on $\tilde{\mathbb{C}}$ except for a simple pole at $-d/c$ and maps $\tilde{\mathbb{C}}$ onto $\tilde{\mathbb{C}}$. Moreover given $w \in \tilde{\mathbb{C}}$ the point

$$z = \frac{dw-b}{-cw+a}$$

has the property that $f(z) = w$. Thus

$$g(z) = \frac{dw-b}{-cw+a}$$

is the inverse of f and f is a bijection from $\tilde{\mathbb{C}}$ to itself. We have

$$\frac{f(w)-f(z)}{w-z} = \frac{ad-bc}{(cw+d)(cz+d)} \tag{2}$$

and letting $w \to z$ gives

$$f'(z) = \frac{ad-bc}{(cz+d)^2}.$$

This is non-zero. Thus f is conformal except possibly at $z = -d/c$.

Consider the equation

$$Az\overline{z} + Bz + \overline{B}\overline{z} + C = 0$$

where A and C are real. The points on any circle satisfy such an equation with $A \neq 0$ ($A|z + B/A|^2 = |B|^2/A - C$) and the points on any line satisfy such an equation with $A = 0$. Suppose that

$$z = \frac{aw+b}{cw+d}.$$

Then on substituting in the above equtions, clearing the denominators $cw+d$. $c\overline{w}+d$ and collecting together coefficients of $w\overline{w}$, w and $\overline{w}$ gives

$$A'w\overline{w} + B'w + \overline{B'}\overline{w} + C' = 0.$$

Hence every Möbius tranformation maps circles and lines into circles and lines.

Since for any $D \in \mathbb{C}\backslash\{0\}$ we have

$$\frac{az+b}{cz+d} = \frac{(a/D)z+b/D}{(c/D)z+d/D}$$

and

$$\frac{a}{D}\cdot\frac{d}{D}-\frac{b}{D}\cdot\frac{c}{D}=\frac{ad-bc}{D^2}$$

we can suppose that

$$ad-bc=1.$$

We can associate with

$$f(z)=\frac{az+b}{cz+d}$$

the 2×2 matrix

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\det A=1$. If f and g are Möbius tranformations with associated matrices A and B, then $(f\circ g)(z)=f(g(z))$ has associated matrix AB. The identity matrix $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ corresponds to $f(z)=z$ and the inverse matrix

$$A^{-1}=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (\text{note } \det A^{-1}=da-bc=1)$$

is associated with $f^{-1}(z)$.

9.3. **The modular group.** The set of all Möbius transforms form a group under composition, and this is associated with $\mathrm{SL}_2(\mathbb{C})$. We will mostly be concerned with the subgroup $\mathrm{SL}_2(\mathbb{Z})$. When a, b, c, d are real one has

$$\Im f(z)=\Im\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}=\Im\frac{z+bc(z+\overline{z})}{|cz+d|^2}=\Im\frac{z+bc(z+\overline{z})}{|cz+d|^2},$$

so

$$\Im f(z)=\frac{\Im z}{|cz+d|^2}. \tag{3}$$

Thus f maps the upper half–plane

$$\mathbb{H}=\{z:\Im z>0\}$$

bijectively to $\mathbb{H}$.

Another important remark is that

$$\frac{az+b}{cz+d}=\frac{(-a)z+(-b)}{(-c)z+(-d)}.$$

In other words,

$$A \quad\text{and}\quad A\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

give identical maps. Thus it is normal to restrict ones attention to

$$\mathrm{PSL}_2(\mathbb{R})=\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$$

and

$$\mathrm{PSL}_2(\mathbb{Z})=\mathrm{SL}_2(\mathbb{Z})/\{\pm I\}.$$

Since $\mathrm{PSL}_2(\mathbb{Z})$ is a handful to write one tends to use a shorthand. Serre uses G and Apostol and many others use Γ, and we will follow the herd. This group is called the modular group.

Theorem 9.1. *The modular group Γ is generated by*

$$T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

i.e. every $A\in\Gamma$ can be expressed in the form

$$A=T^{n_1}ST^{n_2}S\ldots ST^{n_k}$$

where the $n_j \in \mathbb{Z}$*.*

Remark. The matrices S and T correspond to $z \to -1/z$ and $z \to z+1$ respectively.

Proof. Since we are working modulo $\pm I$ we need only consider the

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c \geq 0$. We argue by induction on c. If $c = 0$, then $ad = 1$ so $a = d = \pm 1$ and

$$A = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} = T^{\pm b}.$$

If $c = 1$, then $ad - bc = 1$, so $b = ad - 1$ and

$$A = \begin{pmatrix} a & ad-1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = T^a S T^d.$$

Now suppose that $c > 1$ and assume the conclusion for all

$$A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

with $0 \leq c' < c$. Since $ad - bc = 1$ we have $(d, c) = 1$. Hence $d = cq + r$ where $0 < r < c$. Then

$$AT^{-q} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - aq \\ c & r \end{pmatrix}$$

and

$$AT^{-q}S = \begin{pmatrix} a & b - aq \\ c & r \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b - aq & -a \\ r & -c \end{pmatrix}.$$

The only other observation we need is that $S^2 = -I \equiv I$.

9.3 Fundamental Domains. We are interested in the behaviour of the modular group acting on points in $\mathbb{H}$.

Definition 9.1. *Let G be a subgroup of Γ. Two points z, $w \in \mathbb{H}$ are equivalent under G when $z = Aw$ for some A in G. This equivalence relation partitions $\mathbb{H}$ into equivalence classes called orbits (of G), i.e for a given $z \in \mathbb{H}$ an orbit is the set of all Az with $A \in G$.*

Definition 9.2. *Let G be a subgroup of Γ. Any simply connected subset $\mathbb{D}_G$ of $\mathbb{H}$ is called a fundamental domain (or region) of G when it satisfies the following.*
(i) No two distinct points of $\mathbb{D}_G$ are in the same orbit of G.
(ii) Every orbit of G contains a point of $\mathbb{D}_G$.

When $G = \Gamma$ we simplify the notation by writing $\mathbb{D}$ for $\mathbb{D}_\Gamma$.

Theorem 9.2. *Let*

$$\mathbb{D} = \{z : \text{either } |z| > 1, -\tfrac{1}{2} \leq \Re z < \tfrac{1}{2} \text{ and } \Im z > 0, \text{ or } |z| = 1, -\tfrac{1}{2} \leq \Re z \leq 0 \text{ and } \Im z > 0\}.$$

Then $\mathbb{D}$ is a fundamental domain for Γ.

Proof. Suppose that $z \in \mathbb{H}$. Let N denote the number of integers c and d such that $|cz+d| \leq 1$. Since $\Im z > 0$ we have $|c|\Im z = |\Im(cz+d)| \leq |cz+d| \leq 1$, so that $|c| \leq 1/\Im z$ and $|d| = |cz+d-cz| \leq |cz+d|+|cz| \leq 1+|z|/\Im z$. Thus $N \leq (1 + 2/\Im z)(3 + 2|z|/\Im z)$. Thus for all but N choices of c and d we have $|cz + d| > 1$ and so

$$\Im(Az) = \frac{\Im z}{|cz + d|^2} < \Im z.$$

Thus there is an $A \in \Gamma$ for which $\Im(Az)$ is maximal. Now choose $n \in \mathbb{Z}$ so that $-\frac{1}{2} \le \Re Az + n < \frac{1}{2}$. In other words $-\frac{1}{2} \le \Re T^n Az < \frac{1}{2}$. Then $\Im T^n Az = \Im Az$ is also maximal. If $|T^n Az| < 1$, then $|ST^n Az| = |-1/(T^n Az)| > 1$ so that $\Im(ST^n Az) = \Im(T^n Az)|T^n Az|^2 > \Im(T^n Az) = \Im(Az)$ which would contradict the maximality of $\Im(Az)$. Hence $|T^n Az| \ge 1$. If $|T^n Az| > 1$ or $|T^n Az| = 1$ and $-\frac{1}{2} \le \Re T^n Az \le 0$, then $T^n Az \in \mathbb{D}$. If $|T^n Az| = 1$ and $0 < \Re T^n Az < \frac{1}{2}$, then $ST^n Az \in \mathbb{D}$.

We complete the proof by showing that if $z, w \in \mathbb{D}$, $A \in \Gamma$, $z = Aw$, then $z = w$. As usual we associate A with the element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\mathrm{SL}_2\mathbb{Z}$. If $c = 0$, then $ad = 1$, $a = d = \pm 1$ and $w = Az = z \pm b$. Hence $b = 0$ and $w = z$. Now suppose that $c \ne 0$. We have (3). Since $A^{-1}w = z$ we also have

$$\Im z = \Im(A^{-1}w) = \frac{\Im z}{|-cw + a|^2}. \tag{4}$$

Moreover

$$|cz + d|^2 = c^2|z|^2 + 2cd\Re z + d^2 \ge c^2|z|^2 - |cd| + d^2 \ge c^2 - |cd| + d^2.$$

Since $c \ne 0$ and $u^2 - u + 1$ has no real roots we have

$$|cz + d|^2 \ge c^2|z|^2 - |cd| + d^2 \ge 1. \tag{5}$$

Likewise

$$|-cw + a|^2 \ge c^2|w|^2 - |ca| + a^2 \ge 1. \tag{6}$$

Note that equality could only occur in these last two inequalities if $|z| = |w| = 1$. By (2) and (5), $\Im w \le \Im z$ and by (4) and (6), $\Im z \le \Im w$, so $\Im z = \Im w$. But then we have equality in (5) and (6), so $|z| = |w|$, and hence $|\Re z| = |\Re w|$. But on that part of $\mathbb{D}$ with $|z| = 1$ we have $\Re z \le 0$ hence $\Re z = \Re w$.

Exercises 9.1.

Γ denotes the modular group and S, T are its generators, $S(z) = -1/z$, $T(z) = z + 1$. Given a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with real coefficients, $d = d_Q = b^2 - 4ac$ is called the discriminant of Q.

1. (i) Find all elements A of Γ which commute with S.
(ii) Find all elements A of Γ which commute with T.
(iii) Find the smallest $n > 0$ such that $(ST)^n = I$.
(iv) Determine all A in Γ which leave i fixed.
(v) Determine all A in Γ which leave $\rho = e(1/3)$ fixed.

2. Prove that if $A \in \Gamma$, and $(x, y)^T = A(x', y')^T$, then the quadratic form Q' defined by $Q'(x', y') = Q(x, y)$ satisfies $d_{Q'} = d_Q$. Two forms related in this way are called equivalent. This relation separates all forms into equivalence classes. The forms in the same class have the same discriminant and the ranges $Q(\mathbb{Z}^2)$ coincide.

In the remaining exercises it will be supposed that the quadratic forms have positive coefficients of x^2 and y^2 and negative discriminant. The associated polynomial $Q(z, 1)$ has two complex roots. The one in $\mathbb{H}$ is called the representative of Q.

3. (i) If d is fixed, prove that there is a bijection between the set of forms with discriminant d and the members of $\mathbb{H}$.
(ii) Prove that two quadratic forms with discriminant d are equivalent iff their representatives are equivalent under Γ.

A reduced form is one whose representative lies in the fundamental domain $\mathbb{D}$, the set of z such that either $|z| > 1$ and $-1/2 \le \Re z < 1/2$ or $|z| = 1$ and $-1/2 \le \Re z \le 0$. Thus two reduced forms are equivalent iff they are identical, and moreover each equivalence class contains exactly one reduced form.

4. Prove that $Q(x, y) = ax^2 + bxy + cy^2$ is reduced iff either $-a < b \le a < c$ or $0 \le b \le a = c$.

In questions 5,6 it is assumed that the quadratic forms have integer coefficients.

5. Prove that the number of reduced forms with a given discriminant $d < 0$ is finite. The number of such classes is called the class number and is denoted by $h(d)$.

6. When $d = -3, -4, -7, -8, -11, -15, -19, -20, -23$ determine all reduced forms with discriminant d, and the corresponding class number $h(d)$.

7. (i) Prove that if $p \equiv 1 \pmod 3$, then $\left(\frac{-3}{p}\right)_L = 1$.
(ii) Let $\mathcal{M} = \{n \in \mathbb{N} : p|n \implies p \equiv 1 \pmod 3\}$. Prove that if $n \in \mathcal{M}$, then $x^2 + 3 \equiv 0 \pmod{4n}$ is soluble in x.
(iii) Let $n \in \mathcal{M}$. Prove that there are a, $B \in \mathbb{Z}$ with $a > 0$ such that $B^2 + 12 = 4an$. Let $b = B - 2a$, $c = (b^2 + 12)/4a$. Prove that $b^2 - 4ac = -12$ and $a + b + c = n$.
(iv) Let $h(d)$ be defined as in homework 11. Prove that $h(-12) = 2$.
(v) Prove that if $n \in \mathcal{M}$, then $x^2 + 3y^2 = n$ is soluble in integers x and y.

8. (i) Prove that if $p \equiv 1,\ 4 \pmod 7$, then $\left(\frac{-7}{p}\right)_L = 1$.
(ii) Let $\mathcal{N} = \{n \in \mathbb{N} : p|n \implies p \equiv 1, 4 \pmod 7\}$. Prove that if $n \in \mathcal{N}$, then $x^2 + 7 \equiv 0 \pmod{4n}$ is soluble in x.
(iii) Let $n \in \mathcal{N}$. Prove that there are a, $B \in \mathbb{Z}$ with $a > 0$ such that $B^2 + 7 = 4an$. Let $b = B - 2a$, $c = (b^2 + 7)/4a$. Prove that $b^2 - 4ac = -7$ and $a + b + c = n$.
(iv) Recall from homework 11 that $h(-7) = 1$. Prove that if $n \in \mathcal{N}$, then $x^2 + xy + 2y^2 = n$ is soluble in integers x and y.
(v) Let $n \in \mathcal{N}$. Prove that $x^2 + 7y^2 = 4n$ is soluble in integers x, y. Moreover prove that x and y are both even, and thus $x^2 + 7y^2 = n$ is also soluble in integers x, y.

9.4. Modular functions.

Definition 9.3. *Let $k \in \mathbb{Z}$. Then f is weakly modular of weight $2k$ when f is meromorphic on $\mathbb{H}$ and satisfies*

$$f(z) = (cz + d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Theorem 9.3.. *Let f be meromorphic on $\mathbb{H}$. Then f is weakly modular of weight $2k$ where $k \in \mathbb{Z}$ if and only if*

$$f(z + 1) = z + 1,$$
$$f(-1/z) = z^{2k} f(z)$$

for all $z \in \mathbb{H}$.

Proof. If f is weakly modular of weight $2k$, then at once it must satisfy the above relations. Suppose converesely that it satisfies them. Then we can apply Theorem 9.1 to obtain $f(Az)$ where A is any member of $SL_2(\mathbb{Z})$. We need to show that the correct factor $(cz + d)^{-2k}$ arises. It suffices to show that if $A = S$ or T, so that $a = 1$, $b = 1$, $c = 0$, $d = 1$ or $a = 0$, $b = 1$, $c = -1$, $d - 0$, and

$$B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

then, for example inductively on the number of terms in Theorem 9.1, either

$$\big((c\alpha + d\gamma)z + c\beta + d\delta\big)^{-2k} f(ABz) = (\gamma z + \delta)^{-2k} f(Bz) = f(z)$$

or

$$\big((c\alpha + d\gamma)z + c\beta + d\delta\big)^{-2k} f(ABz) = (\alpha z + \beta)^{-2k} f\left(\frac{-1}{Bz}\right) = (\alpha z + \beta)^{-2k} f\left(\frac{-\gamma z - \delta}{\alpha z + \beta}\right) = f(z).$$

The first of the above relationships tells us that f is periodic with period 1. Thus we can write f as a function of

$$q = e^{2\pi i z}.$$

More precisely we could put $|q| = e^{-2\pi\Im z}$, $\arg q = 2\pi(\Re z - \lfloor z \rfloor)$. Then $z \in \mathbb{Z}$ and z satisfying, say, $-\frac{1}{2} \leq \Im z < \frac{1}{2}$ is equivalent to $0 < |q| < 1$. In other words, regardless of the branch of the logarithm,

$$f(z) = f\left(\tfrac{\log q}{2\pi i}\right) = \widetilde{f}(q)$$

where $\widetilde{f}$ is meromorphic on the punctured disc $\mathcal{A} = \{q : 0 < |q| < 1\}$. If we can extend $\widetilde{f}$ to being meromorphic (or analytic) at 0, then we can say that f is meromorphic (or analytic) at ∞. More precisely this would mean that $\widetilde{f}$ has a Laurent expansion about 0,

$$\widetilde{f}(q) = \sum_{n=-N}^{\infty} a_n q^n.$$

Definition 9.4. *A weakly modular function is called a modular function when it is meromorphic at ∞, and if it is analytic there we write $f(\infty) = \widetilde{f}(0)$. A modular function which is analytic on $\widetilde{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ is called a modular form. If such a function is 0 at ∞, then it is called a cusp form.*

Thus a modular form of weight $2k$ is given by a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \tag{7}$$

which converges for all $q \in \mathcal{D} = \{q : |q| < 1\}$ and satisfies

$$f(-1/z) = z^{2k} f(z).$$

It is a cusp form when $a_0 = 0$. The expansion (7) is called the Fourier expansion of f.

9.5. Lattice functions and modular forms. A lattice Λ can be thought of in various ways. One is that it is a discrete subgroup of a finite dimensional vector space V over $\mathbb{R}$ and there is an $\mathbb{R}$–basis $(e_1, \ldots, e_n)$ of V which is a $\mathbb{Z}$–basis of Λ. Thus when $V = \mathbb{C}$ we could suppose that there are ω_1, $\omega_2 \in \mathbb{C}\backslash\{0\}$ such that $\Im(\omega_1/\omega_2) > 0$ and

$$\Lambda(\omega_1, \omega_2) = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2,$$

i.e.

$$\Lambda(\omega_1, \omega_2) = \{m_1\omega_1 + m_2\omega_2 : m_1, m_2 \in \mathbb{Z}\}.$$

Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \in \mathrm{SL}_2(\mathbb{Z}).$$

Then

$$\begin{aligned} \omega_1' &= a\omega_1 + b\omega_2 \\ \omega_2' &= c\omega_1 + d\omega_2. \end{aligned}$$

is another basis of $\Lambda(\omega_1, \omega_2)$. Since

$$\frac{\omega_1'}{\omega_2'} = \frac{a\omega_1/\omega_2 + b}{c\omega_1/\omega_2 + d} \tag{8}$$

it follows from (2) that $\Im(\omega_1'/\omega_2') > 0$ also. Let

$$\mathcal{M} = \{(\omega_1, \omega_2) \in (\mathbb{C} \setminus \{0\})^2 : \Im(\omega_1/\omega_2) > 0\}.$$

Theorem 9.4. *Two elements of $\mathcal{M}$ define the same lattice if and only if they are congruent modulo* $\mathrm{SL}_2(\mathbb{Z})$.

Proof. In view of the discussion above it suffices to show that if (ω_1, ω_2) and (ω_1', ω_2') define the same lattice, then (8) holds with $\det A = 1$. In fact it suffices to show that (8) holds with $\det A = \pm 1$ for then the positive sign follows from (2) and the facts that $\Im(\omega_1/\omega_2) > 0$ and $\Im(\omega_1'/\omega_2') > 0$.

We have $\boldsymbol{\omega}' = A\boldsymbol{\omega}$ and $\boldsymbol{\omega} = A'\boldsymbol{\omega}'$ where $\mathbf{w}$ denotes the column vector $(w_1, w_2)^T$ and A, $A' \in \mathrm{GL}_2(\mathbb{Z})$. Then $\boldsymbol{\omega} = A'\boldsymbol{\omega}' = A'A\boldsymbol{\omega}$ and since ω_1 and ω_2 are linearly independent over $\mathbb{Z}$ we have $A'A = I$. Thus $\det A' \det A = 1$. But $\det A'$, $\det A \in \mathbb{Z}$. Hence $\det A = \pm 1$.

Let $\mathcal{R}$ denote the set of lattices $\Lambda(\omega_1, \omega_2)$ with $(\omega_1, \omega_2) \in \mathcal{M}$ and suppose that F satisfies

$$F : \mathcal{R} \to \mathbb{C}.$$

Let $k \in \mathbb{Z}$. Then F is of weight $2k$ when

$$F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda)$$

for every $\Lambda \in \mathcal{R}$ and every $\lambda \in \widetilde{\mathbb{C}}$. Now Λ is invariant under the action of $\mathrm{SL}_2\mathbb{Z}$. Moreover

$$\lambda\Lambda(\omega_1, \omega_2) = \Lambda(\lambda\omega_1, \lambda\omega_2).$$

Thus

$${\omega_2}^{2k} F\big(\Lambda(\omega_1, \omega_2)\big) = F\big(\omega_2^{-1}\Lambda(\omega_1, \omega_2)\big) = F\big(\Lambda(\omega_1/\omega_2, 1)\big).$$

Thus there is a function f on $\mathbb{H}$ such that

$$F\big(\Lambda(\omega_1, \omega_2)\big) = {\omega_2}^{-2k} f(\omega_1/\omega_2).$$

Since F is invariant under $\mathrm{SL}_2(\mathbb{Z})$,

$$f(z) = (cz + d)^{-2k} f(Az) \quad \text{for all } A \in \mathrm{SL}_2(\mathbb{Z}),\ z \in \widetilde{\mathbb{H}}.$$

On the other hand given such a function f we can reverse the process and obtain a lattice function of weight $2k$. Thus lattice functions are a fruitful way of creating and identifying modular forms. Perhaps the easiest way is by considering Eisenstein series

$$G_k(\Lambda) = \sum_{\omega \in \Lambda(\omega_1, \omega_2) \setminus \{0\}} \frac{1}{\omega^{2k}} = \sum_{m,n \neq 0,0} \frac{1}{(m\omega_1 + n\omega_2)^{2k}}.$$

The corresponding function on $\mathbb{H}$ is

$$G_k(z) = \sum_{m,n \neq 0,0} \frac{1}{(mz + n)^{2k}}. \tag{9}$$

By the way the above construction would fail if the exponent $2k$ were to be replaced by an odd exponent, for then the function would be identically 0.

Before proceeding further we need to discuss convergence. The following Lemma provides a basis for sufficiency.

Lemma. *Suppose that $\sigma > 2$, $0 < v_1 < v_2$ and $0 < u$, and $\mathcal{H}$ denotes the closed rectangle $\{z \in \mathbb{C} : -u \leq \Re z \leq u, v_1 \leq \Im z \leq v_2\}$. Then*

$$\sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \sup_{z \in \mathcal{H}} \frac{1}{|mz + n|^\sigma}$$

converges.

Proof. For each pair (m, n) which we sum over,

$$|mz + n|^2 = (m\Re z + n)^2 + (m\Im z)^2 \geq v_1^2 m^2$$

and

$$|mz + n|^2 = |z|^2 |m + nz^{-1}|^2 \geq |z|^2 (n\Im(z^{-1}))^2 = n^2 |z|^{-2} (\Im\overline{z})^2 \geq \frac{v_1^2 n^2}{u^2 + v_2^2}.$$

Thus $|mz + n|^{-1} \ll (\max(m, n))^{-1}$ uniformly for $z \in \mathcal{H}$, and so for any real $R > 1$

$$\sum_{\substack{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ |mz+n| \leq R}} \sup_{z \in \mathcal{H}} \frac{1}{|mz + n|^\sigma} \ll \sum_{\substack{n \in \mathbb{Z} \setminus \{0\} \\ n \ll R}} |n|^{-\sigma} + \sum_{\substack{m,n \\ 0 < |m| \leq |n| \ll R}} |n|^{-\sigma} \ll \sum_{n=1}^{\infty} \frac{1}{n^{\sigma - 1}}$$

Theorem 9.5. *Let $k \in \mathbb{N}$, $k > 1$. Then the Eisenstein series $G_k(z)$ given by (9) is a modular form of weight $2k$ and G_k has the Fourier expansion*

$$G_k(z) = 2\zeta(2k) + \frac{2^{2k+1}\pi^{2k}(-1)^k}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n z}.$$

Proof. By the Lemma G_k is uniformly and absolutely convergent on $\mathcal{H}$, and each term of the series is analytic on $\mathbb{H}$. Hence, by a theorem of Weierstrasse G_k is analytic in $\mathcal{H}$, and hence at every point of $\mathcal{H}$.

We have $m(z+1)+n = mz+m+n = 0\cdot z+0$ if and only if $m = n = 0$. Thus

$$G_k(z+1) = G_k(z).$$

Obviously $m(-1/z)+n = (-1/z)((-n)z+m)$, so

$$G_k(-1/z) = z^{2k}G_k(z).$$

Thus, by Theorem 9.3, G_k is weakly modular. We have to show that G_k is analytic at ∞. We establish this by exhibiting a Fourier series for G_k that is analytic at $q = 0$. We start from the partial fraction decomposition

$$\pi\cot\pi z = \frac{1}{z} + \sum_{\substack{n=-\infty\\ n\neq 0}}\left(\frac{1}{z+n} - \frac{1}{n}\right) \tag{10}$$

which is valid for all $z \in \mathbb{C}\setminus\mathbb{Z}$ and converges locally uniformly and absolutely in that domain. For $z \in \mathbb{H}$ we have

$$\pi\cot\pi z = \pi i\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1} = \pi i\frac{q+1}{q-1}.$$

Thus

$$\frac{1}{z} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) = -\pi i\left(1 + 2\sum_{r=1}^{\infty}q^r\right).$$

Differentiating both sides l times gives

$$\frac{(-1)^l l!}{z^{l+1}} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^l l!}{(z+n)^{l+1}} = -(2\pi i)^{l+1}\sum_{r=1}^{\infty}r^l e^{2\pi i r z}.$$

Now for $m \in \mathbb{N}$, we have $z \in \mathbb{H}$ if and only if $mz \in \mathbb{H}$. Thus

$$\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^l l!}{(mz+n)^{l+1}} = -(2\pi i)^{l+1}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}r^l e^{2\pi i r m z} = -(2\pi i)^{l+1}\sum_{n=1}^{\infty}\sigma_l(n)e^{2\pi i n z}$$

where

$$\sigma_l(n) = \sum_{d|n}d^l.$$

When l is odd, say $l = 2k-1 \geq 3$,

$$\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(2k-1)!}{(mz+n)^{2k}} = (2\pi i)^{2k}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi i n z}.$$

Moreover

$$\sum_{m=-\infty}^{-1}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^{2k}} = \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^{2k}}.$$

Hence

$$\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^{2k}}=\frac{2^{2k+1}\pi^{2k}(-1)^k}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi inz}.$$

Adding in the terms with $m=0$ (and $n\neq 0$) gives an extra $2\zeta(2k)$.

Recall that

$$\zeta(2k)=(-1)^{k-1}2^{2k-1}\pi^{2k}B_{2k}/(2k)!. \tag{11}$$

where B_l is the l–th Bernoulli number, and

Table 1

k	B_k	
0	$1/1=$	1.00000 00000
1	$-1/2=$	-0.50000 00000
2	$1/6=$	0.16666 66667
4	$-1/30=$	-0.03333 33333
6	$1/42=$	0.02380 95238
8	$-1/30=$	-0.03333 33333
10	$5/66=$	0.07575 75758
12	$-691/2730=$	-0.25311 35531
14	$7/6=$	1.16666 66667
16	$-3617/510=$	-7.09215 68627
18	$43867/798=$	54.97117 79449
20	$-174611/330=$	-529.12424 24242

Thus $\zeta(4)=\frac{\pi^4}{90}$, $\zeta(6)=\frac{\pi^6}{945}$. There are various standard notations. For example

$$g_2(z)=60G_2(z),\ g_3(z)=140G_3(z)$$

and then it follows that the Fourier expansion of

$$\Delta(z)=g_2(z)^3-27g_2(z)^2 \tag{12}$$

has no constant term. Thus Δ is a cusp form of weight 12. By multiplying out the series and collecting together like powers of q it follows that

$$\Delta(z)=(2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e^{2\pi inz}$$

where the $\tau(n)$ are integers with $\tau(1)=1$, $\tau(2)=-24$. This function was first studied by Ramanujan, and we will come back to it in Chapter 10.

Other standard notation is

$$E_k(z)=G_k(z)/(2\zeta(2k))$$

and then the Fourier expansion has constant term 1. Moreover, by (11),

$$\frac{2^{2k+1}\pi^{2k}(-1)^k}{(2k-1)!2\zeta(2k)}=\frac{2^{2k+1}\pi^{2k}(-1)^k(2k)!}{(2k-1)!(-1)^{k-1}2^{2k}\pi^{2k}B_{2k}}=-\frac{4k}{B_{2k}}.$$

Thus

$$E_k(z)=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)e^{2\pi inz}.$$

It should be born in mind that some authors write G_{2k} and E_{2k} for G_k and E_k respectively.

9.6. Zeros and poles of modular functions. For a function f , meromorphic on $\widetilde{\mathbb{H}}$ and not identically 0 we define, for each $w\in\mathbb{H}$, $v=v_w(f)$ so that $f(z)(z-w)^{-v}$ is analytic and non–zero at w. $v_w(f)$ is called the order of f at w. If $v_w(f)$ is positive, then it is the order of the zero of f at w. Likewise if $v_w(f)$ is negative, then $-v_w(f)$ is the order of the pole at w. When f is a modular function of weight $2k$ and w and Aw are both finite, then the relationship

$$f(z)=(cz+d)^{-2k}f(Az)$$

shows that $v_w(f)=v_{Aw}(f)$. For points at ∞ we define v_∞ to be the order (in q) of $\widetilde{f}(q)$.

Theorem 9.6. *Let f be a modular function of weight $2k$, not identically 0. Then*

$$v_\infty + \tfrac{1}{2}v_i(f) + \tfrac{1}{3}v_\rho(f) + \sum_{w\in\mathbb{D}^*} v_w(f) = \tfrac{k}{6}$$

where $\rho = e^{2\pi i/3}$ and $\mathbb{D}^* = \mathbb{D}\setminus\{i,\rho\}$.

Proof. We consider

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(z)}{f(z)}dz$$

where $\mathcal{C}$ is, with some provisos, the contour consisting of the horizontal line L from $\frac{1}{2}+iY$ to $-\frac{1}{2}+iY$ (where $Y>1$), the vertical line segment L_- from $-\frac{1}{2}+iY$ to ρ, the circular arc C of radius 1, centred at 0 from ρ to $-\overline{\rho}$ through i and the vertical line segment L_+ from $-\overline{\rho}$ to $\frac{1}{2}+iY$. The provisos are (i) that Y is chosen so that L avoids any singularity of the integrand, and (ii) if the integrand has a singularity on the remaining path, then the contour traverses a small detour consisting of a circular arc of small radius centred at the singularity and oriented so that singularities in $\mathbb{D}^*$ are included in the interior and those not in $\mathbb{D}^*$ are excluded from the interior. The integrand has singularities precisely at the zeros and poles of f and the residue at such points is the order of f at that point. Thus, by Cauchy's integral formula,

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f'(z)}{f(z)}dz = \sum_{w\in\mathbb{D}^*(Y)} v_w(f)$$

where $\mathbb{D}^*(Y) = \{z\in\mathbb{D}^* : \Im z \le Y\}$.

Since $f(z) = f(z+1)$ we have

$$\frac{f'}{f}(z) = \frac{f'}{f}(z+1) \tag{13}$$

and thus

$$\int_{L_-} \frac{f'(z)}{f(z)}dz = -\int_{L_+} \frac{f'(z)}{f(z)}dz$$

where any possible detours, except any which might occur at ρ and $-\overline{\rho}$, are included in the paths. In view of the relationship (13) such detours will match exactly. We also have

$$f(z) = z^{-2k}f(-1/z) \tag{14}$$

so

$$f'(z) = -2kz^{-2k-1}f(-1/z) - z^{-2k-2}f'(-1/z).$$

Thus

$$\frac{f'}{f}(z) = -\frac{2k}{z} - \frac{f'}{z^2 f}(-1/z).$$

Let C_- be the subpath of C from ρ to i and C_+ the subpath from i to $-\overline{\rho}$, with the poviso that we exclude any possible detours around ρ, i and $-\overline{\rho}$. Then

$$\int_{C_-} \frac{f'}{f}(z)dz = \int_{C_-} -\frac{2k}{z} - \frac{f'}{z^2 f}(-1/z)dz$$

and by the change of variable $w=-1/z$ this is

$$-2k\left(-\frac{2\pi i}{12}\right) - \int_{C+} \frac{f'}{f}(w)dw.$$

Thus

$$\int_C \frac{f'}{f}(z)dz = \frac{2\pi i k}{6}.$$

There may be detours around ρ, i and $-\overline{\rho}$. If f has a zero or pole at ρ, then by there will be one at $-\overline{\rho}$ of the same order. Letting the radius of the detour at ρ tend to 0 we pick up the i times the residue times minus the angle subtended by the paths L_- and C at ρ, which is $-2\pi/6$. Hence the contribution from ρ and $-\overline{\rho}$ to the integral along the path is

$$-2\pi i v_\rho(f)/3.$$

A detour around i likewise will pick up i times the residue times minus the angle subtended by the path C at i, which is $-\pi$. Thus the contribution from i to the integral along the path is

$$-\pi i v_i(f).$$

It remains to deal with the contribution from L. To summarise so far

$$\int_L \frac{f'}{f}(z)dz + \frac{2\pi i k}{6} - 2\pi i v_\rho(f)/3 - \pi i v_i(f) = 2\pi i \sum_{w\in\mathbb{D}^*(Y)} v_w(f).$$

In the integral along L we make the substitution $q = e^{2\pi i z}$. Then L is tranformed into the circle C_0 centred at 0 of radius $e^{-2\pi Y}$ and traversed in the clockwise direction. Moreover as $\widetilde{f}(q) = f(z)$, we have $\frac{\widetilde{f}'}{\widetilde{f}}(q)\frac{dq}{dz} = \frac{f'(z)}{f(z)}$. Hence

$$\int_L \frac{f'}{f}(z)dz = \int_{C_0} \frac{\widetilde{f}'}{\widetilde{f}}(q)dq.$$

Since $\widetilde{f}(q)$ is meromorphic at 0 there will be a punctured disc $\mathcal{A}$ centred at 0 on which $\widetilde{f}$ is analytic. Thus if Y is large enough $C_0 \subset \mathcal{A}$. Hence by Cauchy's integral formula

$$\int_{C_0} \frac{\widetilde{f}'}{\widetilde{f}}(q)dq = -2\pi i v_\infty(f).$$

Moreover

$$\sum_{w\in\mathbb{D}^*(Y)} v_w(f) = \sum_{w\in\mathbb{D}^*} v_w(f).$$

This completes the proof of the theorem.

When $k \in \mathbb{Z}$, let M_k denote the vector space over $\mathbb{C}$ of modular forms of weight $2k$, and let M_k^0 denote the subspace of cusp forms of weight $2k$. Let f be a non-cusp member of M_k. If g is another, then for some scalar c, $f - cg$ will be a cusp form. Thus every non-cusp member of M_k is a linear combination of f and a cusp form. Thus

$$\dim(M_k \setminus M_k^0) \le 1. \tag{15}$$

Indeed a concomitant argument shows that if M_k^j denotes the subspace of $f \in M_k$ in which $v_\infty(f) \ge j+1$ in q, then

$$\dim(M_k^{j-1} \setminus M_k^j) \le 1. \tag{16}$$

When $k \ge 2$, $G_k \in M_k$ but $G_k \notin M_k^0$. Thus

$$M_k = \mathbb{G}_k \oplus M_k^0 \quad (k \ge 2). \tag{17}$$

Let $f \in M_k$, so that f is analytic on $\widetilde{\mathbb{H}}$. In Theorem 9.6 each $v_z(f)$ is non–negative. Hence $k \ge 0$. Thus M_k is empty when $k < 0$. When $k = 1$ there is no solution to $l + \frac{1}{2}m + \frac{1}{3}n = \frac{k}{6}$ with l, m, n non–negative. Hence

$$M_1 = \emptyset.$$

When $k = 6$, we have seen that Δ is a cusp form of weight 12. Thus $v_\infty(\Delta) \ge 1$. Hence all other $v_z(\Delta)$ are 0. Thus Δ does not vanish on $\mathbb{H}$ and has a simple zero at ∞. Let k be arbitrary and $f \in M_k^0$. Then $g = f/\Delta$ has weight $2k - 12$ and

$$v_z(g) = v_z(f) - v_z(\Delta) = \begin{cases} v_z(f) - 1 & (z = \infty), \\ v_z(f) & (z \ne \infty). \end{cases}$$

Thus $v_z(g) \geq 0$ and is analytic on $\widetilde{\mathbb{H}}$ and thus belongs to M_{k-6}. In fact the relationship $f \to f/\Delta$ give an isomorphism between the vector spaces M_k^0 and M_{k-6}. More generally this relationship gives an isomorphism between M_k^{j+1} and M_{k-6}^j. We have seen that M_k^0 is empty when $k < 6$ or $k = 1$. Thus $\dim M_k \leq 1$ when $1 \leq k \leq 5$ and $k = 7$. We have $1 \in M_0$. Hence

$$\dim M_0 = 1.$$

Also, by (17), when $2 \leq k \leq 5$ or $k = 7$,

$$\dim M_k = 1.$$

Theorem 9.7. *For convenience define $G_0(z) = 1$. Then*
(i) M_k is empty when $k < 0$ or $k = 1$.
(ii) when $k \geq 0$,

$$\dim M_k = \begin{cases} \lfloor k/6 \rfloor & k \equiv 1 \pmod 6, \\ \lfloor k/6 \rfloor + 1 & k \not\equiv 1 \pmod 6. \end{cases}$$

(iii) when $k \geq 0$ and $k \neq 1$,

$$M_k = \mathbb{C}G_k \oplus \mathbb{C}\Delta G_{k-6} \oplus + \cdots + \oplus \mathbb{C}\Delta^j G_{k-6j}$$

where

$$j = \begin{cases} \lfloor k/6 \rfloor - 1 & k \equiv 1 \pmod 6, \\ \lfloor k/6 \rfloor & k \not\equiv 1 \pmod 6. \end{cases}$$

Recall that Δ is a linear combination of G_2^3 and G_3^2. In fact it can be shown that every G_k is polynomial in G_2 and G_3, and indeed that every M_k is spanned by the monomials $G_2^u G_3^v$ where u and v run over the solutions to $2u + 3v = k$ with $u \geq 0$, $v \geq 0$.

It can also be shown that

$$G_2(\rho) = 0, \quad G_3(i) = 0,$$

either directly or by utilising Theorem 9.6.

The cusp form Δ has several remarkable properties. One of them is the product formula below.

Theorem 9.8. *Let $z \in \mathbb{H}$. Then*

$$\Delta(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad (q = e^{2\pi i z}).$$

Proof. There is no very simple proof. We know that $\Delta \in M_6^0$, $\dim M_6^0 = 1$, and the coefficient of q in Δ is $(2\pi)^{12}$. Thus it suffices to show that

$$F(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

is of weight 12. Since it is immediate that it is periodic with period 1, it suffices to show that

$$F(-1/z) = z^{12} F(z) \quad (z \in \mathbb{H}).$$

Consider the function

$$G_1(z) = \frac{\pi^2}{3} - 8\pi^2 \sum_{n=1}^{\infty} \sigma(n) q^n. \tag{18}$$

We will show that

$$G_1(z) = \frac{2\pi i}{z} + \frac{1}{z^2} G_1(-1/z) \quad (z \in \mathbb{H}). \tag{19}$$

Then, by logarithmic differentiation,

$$\begin{aligned}
\frac{F'}{F}(z) &= 2\pi i\left(1-\sum_{n=1}^{\infty}\frac{24q^n}{1-q^n}\right)\\
&= 2\pi i\left(1-24\sum_{m=1}^{\infty}\sigma(m)q^m\right)\\
&= \frac{2\pi i.3}{\pi^2}G_1(z)\\
&= \frac{6i}{\pi}G_1(z)\\
&= \frac{6i}{\pi}\left(\frac{2\pi i}{z}+\frac{1}{z^2}G_1(-1/z)\right)\\
&= -\frac{12}{z}+\frac{d}{dz}\log F(-1/z).
\end{aligned}$$

Thus F satisfies

$$z^{12}F(z) = Cf(-1/z)$$

for some $C \in \mathbb{C}$. Since $F(-1/i) = F(i)$ and $i^{12} = 1$ we have $C = 1$.

To complete the proof of the theorem it suffices to show that G_1, given by (18), satisfies (19). Following the proof of Theorem 9.5, with some care as the double series is no longer absolutely convergent, we have

$$G_1(z) = 2\zeta(2) + \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(mz+n)^2}.$$

Then

$$\begin{aligned}
G_1(-1/z) &= 2\zeta(2) + z^2\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m+nz)^2}\\
&= 2\zeta(2) + z^2 2\zeta(2) + z^2\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{(m+nz)^2}.\\
&= z^2 2\zeta(2) + z^2\sum_{m=-\infty}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{(m+nz)^2}.
\end{aligned}$$

Thus it suffices to show that $L(z)$ and $R(z)$ converge and

$$L(z) = -\frac{2\pi i}{z} + R(z) \tag{20}$$

where

$$L(z) = \sum_{m=-\infty}^{\infty}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{(m+nz)^2}$$

and

$$R(z) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{(m+nz)^2}.$$

Note that the sums are different even though they are only interchanged. Let

$$S(z) = \sum_{m=-\infty}^{\infty}\quad\sum_{\substack{n=-\infty\\ (m,n)\neq(0,0),(1,0)}}^{\infty}\frac{1}{(m-1+nz)(m+nz)}$$

and

$$T(z) = \sum_{n=-\infty}^{\infty} \sum_{\substack{m=-\infty \\ (m,n)\neq(0,0),(0,1)}}^{\infty} \frac{1}{(m-1+nz)(m+nz)}.$$

We will show below that these series converge. Then the convergence of L follows from the relationship

$$S(z) - L(z) = \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(m-1+nz)(m+nz)^2} + \sum_{m\neq 0,1} \frac{1}{m(m-1)}.$$

In the last sum the terms with $m > 1$ sum to 1 and those with $m < 0$ sum to -1. Hence the above becomes

$$S(z) - L(z) = \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(m-1+nz)(m+nz)^2}.$$

Similarly the convergence of R follows from

$$\begin{aligned} T(z) - R(z) &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m-1+nz)(m+nz)^2} + \sum_{m\neq 0,1} \frac{1}{m(m-1)} \\ &= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m-1+nz)(m+nz)^2}. \end{aligned}$$

These series are absolutely convergent and hence can be interchanged. Thus they are identical. Therefore not only will the convergence of $L(z)$ and $R(z)$ follow from that of $S(z)$ and $T(z)$ but we will have

$$L(z) - R(z) = S(z) - T(z).$$

Thus to prove (20) it suffices to show that $S(z)$ and $T(z)$ converge and

$$S(z) - T(z) = -\frac{2\pi i}{z} \tag{21}$$

The sum over m in T when $n \neq 0$ is

$$\sum_{m=-\infty}^{\infty} \left(\frac{1}{m-1+nz} - \frac{1}{m+nz} \right).$$

The part with $m \geq 0$ sums to $\frac{1}{-1+nz}$ and the part with $m \leq -1$ sums to $-\frac{1}{-1+nz}$ Hence when $n \neq 0$ the sum over m in T is 0. When $n = 0$ the sum over m is

$$\sum_{m=2}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) + \sum_{m=-\infty}^{-1} \left(\frac{1}{m-1} - \frac{1}{m} \right) = 1 + 1 = 2.$$

Hence $T(z)$ converges to 2.

The series $S(z)$ is more complicated. We will complete the proof of the theorem by showing that it converges to $2 - \frac{2\pi i}{z}$. We have

$$\begin{aligned} S(z) &= \sum_{m=-\infty}^{\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{m-1+nz} - \frac{1}{m+nz} \right) + \sum_{\substack{m=-\infty \\ m\neq 0,1}}^{\infty} \left(\frac{1}{m-1} - \frac{1}{m} \right) \\ &= \frac{1}{z} \sum_{m=-\infty}^{\infty} (U((m-1)/z) - U(m/z)) + 2 \end{aligned}$$

where

$$U(w) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{w+n} - \frac{1}{n} \right).$$

For $m \neq 0$, by (10),

$$U(w) = \pi \cot \pi w - \frac{1}{w}.$$

Clearly

$$S(z) = 2 + \frac{1}{z} \left(\lim_{M' \to -\infty} U(M'/z) - \lim_{M \to \infty} U(M/z) \right)$$

and the convergence of $S(z)$ stands or falls on the existence of the limits above. Obviously

$$\lim_{M \to \pm\infty} U(M/z) = \lim_{M \to \pm\infty} \pi \cot \pi (M/z).$$

Now $\Re 2\pi i M/z = \Re 2\pi i M(x - iy)/|z|^2 = 2\pi M y/|z|^2$ and as $M \to \infty$, $e^{-2\pi i M/z} \to 0$. Thus

$$\pi \cot \pi M/z = \pi i \frac{1 + e^{-2\pi i M/z}}{1 - e^{-2\pi i M/z}} \to \pi i.$$

On the other hand, as $M \to -\infty$

$$\pi \cot \pi M/z \to -\pi i.$$

This establishes the convergence of $S(z)$ and its evaluation, and completes the proof of the theorem.

Exercises 9.2.

1. Let $E_k(z) = G_k(z)/(2\zeta(2k))$, $q = e^{2\pi i z}$. Show that

$$E_2(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_3(z) = 1 - 540 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$E_4(z) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$E_5(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$E_6(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n,$$

2. Prove that $\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n} \sigma_3(m)\sigma_3(n-m)$.

3. Prove that $11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m)$.

4. Prove that $756\tau(n) = 65\sigma_{11}(n) + 691\sigma_5(n) - 691.252 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m)$. Deduce Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$.