1. In class we showed that if \( \hat{f} \in L^1(\mathbb{R}) \) and \( f \) satisfies

\[
f(x) = \int_{-b}^{b} \hat{f}(t)e^{xt}dt \tag{0}
\]

and

\[
f(x) = 0 \tag{1}
\]

holds for \( |x| > a \), then \( f \) is identically 0. Prove that the conclusion follows provided only that (1) holds when \( x \in (a, a') \) where \( a < a' \).

Clearly the integral in (0) is differentiable arbitrarily many times and the \( k \)-th derivative satisfies

\[
f^{(k)}(x) = \int_{-b}^{b} \hat{f}(t)e^{ct}e^{yt}dt = \int_{-b}^{b} \hat{f}(t)e^{ct}e^{yt}(2\pi i)^k dt = \sum_{k=0}^{\infty} \frac{(2\pi iyt)^k}{k!} dt
\]

\[= 0 \text{ for } x \in (a, a'). \]

Let \( c \in (a, a') \) and \( y \in \mathbb{R} \). Then

\[f(y+c) = \int_{-b}^{b} \hat{f}(t)e^{ct}e^{yt}dt = \int_{-b}^{b} \hat{f}(t)e^{ct}e^{yt}\sum_{k=0}^{\infty} \frac{(2\pi iyt)^k}{k!} dt = \sum_{k=0}^{\infty} \frac{y^k}{k!} \int_{-b}^{b} \hat{f}(t)e^{ct}(2\pi it)^k dt = 0. \]

2. An entire function \( f \) is of exponential type \( T < \infty \) when

\[\limsup_{R \to \infty} R^{-1} \log \max_{|z|=R} |f(z)| = T. \]

(i) Prove that \( T < 0 \) iff \( f \equiv 0 \). (ii) Give examples of non-constant entire functions of type 0 and of type 1.

(i) Suppose that \( T < 0 \). Then we can choose a sequence \( \{ R_k \} \) of real numbers \( R_k \) such that \( R_k \to \infty \) as \( k \to \infty \) and \( \log \max_{|z|=R_k} |f(z)| \leq \frac{1}{2}TR_k < 0 \). Hence \( \max_{|z|=R_k} |f(z)| < 1 \) and by the maximum modulus principle \( f \) is bounded. But a bounded entire function is a constant (Liouville), say \( f(z) = c \). But then \( |c| = \max_{|z|=R_k} |f(z)| \leq \exp(\frac{1}{2}TR_k) \to 0 \) as \( k \to \infty \), and so \( c = 0 \). Now suppose that \( f \equiv 0 \). Then, for any \( R > 0 \), \( \log \max_{|z|=R} |f(z)| = -\infty \) and so \( T = -\infty \).

(ii) Any polynomial, not identically 0, is of exponential type 0. Other such functions are expressions like \( \sum_{k=0}^{\infty} \frac{z^k}{(2k)!} \). This is bounded by \( \exp(\sqrt{|z|}) \) and so is also of exponential type 0. The function \( f(z) = e^z \) satisfies, for any \( R > 0 \), \( \max_{|z|=R} |f(z)| = \max_{|z|=R} e^{Rz} = e^R \) and so is of type 1. Multiplying it by any function of order 0 gives another function of order 1.