1. Prove that if \( f, g \in L^1(\mathbb{R}) \), then \( \| f \circ g \|_1 \leq \| f \|_1 \| g \|_1 \).

By definition \((f \circ g)(x) = \int_{\mathbb{R}} f(y)g(x-y)dy\). Hence by the triangle inequality, \( |(f \circ g)(x)| \leq \int_{\mathbb{R}} |f(y)|g(x-y)|dy| \) and \( \| f \circ g \|_1 = \int_{\mathbb{R}} |(f \circ g)(x)|dx \leq \int_{\mathbb{R}} \| f \|_1 |g(x-y)|dy|dx\). By Fubini the RHS is \( \int_{\mathbb{R}} |f(y)| \left( \int_{\mathbb{R}} |g(x-y)|dx \right) dy\). The inner integral is \( \| g \|_1 \) by an obvious change of variable and thus the whole is \( \| f \|_1 \| g \|_1 \) as required. The fact that this is an upper bound for everything justifies the manipulations.

2. Let \( X > 0 \) and define \( f(x) = \max \left( 0, 1 - \frac{|x|}{X} \right) \). Prove that

\[
\hat{f}(t) = \begin{cases} \frac{1}{X} \left( \frac{\sin \pi Xt}{\pi t} \right)^2 & (t \neq 0), \\ X & (t = 0) \end{cases}
\]

and that the inverse Fourier transform of \( \hat{f} \) is \( f \). First a slight simplification – not necessary but it slightly simplifies some formulæ. Put \( f_X(x) = f(x) \). Then a change of variable, \( y = Xx \) gives \( \hat{f}_X(t) = \hat{f}_1(t/X) \), so we can assume \( X = 1 \) henceforward. Extend the function \( f \) to \( \mathbb{C} \setminus \{0\} \) by taking \( f(z) = (\pi z)^{-2} \sin^2 \pi z \). Then \( f \) has a removable singularity at \( z = 0 \) and \( \lim_{z \to 0} f(z) = 0 \). Thus \( f \) is essentially entire. Since \( f(-x) = f(x) \) it suffices to evaluate the integral when \( t \geq 0 \). Let \( L(R, \varepsilon) \), with \( R \) large and \( \varepsilon \) small, denote the path consisting of the line segments from \(-R\) to \(-\varepsilon\) and from \( \varepsilon \) to \( R \), and the semicircle of radius \( \varepsilon \) from \(-\varepsilon\) to \( \varepsilon \) via \(-i\varepsilon\). Then \( \lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_{L(R, \varepsilon)} f(z)e(-zt)dz = \hat{f}(t) \). Write the integrand as

\[
(2e^{-2\pi izt} - e^{2\pi iz(1-t)} - e^{2\pi iz(-1-t)}) \left( \frac{4\pi^2 z^2}{2} \right)^{-1} = g_1(z) - g_2(z) - g_3(z)
\]

where \( g_1(z) = \frac{2e^{-2\pi izt}}{4\pi^2 z^2} \),

\[
g_2(z) = \frac{e^{2\pi iz(1-t)}}{4\pi^2 z^2},
\]

\[
g_3(z) = \frac{e^{2\pi iz(-1-t)}}{4\pi^2 z^2}.
\]

Let \( C^{-}_R \) denote the semicircle of radius \( R \) from \( R \) to \(-R \) via \(-iR\) and let \( C^+_R \) denote the semicircle of radius \( R \) from \(-R \) to \( R \) via \( iR\). Then the \( g_j \) are analytic on and inside the contour \( L(R, \varepsilon) + C^+_R \). Moreover when \( t \geq 0 \), \( \int_{C_R^{-}} g_1(z)dz \) and \( \int_{C_R^{-}} g_3(z)dz \) both \( \to 0 \) as \( R \to \infty \), and likewise for \( \int_{C_R^{-}} g_2(z)dz \) when \( t \geq 1 \). Thus in each case, by Cauchy’s theorem, \( \int_{L(R, \varepsilon)} g_j(z)dz \to 0 \) as \( R \to \infty \). When \( 0 \leq t < 1 \), \( g_2(z) \) is analytic in \( \mathbb{C} \) except at \( z = 0 \) where it has a double pole with residue \( \frac{t-1}{2\pi i} \). Moreover \( \int_{C_R^{+}} g_2(z)dz \to 0 \) as \( R \to \infty \). Thus, by the residue theorem, \( \int_{L(R, \varepsilon)} g_2(z)dz \to -1 \) as \( R \to \infty \).

The inverse transform is

\[
\int_{-\infty}^{\infty} \left( 1 + t/X \right) e(xt)dt + \int_{0}^{\infty} \left( 1 - t/X \right) e(xt)dt
\]

and these integrals are easily computed by integrating by parts.

3. Prove that (i) \( \exp(-x^2) \) belongs to the Schwartz class but that (ii) \( \frac{1}{1+x^2} \) and (iii) \( \exp(-|x|) \) do not.

(i) It is easily proved by induction on \( k \) that \( \frac{d^k}{dx^k} \exp(-x^2) = P_k(x) \exp(-x^2) \) where \( P_k(x) \) is a polynomial of degree \( k \). Hence, for any pair of non–negative integers \( j, k \), \( x^j \frac{d^k}{dx^k} \exp(-x^2) \to 0 \) as \( |x| \to \infty \). (ii) \( e^{-|x|} \) is not differentiable at \( x = 0 \). (iii) \( x^2(1+x^2)^{-1} \not\to 0 \) as \( |x| \to \infty \) (for example).