1. Let the polynomials $B_p(x)$ for $k \in \mathbb{N}$ be defined on $\mathbb{R}$ by $B_1(x) = x - \frac{1}{2}$, $B_{p+1}(x) = \int_0^x B_p(y)dy - \int_0^1 (1-y)B_p(y)dy$ and let $\hat{B}_p(k)$ denote the Fourier coefficient (relative to the family $e^{ikx}$) on $L^2(S_1)$.

(i) Prove that for $p > 1$ we have $B_p(1) = B_p(0)$ and deduce that $B_p$ restricted to $\mathbb{R}/\mathbb{Z}$ is continuous.

(ii) Prove that $B_p(0) = 0$ and $\hat{B}_p(k) = -(2\pi ik)^{-p}$ when $k \in \mathbb{Z}\setminus\{0\}$.

(iii) Prove that when $p$ is even, then $\sum_{k=1}^{\infty} k^{-p} = 2^{p-1}\pi^{p}(\pi^{p/2} - 1)B_p(0)$.

(iv) Prove that $B_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$, $B_4(x) = \frac{1}{24}x^4 - \frac{1}{12}x^3 + \frac{1}{72}x^2 - \frac{1}{720}$ and that generally $B_p$ is a polynomial of degree $p$ with rational coefficients.

(v) Prove that $\zeta(2) = \frac{\pi^2}{6}$ and that $\zeta(4) = \frac{\pi^4}{90}$.

2. In the notation of the previous question, $B_1$ is the most interesting of the functions since it is not continuous on $\mathbb{R}/\mathbb{Z}$. It has a jump discontinuity at 0. Apart from the redefinition at 0 it coincides on $[0, 1)$ with the sawtooth function $s(x)$ defined by $s(0) = 0$, $s(x) = x - \frac{1}{2}$ when $0 < x < 1$ and otherwise by periodicity with period 1.

Let $E_K(x) = s(x) + \sum_{0 \leq |k| \leq K} \frac{e^{ikx}}{2\pi i k}$.

(i) Prove that $E_k(x)$ is an odd function of $x$.

(ii) Prove that if $x \notin \mathbb{Z}$, then $E'_K(x) = 1 + D_K(x)$ (the Dirichlet kernel).

(iii) Prove that if $0 < x < 1$, then $E_k(x) = E_k(x) - E_k(\frac{1}{2}) = \int_{\frac{1}{2}}^{x} D_K(y)dy$.

(iv) Prove that

$$\int_{\frac{1}{2}}^{x} D_K(y)dy = \left[ \frac{1 - \cos(\pi(2K+1)y)}{(2K+1)\pi \sin \pi y} \right]_{1/2}^{x} + \int_{1/2}^{x} \frac{1 - \cos(\pi(2K+1)y)}{(2K+1)\sin^2 \pi y} \cos \pi y dy.$$ 

(v) Prove that if $0 < x < 1$, then $|E_k(x)| \leq \frac{2}{(2K+1)\pi \sin \pi x}$.

(vi) Prove that for all $x$, $|E_k(x)| \leq \frac{1}{2}$. The facts that $E_k$ is odd and that when $0 < x \leq \frac{1}{2}$ we have $2x \leq \sin \pi x \leq \pi x$ are useful here.

(vii) Prove that $-\sum_{0 \leq |k| \leq K} \frac{e^{ikx}}{2\pi i k}$ converges to $s(x)$.

(viii) Prove that $\|E_K\|_2 = O(K^{-1/2})$ (relative to $L^2(\mathbb{R}/\mathbb{Z})$).