MATH 467 FACTORIZATION AND PRIMALITY
TESTING, FALL TERM 2017, PROBLEMS 4

Return by Monday 25th September

Congruences

1. Solve where possible.
   (i) \( 91x \equiv 84 \pmod{143} \)
   (ii) \( 91x \equiv 84 \pmod{147} \)

2. Prove that \( 7n^3 - 1 \) can never be a perfect square.

3. Let \( f(x) \) denote a polynomial of degree at least 1 with integer coefficients and positive leading coefficient.
   (i) Show that if \( f(x_0) = m > 0 \), then \( f(x) \equiv 0 \pmod{m} \) whenever \( x \equiv x_0 \pmod{m} \).
   (ii) Show that there are infinitely many \( x \in \mathbb{N} \) such that \( f(x) \) is not prime.

4. Suppose that \( m_1, m_2 \in \mathbb{N}, (m_1, m_2) = 1, a, b \in \mathbb{Z} \). Prove that \( a \equiv b \pmod{m_1} \)
   and \( a \equiv b \pmod{m_2} \) if and only if \( a \equiv b \pmod{m_1 m_2} \).

5. Prove that when a natural number is written in the usual decimal notation, (i) it is divisible by 3 if and only if the sum if its digits is divisible by 3 and (ii) it is divisible by 9 if and only if the sum if its digits is divisible by 9.

6. Prove that if \( p \) is prime, and \( a, b \in \mathbb{Z} \), then
   \[ (a + b)^p \equiv a^p + b^p \pmod{p}. \]