The first exam is on Wednesday 6th October, at 9:05 in 100 Boucke.

1. (i) Find the absolute value of \( \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} \). \( \left| \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)} \right| = \frac{|3+4i||-1+2i|}{|-1-i||3-i|} = \sqrt{\frac{(3^2+4^2)(1^2+2^2)}{(1^2+1^2)(3^2+1^2)}} = \sqrt{\frac{25.5}{2.10}} = \frac{5}{2}. \) (ii) Show that if \( a \neq 0 \), then \( \frac{1}{a} = \frac{\bar{a}}{|a|^2} \), and find the real part of \( \frac{4-3i}{1+i} \). Two solutions to first part. (1) Let \( a = u + iv \) where \( u \) and \( v \) are real. Then \( a^{-1} = u/(u^2 + v^2) - iv/(u^2 + v^2) = (u - iv)/(u^2 + v^2) = \bar{a}/|a|^2 \). (2) We have \( |a|^2 = a\bar{a} \), and \( |a| = 0 \) iff \( a = 0 \). Thus \( |a| \neq 0 \). Hence we can divide both sides of this by \( a|a|^2 \). \( \Re \left( \frac{4-3i}{1+i} \right) = \Re \left( \frac{(4-3i)(-1-i)}{-1+i} \right) = \Re \left( -\frac{7-i}{2} \right) = -\frac{7}{2} \).

2. Find the image under the Möbius transformation \( z \mapsto w : w = \frac{z-1}{z+1} \), of (a) the circle \( |z+2| = 1 \), (b) the line \( \Re z = 3z \). The inverse is given by \( z = \frac{-w-1}{w-1} \). (a) This is the set of points \( w \) such that \( 1 = \left| \frac{w-2}{w-1} \right| \) and so is a straight line, through the points \( w = 2 \) and \( w = 2 + i \). (b) Another way of writing the line is as the set of \( z \) such that \( |z+1-i| = |z-1+i| \). Substituting for \( w \) gives the set of \( w \) such that \( \left| \frac{w-1-2i}{w+i(2+i)/5} \right| = \sqrt{5} \).

3. Sketch the set of points \( z \) determined by the given condition. (a) \( |z-1-i| \neq |z+1+i| \), (b) \( |z-i-2| > 3 \). Which, if any, of these sets are regions? (a) Two half planes separated by the straight line through 0 and \(-1+i\). A union of two disjoint subsets, so not connected, so not a region. (b) Open annulus centred at \( 2+i \) and inner radius 3, extending to \( \infty \). Polynomially connected, so connected.

4. (a) Prove that \( \{ z : 0 < \Re z < 1 \} \) is an open set in \( \mathbb{C} \). (b) Prove that if \( S \) and \( T \) are closed sets in \( \mathbb{C} \), then so is \( S \cup T \). (a) Let \( S = \{ z : 0 < \Re z \leq 0 \} \) and let \( z \in S \). Let \( \delta = \min \{ \Re z, 1-\Re z \} \). Now let \( w \in D(z, \delta) \). Then \( |\Re w - \Re z| \leq |w-z| < \delta \), and so \( \Re w = \Re z + (\Re w - \Re z) > \Re z - \delta \geq \delta - \delta = 0 \) and \( \Re w = \Re z + (\Re w - \Re z) < \Re z + \delta \leq 1 - \delta + \delta = 1 \). Hence \( w \in S \) and so \( D(z, \delta) \subseteq S \). (b) Let \( U = S \cup T \) and use \( * \) to denote the complement with respect to \( \mathbb{C} \). Since \( S \) and \( T \) are closed, \( S^* \) and \( T^* \) are open. Let \( z \in U^* \). Then \( z \in S^* \) and \( z \in T^* \), and so there are \( r_1 > 0 \), \( r_2 > 0 \) such that \( D(z, r_1) \subseteq S^* \) and \( D(z, r_2) \subseteq T^* \). Let \( r = \min \{ r_1, r_2 \} \). Then \( D(z, r) \subseteq S^* \) and \( D(z, r) \subseteq T^* \) and so \( D(z, r) \subseteq S^* \cap T^* = U^* \). Hence \( U^* \) is open and thus \( U \) is closed.