MATH 401 INTRODUCTION TO ANALYSIS, PRACTICE FINAL EXAMS

The final exam for the course will be on Thursday 2nd May in 114 Steidle. The location can be checked at http://www.campusmaps.psu.edu/print/.

1. (25 points) Find all real numbers \( x \) such that

\[ |x + 1| - |3x - 1| < 1. \]

2. (25 points) Let \( A = \left\{ \frac{3n^2 + n^2}{n^2} : n \in \mathbb{N} \right\} \).
   (i) Prove that \( \inf A \) and \( \sup A \) exist.
   (ii) Prove that \( \inf A = 1 \).
   (iii) Prove that \( \sup A = 5 \).
   (iv) Is \( 1 \in A \)?

3. (25 points) Prove that \( 5^{1/3} \) is irrational.

4. (25 points) (i) Prove that if \( n \geq 6 \), then \( 2^n \leq (n - 1)! \).
   (ii) Prove that \( \lim_{n \to \infty} \frac{2^n}{n!} = 0 \).

5. (25 points) Prove, using the definition of a limit, that \( \lim_{n \to \infty} \frac{n^2 + n}{n^2 + 2n + 1} = 1 \).

6. (25 points) The sequence \( \langle x_n \rangle \) is defined inductively by \( x_1 = 3 \),

\[ x_{n+1} = \frac{1}{2} \left( x_n + \frac{5}{x_n} \right). \]

   (i) Prove that \( x_n > 0 \) for every \( n \in \mathbb{N} \).
   (ii) Prove that \( x_n^2 \geq 5 \) for every \( n \in \mathbb{N} \).
   (iii) Prove that \( \langle x_n \rangle \) is decreasing.
   (iv) Prove that \( \langle x_n \rangle \) converges and find the limit.

7. (25 points) State in each case the values of \( x \) for which the given series converges.

   (i) \( \sum_{n=1}^{\infty} \frac{1 + |x|^n}{2 + |x|^n} \),
   (ii) \( \sum_{n=1}^{\infty} \frac{|x|}{n + |x|} \),
   (iii) \( \sum_{n=1}^{\infty} \frac{n^2}{2^n} x^n \),
   (iv) \( \sum_{n=1}^{\infty} \frac{x^n}{n} \).

8. (25 points) Prove that the quartic equation \( x^4 - 12x^2 + x + 24 = 0 \) has four real roots.
MATH 401 INTRODUCTION TO ANALYSIS-I, SECOND PRACTICE FINAL EXAM

1. Find all real values of \( x \) such that \( \frac{x+1}{x^2+3} < \frac{2}{x} \).

2. Let \( A = \left\{ 1 - \frac{2}{\sqrt{n}} : n \in \mathbb{N} \right\} \).
   (i) Prove that \( \inf A \) and \( \sup A \) exist.
   (ii) Prove that \( \inf A = -1 \).
   (iii) Prove that \( \sup A = 1 \).
   (iv) Is \( 1 \in A \)?

3. Suppose that \( a < b \). Prove that there is an irrational number \( x \) such that \( a < x < b \).

4. Prove, using only the definition of a limit, that the sequence \( \langle \sqrt{n} \rangle \) diverges.

5. The sequence \( \langle x_n \rangle \) is defined by \( x_1 = 1 \) and \( x_n = \frac{n^2+1}{2(n^2-1)} x_{n-1} \) \( (n = 2, 3, 4, \ldots) \). Prove that
   (i) for each \( n \in \mathbb{N} \), \( x_n > 0 \).
   (ii) for each \( n \in \mathbb{N} \), \( x_{n+1} \leq x_n \).
   (iii) \( \lim_{n \to \infty} x_n \) exists, and find its value.

6. The sequence \( \langle x_n \rangle \) is defined by \( x_1 = 1 \), \( x_2 = \frac{5}{6} \) and \( x_n = \frac{1}{3} x_{n-1} + \frac{1}{3} x_{n-2} \) \( (n = 3, 4, 5, \ldots) \). Prove that
   (i) for every \( n \in \mathbb{N} \), \( x_n > 0 \).
   (ii) for every \( n \in \mathbb{N} \), \( x_n \leq \left( \frac{5}{6} \right)^{n-1} \).
   (iii) \( \lim_{n \to \infty} x_n \) exists and find its value.

7. State in each case whether the series below converges, and justify your assertions.
   (i) \( \sum_{n=1}^{\infty} \frac{1}{2n^2 - 1} \), \quad (ii) \( \sum_{n=1}^{\infty} \frac{n+1}{n^2 + 1} \), \quad (iii) \( \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} 5^{-n} \),
   \( (iv) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^{1/2}}{n + 1} \), \quad \( v) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2 + 1} \).

8. Prove, using only the definition of limit, that
   \( \lim_{x \to 1} (x^2 - 3x) = -2 \).