MATHEMATICAL AND NUMERICAL ANALYSIS OF LINEAR PERIDYNAMIC MODELS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the linear bond-based nonlocal peridynamic models with a particular focus on problems associated with nonstandard nonlocal displacement loading conditions. Both stationary and time-dependent problems are considered for a one-dimensional scalar equation defined on a finite bar and for a two-dimensional system defined on a square. The related peridynamic operators and associated functional spaces are analyzed. Applications to the numerical analysis of the finite-dimensional approximations to peridynamic models are also illustrated.

Key words. peridynamics model, nonlocal boundary conditions, well-posedness and regularity, Galerkin–Ritz approximations, error estimates, condition numbers

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1. Introduction. The peridynamic (PD) model proposed by Silling [21] is an integral-type nonlocal continuum model which provides a general theory for problems involving discontinuities or other singularities in the deformation. We refer to [6, 17, 26, 27] for reviews of the recent applications and theoretical development of the PD framework. The effectiveness of PD models has already been demonstrated in several sophisticated applications, including the fracture and failure of composites, crack instability, the fracture of polycrystals, and nanofiber networks. More recently, rigorous mathematical analysis of the PD models is also receiving much attention. For instance, an abstract variational formulation is presented in [18]. The existence and uniqueness of $L^2$ solutions of the PD models associated with bounded integral PD operators have been given in [2, 13, 14]. In [16], a nonlocal vector calculus was developed which can be applied to study boundary value problems (BVPs) related to the linear PD models.

In an earlier work [10], we studied the linear PD models defined in $\mathbb{R}^d$. By defining an appropriate functional analytical framework, we were able to establish the well-posedness and regularity theory for very general linear bond-based PD models. Connections between the nonlocal PD models and the classical elasticity equations are also established in a general setting with minimal regularity assumptions on the weak solutions. In what follows, we extend the earlier study to some nonlocal BVPs related to a linear bond-based PD model studied in many existing works [1, 7, 8, 13, 14, 16, 20, 23]; see section 2 for a brief description of the model equations. For technical simplicity, we focus on boundary conditions that may be viewed as natural nonlocal extensions to the local boundary conditions for classical differential equations. We investigate the variational formulations of the nonlocal BVPs and the existence, uniqueness, and regularity issues for both stationary and time-dependent PD models as well as the convergence of the solutions of PD equations to that of

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the classical differential equations when the PD horizon parameter goes to zero. As illustrations, the results are presented for a scalar equation defined on a finite bar in the one-dimensional (1-D) setting in section 3 and for a system of equations defined on a two-dimensional (2-D) square in section 4. We begin with a detailed analysis on the related PD operators and associated functional spaces and then establish the corresponding properties of the stationary model and the time-dependent equation. These analytical results reveal some interesting features of the nonlocal PD models such as a more general interpretation of the solution regularity for the nonlocal BVPs associated with the PD operators. Indeed, depending on the properties of the micro-modulus functions, it is shown that solutions may not possess any extra regularity over the data in some cases, while they could gain some fractional order regularity in some other cases. Moreover, the natural function spaces associated with the nonlocal PD models may include functions which are less regular than those often used for the corresponding local PDE models. Hence, our analysis further supports the idea that PD models may provide better modeling approaches to treat problems involving singular solutions [6, 21, 25, 27].

To demonstrate that the analytic framework developed here can be useful to the analysis of the numerical discretizations of the PD models, we consider various finite dimensional approximations to the nonlocal BVPs. A number of theorems are given on the convergence and error estimates of these approximations (see section 3 for stationary problems and section 4 for time-dependent problems), which appear to be the first of their kind in the literature. Estimates of the condition numbers of the resulting linear systems of equations are also provided here. Our findings are established in very general settings, and they are consistent with the results derived or observed for various specialized cases in earlier studies [1, 7, 8, 24].

In order to have a better understanding of the PD models, their numerical approximations, and their various applications, detailed analytical studies are becoming increasingly important. The results given in this paper can be considered as another step forward in this direction. We refer to section 5 for some additional discussions on the key conclusions.

2. The PD model. To describe the PD model, let \( x \) and \( x' \) denote positions of the particles in the reference configuration, and let \( u(x) \) and \( u(x') \) be the displacements with respect to the reference positions. Let \( F(u(x'), t) - u(x, t), x' - x) \) denote a pairwise force density function with units of force per volume squared; the PD equation of motion of the particle \( x \) is then

\[
\dot{u}_{tt}(x, t) = \int_{B_{\delta}(x)} F(u(x'), t) - u(x, t), x' - x) dx' + b(x, t),
\]

where \( b \) is the external force density. Here, the mass density is assumed to be a unit constant for simplicity, and with \( V \) being the domain of reference configuration of the material, we have the PD spherical neighborhood of \( x \in V \) given by \( B_{\delta}(x) = \{x' \in V : |x' - x| < \delta\} \) with the horizon parameter \( \delta > 0. \)

Our study here focuses on the isotropic, homogeneous microelastic material with pairwise equilibrated reference configuration, i.e., \( F(0, x' - x) = 0 \), modeled by the linear bond-based PD equation

\[
(2.1) \quad \dot{u}_{tt}(x, t) = \mathcal{L}_{\delta} u(x, t) + b(x, t),
\]
where

\[ L_\delta u(x) = c_\delta \int_{B_\delta(x)} \left( \frac{(x' - x) \otimes (x' - x)}{\sigma(|x' - x|)}(u(x') - u(x)) \right) dx'. \]

Here, \( \otimes \) denotes the Kronecker product, \( C(|x' - x|) = c_\delta \frac{(x' - x) \otimes (x' - x)}{\sigma(|x' - x|)} \) is called a micromodulus function in which \( c_\delta > 0 \) is a positive normalization constant and \( \sigma = \sigma(|x' - x|) \) is its kernel function which uniquely determines the properties of \( C(|x' - x|) \). The PD equation (2.1)–(2.2) models a spring network system [21], which is one of the most popular cases considered in many recent studies of the PD models.

Similar to [10], it is natural to make the following assumptions on the function \( \sigma \). To avoid degeneracy, we first introduce a nonnegative density function \( \rho = \rho(|x|) \) in \( L^1(B_\delta(0)) \), which is strictly positive in at least a measure nonzero subset of \( B_\delta(0) \).

Then we assume that

\[ \frac{|x|^2}{\sigma(|x|)} \geq \rho(|x|), \forall x \in B_\delta(0), \text{ and } \tau_3 := c_\delta \int_{B_\delta(0)} \frac{|x|^4}{\sigma(|x|)} dx < \infty. \]

Note that the nondegeneracy condition, given by \( |x|^2/\sigma(|x|) \) being bounded below by a nonnegative \( L^1(B_\delta(0)) \) function \( \rho(|x|) \), is easily satisfied for all practical choices of the micromodulus functions. In many existing mathematical analyses [2, 14, 16], it is assumed that the function \( |x|^2/\sigma(|x|) \) is itself in \( L^1(B_\delta(0)) \), which is more restrictive.

In fact, as pointed out in the literature (see, for instance, equation (2.10) in [14] and equation (93) in [21]), it is more general to relax the integrability assumption of \( |x|^2/\sigma(|x|) \) and only make the assumption on \( \tau_3 \) be finite in order to have a suitable definition of the elastic modulus for the corresponding material under consideration.

This, in fact, allows much more general micromodulus functions than those considered in the existing mathematical analysis of the PD models. Meanwhile, many interesting properties associated with the continuum description of the elastic materials can still be discussed under the above assumption.

### 3. The linear PD model on a finite bar.

In this section, we mainly discuss the linear PD model equation defined on a finite bar, represented by the interval \((0, \pi)\), with the solution \( u = u(x) \) satisfying either

\[ u \text{ is odd in } (-\delta, \delta) \text{ and } (\pi - \delta, \pi + \delta), \]

or

\[ u \text{ is even in } (-\delta, \delta) \text{ and } (\pi - \delta, \pi + \delta). \]

The condition (3.1) resembles (and thus is called) a Dirichlet-like condition, while (3.2) resembles a Neumann-like condition. Indeed, for smooth enough functions, (3.1) implies \( u(0) = u(\pi) = 0 \), while (3.2) leads to \( u_x(0) = u_x(\pi) = 0 \).

In practice, one often also studies other displacement loading or force loading conditions. We note that the odd and even extensions given in (3.1) and (3.2) allow us to more easily formulate the spectrum of the corresponding PD operator than, for example, the case of force loading condition. We thus focus on the former cases in this work and leave discussions on other general nonlocal BVPs to future works.

#### 3.1. The PD operators and related function spaces.

Similar to the previous work [10], we begin with a definition of the PD operators on \((0, \pi)\).

**Definition 3.1.** For \( \sigma \) satisfying (2.3), the PD operator \(-L_\delta^\sigma\) is defined by

\[ -L_\delta^\sigma u(x) = -c_\delta \int_{x-\delta}^{x+\delta} \frac{|x' - x|^2}{\sigma(|x' - x|)}(u(x') - u(x)) dx' \forall x \in (0, \pi) \]
Similarly, we have the definitions of \( L_0^f \) as finite. The corresponding inner product in \( \mathbb{R}^d \) is defined for functions satisfying the Neumann-like displacement loading condition (3.2). More precisely, with the Fourier sine and cosine series expansions,

\[
u(x) = \sum_k v_k^o \cos(kx) \quad \text{and} \quad v(x) = \sum_k v_k^e \cos(kx)
\]

with the coefficients \( \{u_k^o\} \) and \( \{v_k^e\} \) given by

\[
u_k^o = \frac{2}{\pi} \int_0^\pi u(x) \sin(kx) dx, \quad v_k^e = \frac{2}{\pi} \int_0^\pi v(x) \cos(kx) dx, \quad \forall k \geq 1,
\]

we have the following representations of the PD operators \(-L_0^o\) and \(-L_0^c\):

\[
\begin{align*}
-L_0^o u(x) &= \sum_k \eta_0(k) u_k^o \sin(kx), \\
-L_0^c v(x) &= \sum_k \eta_0(k) v_k^e \cos(kx),
\end{align*}
\]

where

\[
\eta_0(k) = CS \int_\delta^1 (1 - \cos(ky)) \frac{|y|^2}{\sigma(|y|)} dy \quad \forall k \geq 1.
\]

Note that we have used \( \sum_k \) to denote the sum over all positive integers. In the case of the Neumann boundary condition with the Fourier cosine expansion, we have the constant term removed from the expansion so that the corresponding function always has zero mean. We now define the associated functional spaces and explicit spectra and eigenfunctions for the PD operators, similar to the study for the PD equation in \( \mathbb{R}^d \) [10].

**Definition 3.2.** The space \( M^o_\sigma \), which depends on the kernel function \( \sigma \), consists of all functions \( u \) for which the \( M^o_\sigma \)-norm

\[
\|u\|_{M^o_\sigma} = \left( -\frac{2}{\pi} \langle L_0^o u, u \rangle \right)^{1/2} = \left( \sum_k \eta_0(k) u_k^o \right)^{1/2}
\]

is finite. The corresponding inner product in \( M^o_\sigma \) is given by

\[
(u, v)_{M^o_\sigma} := \sum_k \eta_0(k) u_k^o v_k^o \quad \forall u, v \in M^o_\sigma.
\]

In addition, given an exponent \( s \), we let \( M^{\sigma o}_s \) be generalized energy spaces with

\[
M^{\sigma o}_s = \left\{ u : \|u\|_{M^{\sigma o}_s}^2 = \sum_k \eta_0^2(k) u_k^o < \infty \right\}.
\]

Similarly, we have the definitions of \( M^{\sigma e}_s \) and their norms and inner products.

We can see that \( M^{\sigma o}_s \) is a Hilbert space, with its dual space with respect to the standard \( L^2 \) duality pairing given by \( M^{-\sigma o}_s \). Moreover, the PD operator \(-L_0^o\) is a self-adjoint operator, and it is also an isometry from \( M^o_\sigma \) to \( M^{-\sigma o}_s = M^{-1 o}_s \).

Let \( H^s_\sigma \) denote the fractional Sobolev space on \((0, \pi)\) for \( s \in [0, 1) \), which is the closure of \( C^\infty_\sigma \) in \( H^s(0, \pi) \). Here \( C^\infty_\sigma \) is the space of functions which are the restrictions on \((0, \pi)\) of functions in \( C^\infty(\mathbb{R}) \) satisfying (3.1), and \( H^s(0, \pi) \) is the standard fractional
order Sobolev space [28]. Note that $H^0_o = L^2_o = M^0_o$, and one can characterize $H^s_o$ and its norm in the Fourier space as

$$H^s_o = \left\{ v = \sum_k v_k \sin(kx) \mid \|v\|^2_s = \sum_k |v_k|^2 k^{2s} < \infty \right\}.$$  

Similarly, we have the same results for spaces $M^m_o$ related to $L^r_o$ and $H^s_o$. The formulations given in (3.4) are motivated by the explicit eigenfunctions and spectra of the PD-operators $-L^r_o$ and $-L^s_o$. For $-L^r_o$, it has a complete set of eigenfunctions $\{\sin(kx)\}$ with the corresponding spectra $\{\eta_o(k)\}$. Analogously, $-L^s_o$ has a complete set of eigenfunctions $\{\cos(kx)\}$ with spectra $\{\eta_s(k)\}$.

### 3.2. Space equivalence for special micromodulus functions.###

We now establish relations between $M^s_o$ and the Sobolev spaces. Similar results hold for the space $M^s_o$. Our discussion here is similar to that in [10] for functions in $\mathbb{R}^d$. We also refer to [3] for discussions of nonlocal spaces in more general non-Hilbert space settings and its application to nonlinear variational problems.

Under the assumption (2.3) and with the symbol $\mapsto$ being the conventional notation for the continuous embedding between spaces, we first have what follows.

**Lemma 3.3.** Let $\sigma = \sigma(|y|)$ satisfy (2.3). We have for $k \geq 1$,

$$\eta_o(k) \geq \inf_{k \geq 1} c_s \int_{-\delta}^\delta (1 - \cos(ky)) \rho(|y|) dy > 0, \tag{3.8}$$

$$0 \leq \tau_o k^2 - 2\eta_o(k) \leq \frac{k^4 \delta^2 \tau_o}{12}. \tag{3.9}$$

Moreover, $H^1_o \mapsto M^o_o \mapsto L^2_o, H^2_o \mapsto M^{2o}_o \mapsto L^2_o$. Similar results hold for spaces associated with (3.2).

**Proof.** By the assumption on $\sigma = \sigma(|x|)$, we have for any $k \geq 1$ that

$$\eta_o(k) \geq c_s \int_{-\delta}^\delta (1 - \cos(ky)) \rho(|y|) dy > 0.$$  

Since $\rho \in L^1(-\delta, \delta)$, by the Riemann lemma, we have

$$\lim_{k \to \infty} \int_{-\delta}^\delta \rho(|y|) \cos(ky) dy = 0,$$

which gives

$$\lim_{k \to \infty} \int_{-\delta}^\delta (1 - \cos(ky)) \rho(|y|) dy = \int_{-\delta}^\delta \rho(|y|) dy > 0.$$  

Thus, we have the infimum attainable in (3.8), which remains strictly positive, thus implying that $M^o_o \hookrightarrow L^2_o$.

Next, using the inequality $|1 - \cos(x)| \leq |x|^2/2$, we have

$$0 < 2\eta_o(k) \leq c_s \int_{-\delta}^\delta \frac{k^2 |y|^4}{\sigma(|y|)} dy \leq \tau_o k^2 \quad \forall k \geq 1. \tag{3.10}$$

We get the first inequality in (3.9). Using Taylor expansions of the cosine function, we can also get the second inequality in (3.9). The continuous space embedding results
then follow from the definitions of the corresponding spaces. We omit the details here (for a proof of similar results in $\mathbb{R}^d$, we refer to [10]).

The above lemma implies in particular that the solution spaces of the nonlocal BVPs of the PD model are subspaces of $L^2$. In fact, for a micromodulus function $\sigma$ such that $y^2/\sigma(y) \in L^1(0, \delta)$, $L^2$ is the natural solution space. As shown in [2, 10, 14] for problems defined in $\mathbb{R}^d$, the linear PD operator is a bounded linear operator from $L^2$ to itself. For the finite bar case, we also have what follows.

Lemma 3.4. Let $\sigma = \sigma(|y|)$ satisfy the additional condition that

\begin{equation}
\frac{y^2}{\sigma(y)} \in L^1(0, \delta);
\end{equation}

then $M^{2o}_\sigma = M^o_\sigma = L^2_0$ and $-L^o_\sigma$ is a bounded linear operator from $L^2_0$ to $L^2$, and

\begin{equation}
0 < \inf_{k \geq 1} \eta_\delta(k) \leq \eta_\delta(k) \leq 4cs \int_0^\delta \frac{y^2}{\sigma(y)} dy \quad \forall k \geq 1.
\end{equation}

Similar results hold for spaces associated with (3.2).

Using the definition of $\eta_\delta(k)$ given in (3.5), it is obvious that the rightmost inequality of (3.12) holds, while the lower bound can be derived in the same way as (3.8) with $\rho(|y|)$ being replaced by $y^2/\sigma(|y|)$.

We note that while the above lemma implies the equivalence of norms between $M^{2o}_\sigma$, $M^o_\sigma$, and $L^2_0$ and the boundedness of $-L^o_\sigma$ from $L^2_0$ to $L^2$, such equivalence relations and the operator bound of $-L^o_\sigma$ are not uniform with respect to the horizon parameter $\delta$. In fact, the upper bound in (3.12) tends to grow to infinity as $\delta \to 0$. This is not surprising since, based on the analysis given later in the section 3.4, the operator $-L^o_\sigma$ actually converges to a second order differential operator and is thus unbounded from $L^2_0$ to $L^2$ in such a limit.

For more general micromodulus functions, the PD operator $-L_\delta$ may become unbounded in $L^2$, yet the basic existence and uniqueness results remain valid as we demonstrate below, with the discussion taking place in other function spaces defined earlier. For convenience, we define the following function:

\begin{equation}
\omega_\delta(\alpha, \chi) = c_\delta \int_{-\chi}^{\chi} \frac{1 - \cos(z)}{|z|^{1+\alpha}} dz.
\end{equation}

Lemma 3.5. Let $\sigma = \sigma(|y|)$ satisfy, for some positive constant $\gamma_1$,

\begin{equation}
\sigma(|y|) \geq \gamma_1 |y|^{3+\alpha} \quad \forall |y| \leq \delta
\end{equation}

for some exponent $\alpha \in (0, 2)$. Then we have

\begin{equation}
0 \leq \eta_\delta(k) \leq C^1_\delta(\alpha)^2 k^\alpha \quad \forall k \geq 1,
\end{equation}

for $C^1_\delta(\alpha) = (\omega_\delta(\alpha, \infty)/\gamma_1)^{\frac{1}{\alpha}}$, and $H^\alpha/2 \rightarrow M^o_\sigma$ and $H^\alpha \rightarrow M^{2o}_\sigma$. Moreover,

\begin{align}
\|u\|_{M^o_\sigma} &\leq C^1_\delta(\alpha)\|u\|_{\alpha/2} \quad \forall u \in H^\alpha/2, \\
\|u\|_{M^{2o}_\sigma} &\leq C^1_\delta(\alpha)^2\|u\|_{\alpha} \quad \forall u \in H^\alpha.
\end{align}

Similar results hold for spaces associated with (3.2).

Proof. The lemma follows straightforwardly from the observation that for any $k \geq 1$, the coefficient $\eta_\delta(k)$ as defined in (3.5) satisfies

$$0 \leq \eta_\delta(k) \leq \frac{c_\delta}{\gamma_1} \int_{-\delta}^{\delta} (1 - \cos(ky)) \frac{|y|^2}{|y|^{3+\alpha}} dy \leq C^1_\delta(\alpha)^2 k^\alpha,$$
In fact, micromodulus functions of the form of the linear PD models, conditions (3.11), (3.14), and (3.18) are often very relevant. For completeness, parallel conclusions are also stated (without proof) on the solution in (3.19)

\[ \sigma(|y|) \leq \gamma_2 |y|^{3+\beta} \quad \forall |y| \leq \delta \]

for some exponent \( \beta < 2 \). Then we have

\[ \eta_\beta(k) \geq C_2^\delta (\beta)^2 k^\beta \quad \forall k \geq 1, \]

for \( C_2^\delta (\beta) = (\omega_\delta(\beta, \delta)/\gamma_2)^{\frac{1}{2}} \), and \( M^\sigma_\gamma \hookrightarrow H^\beta_0 \) and \( M^{2\sigma}_\gamma \hookrightarrow H^\beta_0 \). Moreover,

\[ C_2^\delta (\beta) \| u \|_{\beta/2} \leq \| u \|_{M^\sigma_\gamma} \quad \forall u \in M^\sigma_\gamma, \]

\[ C_2^\delta (\beta)^2 \| u \|_{\beta} \leq \| u \|_{M^{2\sigma}_\gamma} \quad \forall u \in M^{2\sigma}_\gamma. \]

Similar results hold for spaces associated with (3.2).

The proof again follows from similar estimates on \( \eta_\beta(k) \) for any \( k \geq 1 \):

\[ \eta_\beta(k) \geq \frac{c_\delta}{\gamma_2} \int_{-\delta}^{\delta} \frac{|y|^2}{(1 - \cos(ky)) |y|^{3+\beta}} dy \geq C_2^\delta (\beta)^2 k^\beta. \]

We note that for \( \beta < 0 \), Lemma 3.4 can be applied, which gives stronger results.

Based on the above discussion, we see that under suitable conditions on the micromodulus function, for example, if (3.14) and (3.18) are satisfied with \( \beta = \alpha \in (0, 2) \), then the spaces \( M^\sigma_\gamma \) and \( M^{2\sigma}_\gamma \) are equivalent to standard fractional Sobolev spaces \( H^\alpha_{\gamma/2} \) and \( H^\alpha_\gamma \), respectively. We caution again that the equivalence relations are not uniform as \( \delta \to 0 \).

Let us also note that for many micromodulus functions used in the existing studies of the linear PD models, conditions (3.11), (3.14), and (3.18) are often very relevant. In fact, micromodulus functions of the form \( \sigma(|y|) = \gamma_2 |y|^{3+\beta} \) with \( \beta < 2 \), especially the case with \( \beta = 0 \) and \( |y|^2/\sigma(|y|) = \gamma_2^{-1} |y|^{-1} \), have been frequently used in the literature [1, 8, 20].

3.3. Properties of the stationary PD model. We now discuss the existence and uniqueness of weak solutions to the stationary (equilibrium) PD model. We again focus on the solution in \( M^\sigma_\gamma \) of the equation

\[ -\mathcal{L}^\delta_\sigma u = f. \]

For completeness, parallel conclusions are also stated (without proof) on the solution in \( M^{2\sigma}_\gamma \) of the equation

\[ -\mathcal{L}^{2\sigma}_\delta u = f. \]

Similar to elliptic equations, we may establish the variational theory and regularity properties for the stationary PD model. First, we obviously have what follows.

Lemma 3.7. Let \( \sigma = \sigma(|y|) \) satisfy (2.3); then for any \( f \in M^{-\sigma}_\gamma \), the problem (3.22) has a unique solution \( u \in M^\sigma_\gamma \) which is the minimizer of the functional

\[ \frac{1}{2} \| u \|_{M^\sigma_\gamma}^2 - (u, f)_{L^2} = \frac{1}{2} \sum_k \eta_\beta(k) u_k^2 - (u, f)_{L^2} \]

in \( M^\sigma_\gamma \). A similar result holds for (3.23).
As for the regularity, we have for some special micromodulus functions what follows.

**Lemma 3.8.** Let $\sigma = \sigma(|y|)$ satisfy (2.3) and (3.18). For $\beta \in (0, 2]$, the problem (3.22) has a unique solution $u = u(x) \in H^{m+\beta}_o$ whenever $f \in H^{m}_o$ for any $m \geq -\beta$. Moreover,

$$\|u\|_{m+\beta} \leq C_2(\beta)^{-2} \|f\|_m.$$

Similar results hold for (3.23).

**Proof.** Considering (3.22), with $\{f^\sigma_k\}$ being the coefficients of the Fourier sine expansion of $f$, obviously we have a unique solution $u = u(x)$ with $u^\sigma_k = f^\sigma_k/\eta^\sigma(k)$ being the coefficients of its Fourier sine expansion. Moreover,

$$\sum_k C^\sigma_2(\beta)^4k^{2m+2\beta}u^\sigma_k^2 \leq \sum_k \eta^\sigma_2(k)k^{2m}u^2_k = \sum_k k^{2m}f^\sigma_k^2 < \infty.$$

So, we have the solution $u = u(x) \in H^{m+\beta}_o$ with the desired bound.

The above lemma implies a lifting in regularity on the order of $\beta \in (0, 2]$ for micromodulus functions satisfying (3.14) and (3.18). We now go back to consider the regularity of solutions of (3.22) and (3.23) for more general cases.

**Lemma 3.9.** Let $\sigma = \sigma(|y|)$ satisfy (2.3); The problem (3.22) has a unique solution $u \in M^2_o$ for $f \in M^2_o$, and $\|u\|_{M^2_o} = \|f\|_{M^2_o}$. Moreover, if $f \in L^2_o$, we have $u \in M^{2\sigma}_o$ and $\|u\|_{M^{2\sigma}_o} = \|f\|_{L^2_o}$. Similar results hold for (3.23).

Using the linearity of the model, we have the following corollary, similar as in [10]. We now let $D_y$ be a difference operator given by $D_yv(x) = v(x + y) - v(x)$ for any function $v$ defined in a suitable space. Then let $P$ be a linear operator which commutes with $D_y$ for any $y$ with $P^{-1}$ its inverse (defined in a suitable space). Moreover, for any measurable function $v$ that is odd in $(-\delta, \delta)$ and $(\pi - \delta, \pi + \delta)$, we assume that $Pv$ in the sense of distributions, remains odd in $(-\delta, \delta)$ and $(\pi - \delta, \pi + \delta)$. Examples of such operators include linear differential operators of even orders with constant coefficients.

**Corollary 3.10.** Under the above assumptions on $P$, we have $P\mathcal{L}_o^\sigma = \mathcal{L}_o^\sigma P$ and $P^{-1}\mathcal{L}_o^\sigma = \mathcal{L}_o^\sigma P^{-1}$. This in turn implies that if $u$ is a solution of (3.22) with a given function $f$, then $Pu$ and $P^{-1}u$ are the solutions of (3.22) with the right-hand side being $ Pf$ and $P^{-1}f$, respectively. A similar conclusion holds for (3.23).

Although the above corollary is trivial to derive, we note that problems with singular data are indeed of practical interests; see [7] for examples where Dirac-delta functions are used for $f$ as concentrated forces.

While $-\mathcal{L}_o^\sigma$ is a nonlocal integral operator, it shares some properties of an elliptic differential operator such as those associated with the elliptic regularity, as shown earlier. As another illustration, we know that if $u$ satisfies (3.22), with $\sigma$ satisfying (3.11) and $f$ being nonnegative, then

$$\left(\int_{-\delta}^{\delta} \frac{y^2}{\sigma(|y|)} dy \right) u(x) \geq \int_{-\delta}^{\delta} u(x + y) \frac{y^2}{\sigma(|y|)} dy \quad \forall x \in (0, \pi).$$

Thus, we can readily get the following maximum principle.

**Lemma 3.11.** Let $\sigma = \sigma(|y|)$ satisfy (3.11), let $f$ be nonnegative almost everywhere, and let $u$ be a solution of (3.22). Then $u$ is nonnegative in $(0, \pi)$, and its maximum is attained in the interior or it is identically zero. A similar conclusion holds for (3.23).
3.4. Convergence to differential equations. For \( f = f(x) \in L^2 \), we denote \( u^o \in H_0^2 \) by the solution of
\[
-\nu_{xx} = f \text{ in } (0, \pi), \quad \text{with } u(0) = u(\pi) = 0,
\]
and let \( u^e \in H_0^2 \) be the solution of the same equation with \( u_x(0) = u_x(\pi) = 0 \) and mean zero. Similarly, we let \( u_3^o = u_3^o(x) \) and \( u_3^e = u_3^o(x) \) be the solutions of (3.22) and (3.23) as defined, respectively, before. Then we have the following theorem.

**Theorem 3.12.** Let \( \sigma = \sigma(|y|) \) satisfy (2.3). Then, as \( \delta \to 0 \), we have for any \( u \in H^1_0, \|u\|_{M^2_{\sigma}} \to \|u\|_1 \). We also have the solutions \( \{u_3^o\} \) of (3.22) converge to the solution \( u^o \) of (3.24) in \( M^{20}_{\sigma} \), provided that \( f \in L^2 \) and
\[
(3.25) \quad \tau_\delta \to 2
\]
for \( \tau_\delta \) defined in (2.3). If we assume in addition that \( \sigma \) satisfies (3.14) and (3.18) for \( \alpha = \beta \in [0, 2] \) and constants \( \gamma_1 \) and \( \gamma_2 \), all independent of \( \delta \), then we have also \( \|u_3^o - u^o\|_{\beta} \to 0 \) as \( \delta \to 0 \). Similar results hold for the convergence of \( \{u_3^o\} \) to \( u^e \).

**Proof.** For the solutions of (3.22) and (3.24), the coefficients of the Fourier sine series satisfy
\[
\begin{align*}
&u_{3,k}^o = \frac{f_k^o}{\eta_\delta(k)}, \quad u_{3,k}^e = \frac{f_k^e}{k^2} \quad \text{respectively, with } \{f_k^e\} \text{ being the Fourier sine coefficients of } f = f(x).
\end{align*}
\]
Now
\[
\|u_{3,k}^o - u_{3,k}^e\|_{M^2_{\sigma}}^2 = \sum_k \eta_\delta(k)(u_{3,k}^o - u_{3,k}^e)^2 = \sum_k \left(1 - \frac{\eta_\delta(k)}{k^2}\right)^2 (f_k^o)^2.
\]
First, by the assumption (3.25) and the inequality (3.9), we have for any given \( k \), \( \eta_\delta(k) \to k^2 \) and
\[
\left(1 - \frac{\eta_\delta(k)}{k^2}\right)^2 (f_k^o)^2 \to 0
\]
as \( \delta \to 0 \). Moreover,
\[
\sum_k \left(1 - \frac{\eta_\delta(k)}{k^2}\right)^2 (f_k^o)^2
\]
is uniformly bounded by \( (1 + \tau_\delta)^2 \|f\|_2^2 \). Thus, we can apply the dominant convergence theorem to deduce
\[
\|u_{3,k}^o - u_{3,k}^e\|_{M^2_{\sigma}}^2 \to 0 \quad \text{as } \delta \to 0.
\]
Similarly, for any \( u \in H^1_0 \), since \( \|u\|_{M^2_{\sigma}} \leq \tau_\delta \|u\|_1^2 \), we also get \( \|u\|_{M^2_{\sigma}} \to \|u\|_1 \) by the dominant convergence theorem. Furthermore, with (3.18), for \( C_2^2(\beta)^2 = \omega_\delta(\beta, \delta)/\gamma_2 \), by (3.21), we have
\[
C_2^2(\beta)^2 \|u_{3,k}^o - u_{3,k}^e\|_{\beta} \leq \|u_{3,k}^o - u_{3,k}^e\|_{M^2_{\sigma}}.
\]
Now, by the definitions of \( \omega_\delta(\beta, \delta), \tau_\delta \), and condition (3.14), we have
\[
(3.26) \quad \lim_{\delta \to 0} \frac{\omega_\delta(\beta, \delta)}{\tau_\delta} \geq \lim_{\delta \to 0} \gamma_1 \left(\int_{-\delta}^{\delta} \frac{1 - \cos(y)}{|y|^{1+\beta}} dy\right) \left(\int_{-\delta}^{\delta} |y|^{1-\beta} dy\right)^{-1} = \frac{\gamma_1}{2},
\]
so that \( C^2(\beta)^2 \geq \gamma_1/\gamma_2 \) for \( \delta \) small, since \( \tau_\delta \to 2 \) as \( \delta \to 0 \). Thus, \( C^2(\beta)^2 \) is uniformly bounded below, which then implies that \( \|u^\delta_n - u^\delta\|_\beta \to 0 \). Similar proof can be given for the other conclusions given in the theorem.

**Remark 3.13.** The condition (3.25) is a natural assumption that leads to a consistent definition of the elastic constant of the material under consideration [14]. Unlike discussions in the literature \([2, 14]\), the above convergence theorem does not require extra assumptions on the regularities of the solutions to the PD model beyond those already established in this work and remains valid for weak solutions. Moreover, using the properties stated in the Corollary 3.10, one may establish convergence in the distribution sense.

**Remark 3.14.** From (3.12), we see that the constant \( C^2(\beta)^{-2} \) is uniformly bounded as \( \delta \to 0 \), provided that \( \tau_\delta \to 2 \).

To conclude the discussion here, let us note that if additional regularities on the function \( f \) can be assumed together with the order of convergence of \( \tau_\delta \) to 2, then one may also establish the order of convergence of \( u^\delta - u^\delta_0 \) in suitable norms using techniques similar to that in the proof of the theorem. One can also examine further the higher order expansions of the PD operator in \( \delta \); see for instance the formal derivations given in [29] for smoothly defined solutions.

### 3.5. Finite-dimensional approximations

In practice, finite difference, finite element, quadrature, and particle-based methods have all been used to solve PD models \([1, 7, 8, 9, 13, 15, 19, 20, 24]\). Given the nonlocal nature, in all of these approximations, the discrete schemes are also nonlocal. The functional setting presented earlier can be readily used to analyze the convergence of finite-dimensional approximations and numerical solutions. We now present a few illustrative examples.

First, we consider the problem (3.22). Let \( \{V_n\} \) be finite-dimensional subspaces of \( M^\sigma_\omega \), and assume that as \( n \to \infty \), \( \{V_n\} \) is dense in \( M^\sigma_\omega \); that is, for any \( u \in M^\sigma_\omega \), there exists a sequence of \( u_n \in V_n \) such that

\[
(3.27) \quad \|u - u_n\|_{M^\sigma_\omega} \to 0 \quad \text{as} \quad n \to \infty.
\]

The same discussion can be made for \( M^\sigma_\omega \) and (3.23) as well.

We seek approximations to the PD model in \( V_n \), which fall into the so-called internal or conforming approximations. Examples include the space formed by the first \( n \) Fourier sine modes or the space formed by continuous piecewise finite element polynomials, satisfying (3.1), on a mesh having \( n \) grid points and the mesh parameter \( h_n \to 0 \) as \( n \to \infty \). In cases where the micromodulus function satisfies the suitable conditions so that \( H^s_\omega = M^\sigma_\omega \) for some \( s < 1/2 \), we may also take the space of odd piecewise polynomials which are not required to be continuous across grid points.

We let \( u_n \) be Galerkin–Ritz approximation to the problem (3.22); that is,

\[
(3.28) \quad \min_{v_n \in V_n} E(v_n) = \min_{v_n \in V_n} \frac{1}{2} \|v_n\|_{M^\sigma_\omega}^2 - (v_n, f)_{L^2}.
\]

It follows that \( u_n \) is in fact the best approximation of \( u \) in the subspace \( V_n \), measured by the norm of \( M^\sigma_\omega \). Thus, we easily get the following theorem.

**Theorem 3.15.** If \( \sigma = \sigma(|y|) \) satisfies (2.3), then for any \( f \in M^{-\sigma}_{-\infty} \), we have

\[
(3.29) \quad \|u - u_n\|_{M^\sigma_\omega} \leq \min_{v_n \in V_n} \|u - v_n\|_{M^\sigma_\omega} \to 0 \quad \text{as} \quad n \to \infty.
\]

A similar result holds for (3.23).
Given appropriate regularity of the solution $u$, one could also use the best approximation property to derive error estimates for various different approximation methods. For example, for the nonlocal BVPs considered here, Fourier spectral methods are quite natural and can be easily implemented. In this regard, we have what follows.

**Theorem 3.16.** Let $\sigma = \sigma(|y|)$ satisfy (2.3) and (3.18) with $\beta < 2$. With $f \in H^m_\sigma$ for some $m \geq \min\{0, -\beta\}$ and $V_n$ is the subspace spanned by the first $n$ Fourier sine modes, then

$$\|u - u_n\|_\gamma \leq C_2^\delta(\beta)^{-2} n^{-m - \beta + \gamma} \|f\|_m \quad \text{as} \ n \to \infty$$

for any $\gamma \leq \min\{m + \beta, m\}$. A similar result holds for (3.23).

**Proof.** Under the assumptions on $\sigma$, we have from (3.19) that

$$\eta_\delta(k) \geq C_2^\delta(\beta)^2 k^\beta \quad \forall k \geq 1.$$ 

We also have the identity

$$\|u - u_n\|_\alpha = \left\{ \sum_{k=n+1}^\infty \frac{k^{2\gamma}}{\eta_\delta^2(k)} f_k^2 \right\}^{1/2} = \left\{ \sum_{k=n+1}^\infty \frac{1}{k^{2m+2\beta-2\gamma}} \frac{k^{2\beta}}{\eta_\delta^2(k)} (k^m f_k)^2 \right\}^{1/2}.$$ 

The error estimates then follow by applying the regularity on $f$ to get

$$\|u - u_n\|_\gamma \leq \frac{1}{C_2^2(\beta)^2} n^{-m - \beta} \left\{ \sum_{k=n+1}^\infty (k^m f_k)^2 \right\}^{1/2} \leq C_2^\delta(\beta)^{-2} n^{-m + \gamma - \beta} \|f\|_m.$$ 

The case for (3.23) can be considered similarly.  

The condition $m \geq \min\{0, -\beta\}$ is stated in the above theorem only for assuring that the solution to the PD model has enough regularity. We note in addition that if $\sigma = \sigma(|y|)$ satisfy (3.14) and (3.18) with $\alpha = \beta \in [0, 2)$, then by (3.26), the constant $C_2^\delta(\beta)^{-2}$ is uniformly bounded as $\delta \to 0$, provided that $\tau_\delta \to 2$. Thus, the estimate (3.30) gives dependence on both the number of Fourier modes and the horizon parameter.

Similarly, for finite element methods, we have what follows.

**Theorem 3.17.** Let $\sigma = \sigma(|y|)$ satisfy (2.3) and additional conditions (3.14) and (3.18) with $0 \leq \beta \leq \alpha \in (0, 2)$. Given a regular and quasi-uniform mesh having the mesh parameter $h \to 0$, let $V_n$ be made up either by continuous piecewise finite element polynomials that are of degree $m \geq 1$ and odd in $(-\delta, \delta)$ and $(\pi - \delta, \pi + \delta)$, or, for $\alpha \leq 1$, $V_n$ can also be made by piecewise polynomials that are of degree $m \geq 0$, not necessarily continuous across grid points, and odd in $(-\delta, \delta)$ and $(\pi - \delta, \pi + \delta)$. Then for $f \in H^m_{\sigma^\alpha}$ with $0 \leq m' \leq m + 1$ and $s \in [0, \beta/2]$, we have, as $h \to 0$,

$$\|u - u_n\|_s \leq c_s C_2^\delta(\alpha)^{1 + s'} C_2^\delta(\beta)^{-3 - s'} h^{m' - s'/2 + (\beta - \alpha)/2} \|f\|_{m' - \beta}$$

for $s' = 1 - 2s/\beta$ and some generic constant $c_s$ independent of $h$, $\delta$, and $f$. A similar result holds for (3.23).

**Proof.** For $s = \beta/2$, we first use the best approximation property and the norm equivalence to get that

$$\|u - u_n\|_{\beta/2} \leq C_2^\delta(\beta)^{-1} \|u - u_n\|_{M_\delta} \leq C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \|u - v_n\|_{M_\delta} \leq C_1^\delta(\alpha) C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \|u - v_n\|_{\alpha/2.}$$
By the standard finite element approximation theory in the fractional Sobolev space norms, which are consequences of the approximation theory in the integer order Sobolev space norm based on space interpolation techniques (see a direct derivation in [12]), we have, for $h$ small,

$$
\| u - u_n \|_{\beta/2} \leq C_1^\delta(\alpha) C_2^\delta(\beta)^{-1} \min_{v_n \in V_n} \| u - v_n \|_{\alpha/2} \\
\leq C_1^\delta(\alpha) C_2^\delta(\beta)^{-1} c h^{m'-\alpha/2} \| u \|_{m'}
$$

for some constant $c$, independent of $h$, $\delta$, and $u$. Now using the regularity result in Lemma 3.8, we have that for $m' \geq \beta$,

$$
\| u \|_{m'} \leq C_2^\delta(\beta)^{-1} \| f \|_{m'-\beta},
$$

which thus implies

\begin{equation}
(3.32) \quad \| u - u_n \|_{\beta/2} \leq c C_1^\delta(\alpha) C_2^\delta(\beta)^{-3} h^{m'-\alpha/2} \| f \|_{m'-\beta}
\end{equation}

for a generic constant $c$, independent of $h$, $\delta$, and $f$. This gives (3.31) for $s = \beta/2$.

By the choice of $m$, we can always take $m' = \beta \leq m + 1$ in the above derivation to get

\begin{equation}
(3.33) \quad \| u - u_n \|_{M_\sigma^\beta} \leq c C_1^\delta(\alpha) C_2^\delta(\beta)^{-2} h^{\beta-\alpha/2} \| f \|_0.
\end{equation}

Next, we may invoke the duality argument to prove (3.31) for $s = 0$. For $\beta = 0$, we already have the estimate. So we consider only $\beta > 0$. Let $w$ be the solution of

$$
-L_\beta^\omega w = u - u_n,
$$

and let $w_n$ be the finite element approximation of $w$ in $V_n$; from (3.33), we have

$$
\| w - w_n \|_{M_\sigma^\beta} \leq c C_1^\delta(\alpha) C_2^\delta(\beta)^{-2} h^{\beta-\alpha/2} \| u - u_n \|_0.
$$

On the other hand,

$$
\| u - u_n \|_{M_\sigma^\beta} \leq c C_1^\delta(\alpha) C_2^\delta(\beta)^{-2} h^{m'-\alpha/2} \| f \|_{m'-\beta}.
$$

So, we get that

$$
\| u - u_n \|_0^2 = ( w - w_n, u - u_n )_{M_\sigma^\beta} \leq c C_1^\delta(\alpha)^2 C_2^\delta(\beta)^{-4} h^{m'+\beta-\alpha} \| u - u_n \|_0 \| f \|_{m'-\beta}
$$

for a constant $c$, independent of $h$, $\delta$, and $f$. This leads to (3.31) for $s = 0$. The general conclusion (3.31) follows from these two special cases of $s = 0$ or $s = \beta/2$ based on the standard space interpolation techniques.

Note that for $\alpha \in (0, 1)$, all piecewise polynomial spaces, discontinuous or not, automatically become conforming elements for the internal discretization of the PD nonlocal BVPs considered here. In addition, for the case $\beta = 0$, we simply get from (3.32) that, for any $0 \leq m' \leq m + 1$,

\begin{equation}
(3.34) \quad \| u - u_n \|_0 \leq c C_1^\delta(\alpha) C_2^\delta(0)^{-3} h^{m'-\alpha/2} \| f \|_{m'},
\end{equation}

which gives sharper dependence on $\delta$. Since $\alpha$ can be arbitrarily small, the estimate shows the almost optimal order error estimates, with respect to the mesh size $h$, in $L^2$ even for $m = 0$, that is, $V_n$ being the space of piecewise constant functions.
Due to the use of the various bounds on the different norms in the derivation, the estimates in (3.31) provide only some conservative bounds that may only be tight in the worst-case scenario. In fact, if the bound on the best approximation \( \| u - v_n \|_{H^2} \) can be directly estimated instead of using the known results on the Sobolev norms, then we may avoid the explicit dependence on the parameter \( C_1^2(\alpha) \). Yet, the above analysis does provide some upper bounds on how the error might be dependent on the mesh size \( h \), horizon parameter \( \delta \), and the regularity of the solution and data in the model. For example, with \( \sigma(\| y \|) = \| y \|^\beta \), we can take \( \beta = 0, \gamma_2 = 1 \), and for any \( \alpha \in (0, 2), \gamma_1 = \delta^{-\alpha} \). Provided that \( \tau_\delta \rightarrow 2 \), we get \( c_3 = O(\delta^{-2}) \). Moreover, from (3.26), we have that \( C_2^0(0) \) is uniformly bounded from below. Direct calculation shows that

\[
C_1^0(\alpha)^2 = 2c_3\delta^\alpha \int_0^\infty \frac{1 - \cos(z)}{|z|^{1+\alpha}} dz = O(\delta^{-2+\alpha}),
\]

so that \( C_1^0(\alpha) \) is uniformly bounded from the above by \( a\delta^{-1+\alpha/2} \) as \( \delta \rightarrow 0 \). Then the error bound for \( \| u - u_n \|_0 \) becomes \( O(h^m - \alpha/2 \delta^{-1+\alpha/2}) \). In turn, this implies that one can expect the convergence of the finite element approximation when both \( h \) and \( \delta \) approach to zero, as long as one keeps \( h^m - \alpha/2 \delta^{-1+\alpha/2} \rightarrow 0 \). Naturally, in this case, the numerical solutions actually converge to the solutions of the classical differential equations by the results of section 3.4. We refer to [7, 8] for numerical results.

Concerning the finite-dimensional approximations, an issue of interests is the condition number estimation for the resulting stiffness matrices associated with the full finite element approximations; see [1, 7] for related discussions. The analytic framework provided here can again be readily applied. We give the following as an illustration.

**Theorem 3.18.** Let \( \alpha = \sigma(\| y \|) \) satisfy (2.3) and additional conditions (3.14) and (3.18) for some \( 0 \leq \beta \leq \alpha \in (0, 2) \). Given a regular and quasi-uniform mesh having the mesh parameter \( h \rightarrow 0 \), let \( V_n \) be as defined in the Theorem 3.17 with a set of nodal basis functions \( \{ \phi_j \}_{j=1}^n \). Let \( A^o = (A^o_{ij})_{1 \leq i,j \leq n} \) be the \( n \times n \) stiffness matrix corresponding to the finite element approximation of (3.22) with \( \lambda_1 \) and \( \lambda_n \) being the smallest and largest eigenvalues. Then, for \( h \) small, we have

\[
0 < c_1 C_2^0(\beta)^2 h \leq \lambda_1 \leq \lambda_n \leq c_2 C_1^0(\alpha)^2 h^{1-\alpha}
\]

and

\[
\text{cond}(A^o) \leq c\left(\frac{C_1^0(\alpha)}{C_2^0(\beta)}\right)^2 h^{-\alpha}
\]

for some generic constant \( c \), \( c_1 \), and \( c_2 \), independent of \( h \) and \( \delta \). A similar result holds for (3.23).

**Proof.** The result is a simple consequence of some space and norm equivalences and the inverse inequality. We first have, for the given finite element nodal basis, that there are some generic constants \( c_1 \geq c_2 > 0 \) such that \( |4, 11| \)

\[
c_2 h |z|^2 \leq \| u_n \|^2_0 \leq c_1 h |z|^2 \quad \forall u_n = \sum_{j=1}^n z_j \phi_j \in V_n,
\]

where \( z \in \mathbb{R}^n \) is the vector with components \( \{ z_j \} \). Then, for \( \alpha > 0 \),

\[
c_2 h |z|^2 \leq \| u_n \|^2_0 \leq \| u_n \|_{H^2}^2
\]

\[
\leq C_2^0(\beta)^{-2} \| u_n \|^2_{H^2} = C_2^0(\beta)^{-2} z^T A^o z
\]

\[
\leq (C_1^0(\alpha)/C_2^0(\beta))^2 \| u_n \|^2_{H^2} \leq c(C_1^0(\alpha)/C_2^0(\beta))^2 h^{-\alpha} \| u_n \|^2_0
\]

\[
\leq c c_1 (C_1^0(\alpha)/C_2^0(\beta))^2 h^{1-\alpha} |z|^2
\]
for a generic constant $c$, independent of $h$ and $\delta$. The eigenvalue and condition number estimates then follow immediately.

We pay special attention to the dependence of the condition number bound on the value of $\delta$. Take $\sigma(y) = |y|^2$ with $\tau_\delta = 1$ as an illustration (a case studied in [1]); (3.10) and a direct calculation then show that
\[
\eta_\delta(k) = \frac{3}{\delta^2} \int_0^\delta (1 - \cos(kx))dx = \frac{3}{\delta^2} \left( -\sin(k\delta) \right),
\]
which satisfies, for small $\delta < 1$,
\[
(3.37) \quad \tilde{c} \leq 3\delta^{-3}(\delta - \sin(\delta)) \leq \eta_\delta(k) \leq \min\{6\delta^{-2}, \tau_\delta k^2\} \quad \forall k \geq 1,
\]
where $\tilde{c}$ is a positive constant independent of $\delta$. We thus can take, for this special $\sigma = \sigma(|y|)$, $\beta = 0$. Then, using (3.37) and the inverse inequality for finite element functions, we get
\[
\tilde{c}||u_n||^2_{L^2} \leq ||u_n||^2_{M^2} \leq \min\{6\delta^{-2}||u_n||^2_{L^2}, \tau_\delta ||u_n||^2_{H^1}\} \leq \min\{6\delta^{-2}, c\delta^{-2}\} ||u_n||^2_{L^2},
\]
so that
\[
\text{cond}(A^0) \leq c \min\{\delta^{-2}, h^{-2}\},
\]
which, in the limit $h \to 0$, is consistent with the bound $c\delta^{-2}$ established both rigorously and computationally in [1] for $\sigma(|y|) = |y|^2$ with a finite $\delta$. This property can also be easily interpreted from the fact that this particular micromodulus function satisfies (3.11) so that the PD operator is bounded from $L^2$ to $L^2$ for any finite $\delta$. Meanwhile, for $\delta \to 0$, the above estimate also recovers the standard bound of $c\delta^{-2}$ for the finite element stiffness matrices associated with second order elliptic equations.

Theorem 3.18 gives a more general estimate, which also covers other interesting cases. For example, if we take instead $\sigma(|y|) = |y|^{1+\beta}$, for some $\beta \in [0, 2)$, by Theorem 3.18, we notice that the crucial quantity is $(C_1^\beta(\alpha)/C_2^\beta(\beta))^2$. Let us consider the case $\beta = 0$ in more detail which, as mentioned in the discussion of error estimate, corresponds to a choice of common interests. Following the estimates on $C_1^\beta(\alpha)$ given earlier, we have that, for any arbitrarily small $\alpha > 0$,
\[
\text{cond}(A^0) \leq c \min\{h^{-\alpha}, \delta^{-2}, h^{-2}\}.
\]
This shows a very mild dependence on the mesh size for any finite $\delta$. Furthermore, we note that this line of discussion may also have other implications. For example, for time-dependent PD models solved via explicit marching schemes, the time step constraint for the numerical stability may also be tied to the extreme eigenvalues of $A^0$. We then expect that similar conclusions concerning the mild dependence of the time-step stability conditions on the spatial mesh size hold for the micromodulus function with $\sigma(|y|) = |y|^2$. This is indeed the case as illustrated in [20] where more detailed discussions can be found.

### 3.6. The time-dependent PD model on a finite bar

Similar results can be established for the time-dependent PD models by adapting the approaches given in [10] to the setting of nonlocal BVPs as the case for the equilibrium models. We thus simply state a result without detailed derivation.

**Theorem 3.19.** Let $\sigma = \sigma(|y|)$ satisfy (2.3), and let $P$ be a time-independent operator that satisfies the same assumptions as those made in the Corollary 3.10. If
\( P_0 \in M_o, P_1 \in L^2, \) and \( P b \in L^2(0, T; L^2) \), then (2.1), with the nonlocal boundary condition (3.1), the initial conditions \( u(x, 0) = g_0(x) \), and \( u_t(x, 0) = g_1(x) \) and the forcing term \( b = b(x, t) \) has a unique solution \( u = u(x, t) \) with \( P u \in C([0, T], M_o) \cap H^1(0, T; L^2) \). In particular, for \( P = I \), we get \( u \in C([0, T], M_o) \cap H^1(0, T; L^2) \) with \( g_0 \in M_o, g_1 \in L^2, \) and \( b \in L^2(0, T; L^2) \). We also have \( u \in C([0, T], L^2) \cap H^1(0, T; M^{\sigma-o}) \) with \( g_0 \in L^2, g_1 \in M^{\sigma-o}, \) and \( b \in L^2(0, T; M^{\sigma-o}) \). Similar results hold for problems associated with the nonlocal boundary condition (3.2).

Similar to the discussion given in the previous section, we may also establish the convergence property of the solution of PD equation (3.1) to the standard wave equation in the limit when \( \delta \rightarrow 0 \). We skip the detailed discussion here.

4. PD model on a 2-D square. The framework established in section 3 is not limited to the 1-D problems but can be generalized to multidimensional cases as well.

As an interesting illustration, we consider a special 2-D example corresponding to the linear bond-based PD model on \( \Omega = (0, \pi)^2 \) with a greased-wall–like displacement loading condition.

Let \( L_\delta \) be as defined in (2.2); we consider function \( u = (u, v) \) with the Fourier expansion

\[
\begin{align*}
        u &= \sum_{k,l=1}^{\infty} \left( u_{kl} \sin(kx_1) \cos(lx_2) \cos(kx_1) \sin(lx_2) \right) , \\
        & \quad \text{with}
        \begin{align*}
            \left( u_{kl} \right) &= \frac{4}{\pi^2} \int_{\Omega} \left( u(x) \sin(kx_1) \cos(lx_2) \cos(kx_1) \sin(lx_2) \right) dx .
        \end{align*}
    \end{align*}
\]

Then

\[
\begin{align*}
    -L_\delta u &= \sum_{k,l=1}^{\infty} \left( \begin{array}{cc}
        \sin(kx_1) \cos(lx_2) & 0 \\
        0 & \cos(kx_1) \sin(lx_2)
    \end{array} \right) M_{\delta,kl} \left( \begin{array}{c}
        u_{kl} \\
        v_{kl}
    \end{array} \right) ,
\end{align*}
\]

where

\[
M_{\delta,kl} = c_\delta \int_{B_\delta(0)} \frac{1 - \cos(k\xi_1 + l\xi_2)}{\sigma(\xi)} \left( \begin{array}{cc}
        \xi_1^2 & \xi_1 \xi_2 \\
        \xi_1 \xi_2 & \xi_2^2
    \end{array} \right) d\xi .
\]

For convenience, we take the notation that \( A > B \) implies \( A - B \) is positive definite and \( A \geq B \) implies \( A - B \) is positive semidefinite for any symmetric matrices \( A \) and \( B \).

Similar to the 1-D case, we let the space \( M^{oe}_\sigma \) be one that consists of \( u \) of the form (4.1) associated with the norm and inner product given by

\[
\|u\|_{M^{oe}_\sigma} = \sqrt{(u, u)_{M^{oe}_\sigma}} \quad \text{and} \quad (u, \bar{u})_{M^{oe}_\sigma} = \sum_{k,l=1}^{\infty} (\hat{u}_{kl}, \hat{v}_{kl})_{M^{oe}_\delta} \left( \begin{array}{c}
        u_{kl} \\
        v_{kl}
    \end{array} \right)
\]

for any \( u, \bar{u} \in M^{oe}_\sigma \). Similar as before, we let \( M_\sigma^{oe} = M^{oe}_1 \). Moreover, the PD operator \( -L_\delta \) represented by (4.3) is self-adjoint, and it is also an isometry from \( M^{oe}_\sigma \) to \( M^{oe}_\sigma \). Define \( M_\rho^\delta \) by

\[
M_\rho^\delta = c_\delta \inf_{k,l \geq 1} \int_{B_\delta(0)} \rho(\xi) \frac{1 - \cos(k\xi_1 + l\xi_2)}{\xi^2} \left( \begin{array}{cc}
        \xi_1^2 & \xi_1 \xi_2 \\
        \xi_1 \xi_2 & \xi_2^2
    \end{array} \right) d\xi ,
\]
for the function $\rho$ (4.5) as given in (2.3), we can see that, again by the Riemann lemma, $M_\rho$ is positive definite. Similar to the 1-D case, we have a general embedding result.

**Lemma 4.1.** Let $\sigma = \sigma(|y|)$ satisfy the condition (2.3) and define

\[
M_{0,kl} = \begin{pmatrix} (2\mu + \lambda)k^2 + \mu l^2 & (\mu + \lambda)kl \\ (\mu + \lambda)kl & \mu k^2 + (2\mu + \lambda)l^2 \end{pmatrix},
\]

which is the symbol of a 2-D Navier operator with the greased-wall boundary condition. Then, for any $k \geq 1$ and $l \geq 1$, we have

\[
0 < M_\rho \leq M_{\delta,kl} \leq \frac{\tau_3}{2\mu} M_{0,kl}.
\]

Consequently, we have $H^s_{\rho,c} \hookrightarrow M^s_{\rho,c} \hookrightarrow L^2$, and $H^2_{\rho,c} \hookrightarrow M^2_{\rho,c} \hookrightarrow L^2$.

**Proof.** By (2.3), we have $M_{\delta,kl} \geq M_\rho > 0$. Using the inequality

\[
0 \leq 1 - \cos(k\xi_1)\cos(l\xi_2) \leq 1 - \left(1 - \frac{k^2\xi_1^2}{2}\right)\left(1 - \frac{l^2\xi_2^2}{2}\right) \leq \frac{(k^2 + l^2)|\xi|^2}{2}
\]

the fact that $M_{\delta,kl} > 0$ and $M_{0,kl} > 0$, and that $M_{0,kl}$ has two eigenvalues $\lambda_{\min} = \mu(k^2 + l^2)$, $\lambda_{\max} = (2\mu + \lambda)(k^2 + l^2)$, we have

\[
M_{\delta,kl} \leq c_\delta \left(\int_{B_1(0)} \frac{1 - \cos(k\xi_1 + l\xi_2)}{\sigma(|\xi|)} |\xi|^2 d\xi\right) I
\]

\[
= c_\delta \left(\int_{B_1(0)} \frac{1 - \cos(k\xi_1)\cos(l\xi_2)}{\sigma(|\xi|)} |\xi|^2 d\xi\right) I
\]

\[
\leq c_\delta \left(\int_{B_1(0)} \frac{(k^2 + l^2)}{2\sigma(|\xi|)} |\xi|^4 d\xi\right) I \leq \frac{\tau_3}{2\mu} M_{0,kl}.
\]

So we get (4.7). The rest of the conclusions follow simply from the above inequality and the fact that $M_{\delta,kl}$ is semipositive definite.

**4.1. Space equivalence for special micromodulus functions in two dimensions.** We now consider micromodulus functions with special relations between $M_\rho$ and the conventional Sobolev spaces in two dimensions. We let $H^s_{\rho,c}$ denote the fractional order Sobolev space on $\Omega = (0, \pi)^2$, which consists of distributions of the form (4.1) with

\[
||u||^2_2 = \sum_{k,l=1}^{\infty} (k^2 + l^2)^s (u_{kl}^2 + v_{kl}^2) < \infty.
\]

We let $L^2_{\rho,c} = H^0_{\rho,c}$, and for convenience we define

\[
\mathcal{L}_s(\alpha, \chi) = \frac{c_\delta}{2} \int_{B_1(0)} \frac{2 - \cos(z) - \cos(w)}{(z^2 + w^2)^{2+\alpha/2}} \min(z^2, w^2) dz dw,
\]

\[
\mathcal{L}_s(\alpha, \chi) = \frac{c_\delta}{2} \int_{B_1(0)} \frac{2 - \cos(z) - \cos(w)}{(z^2 + w^2)^{2+\alpha/2}} \max(z^2, w^2) dz dw.
\]

Similar to the 1-D case, we have what follows.
LEMMA 4.2. Let \( \sigma = \sigma(|y|) \) satisfy that \( |y|^2/\sigma \in L^1(B_3(0)) \). We then have \( M_\sigma^{2\alpha} = M_\sigma^{\alpha} = L^2_{\alpha\gamma} \), and \( -\mathcal{L}_3 \) is a bounded linear operator from \( L^2_{\alpha\gamma} \) to itself. On the other hand, let \( \sigma = \sigma(|y|) \) satisfy (2.3) and the condition that

\[
\gamma|y|^{4+\alpha} \leq \sigma(|y|) \leq \gamma_1 |y|^{4+\beta} \quad \forall |y| \leq \delta
\]

for some exponents \( 0 \leq \beta \leq \alpha \in (0, 2) \) and positive constants \( \gamma_1 \) and \( \gamma_2 \); then we have \( H^{\beta/2}_{\alpha\gamma} \hookrightarrow M_\sigma^{\alpha} \hookrightarrow H^{\beta/2}_{\alpha\gamma} \) and \( H^{\alpha/2}_{\alpha\gamma} \hookrightarrow M_\sigma^{2\alpha} \hookrightarrow H^{\alpha/2}_{\alpha\gamma} \).

Proof. First, for \( |y|^2/\sigma \in L^1(B_3(0)) \), we have

\[
0 < M^{\delta} \leq M_{\delta,kl} \leq 2c_3 \left( \int_{B_3(0)} \frac{|y|^2}{\sigma(|y|)} dy \right) I.
\]

Thus, \( M_\sigma^{2\alpha} = M_\sigma^{\alpha} = L^2_{\alpha\gamma} \), and \( -\mathcal{L}_3 \) is a bounded operator from \( L^2_{\alpha\gamma} \) to \( L^2_{\alpha\gamma} \). Next,

\[
M_{\delta,kl} \geq \frac{c_3}{\gamma_1} \int_{B_{3}(0)} \frac{1 - \cos(k\xi_1 + l\xi_2)}{(\xi_1^2 + \xi_2^2)^{\beta/2}} \begin{pmatrix} \xi_1 \xi_2 \\ \xi_1^2 \xi_2^2 \end{pmatrix} d\xi.
\]

Under (4.8), by a change of variable, \( z = k\xi_1 + l\xi_2, w = -l\xi_1 + k\xi_2 \), and we have, for \( \sigma \) satisfying (4.8), that

\[
M_{\delta,kl} \geq \frac{c_3}{\gamma_1} \left( k^2 + l^2 \right)^{\beta/2} \int_{B_{3}(0)} \frac{1 - \cos(z)}{(z^2 + w^2)^{\beta/2}} M(k, l, w, z) dz dw,
\]

where

\[
M(k, l, w, z) = \frac{1}{k^2 + l^2} \begin{pmatrix} k^2 z^2 + l^2 w^2 & kl(z^2 - w^2) \\ kl(z^2 - w^2) & l^2 z^2 + k^2 w^2 \end{pmatrix}.
\]

Since for any \( k, l \), we have \( \min(z^2, w^2) I \leq M(k, l, w, z) \). So,

\[
\frac{c_3}{\gamma_1} \left( k^2 + l^2 \right)^{\beta/2} I \leq M_{\delta,kl}.
\]

Similarly, noticing that \( M(k, l, w, z) \leq \max(z^2, w^2) I \), by using the lower bound on \( \sigma \), we can get

\[
M_{\delta,kl} \leq \frac{c_3(\alpha, \infty)}{\gamma_2} \left( k^2 + l^2 \right)^{\alpha/2} I.
\]

We thus have the corresponding space embedding results. \( \square \)

4.2. Properties of stationary PD model. For (4.3) and \( f \in M_\sigma^{-\alpha\gamma} \), we consider

(4.9) \[ -\mathcal{L}_3 u = f, \]

with the solution in \( M_\sigma^{-\alpha\gamma} \). First, we have what follows.

LEMMA 4.3. Let \( \sigma = \sigma(|y|) \) satisfy (2.3). Then, for any \( f \in M_\sigma^{-\alpha\gamma} \), the problem (4.9) has a unique solution \( u \in M_\sigma^{-\alpha\gamma} \) which minimizes

\[
\frac{1}{2} \| u \|_{M_\sigma^{-\alpha\gamma}}^2 - \langle u, f \rangle_{L^2_{\alpha\gamma}} = \frac{1}{2} \sum_{k,l=1}^{\infty} (u_{kl}, v_{kl}) M_{\delta,kl} \begin{pmatrix} u_{kl} \\ v_{kl} \end{pmatrix} - \langle u, f \rangle_{L^2_{\alpha\gamma}}.
\]
in \(M_{\sigma}^{\infty}\). Moreover, if \(f \in L_{\sigma}^{2}\), then \(u \in M_{\sigma}^{2\infty}\) with \(\|u\|_{M_{\sigma}^{2\infty}} = |f|_{0}\). In addition, if \(\sigma\) satisfies (4.8) \(\alpha = \beta\), then \(u \in H_{\sigma}^{m+\alpha}\) whenever \(f \in H_{\sigma}^{m}\) for any \(m \geq -\alpha\).

The regularity of the solution, for \(\sigma\) satisfying (4.8), can be obtained by considering the Fourier version of (4.9),

\[
\sum_{k,l=1}^{\infty} \begin{pmatrix}
\sin(kx_1)\cos(lx_2) & 0 \\
0 & \cos(kx_1)\sin(lx_2)
\end{pmatrix} M_{\delta,kl}(u_{kl}) = f(x).
\]

Noticing that

\[
f = \sum_{k,l=1}^{\infty} \begin{pmatrix}
f_{1,kl} \sin(kx_1) \cos(lx_2) \\
f_{2,kl} \cos(kx_1) \sin(lx_2)
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
u_{kl} \\
v_{kl}
\end{pmatrix} = M_{\delta,kl}^{-1} \begin{pmatrix}
f_{1,kl} \\
f_{2,kl}
\end{pmatrix},
\]

we readily have

\[
\sum_{k,l=1}^{\infty} (k^2 + l^2)^m \begin{pmatrix}
u_{kl} \\
v_{kl}
\end{pmatrix} M_{\delta,kl}^2 \begin{pmatrix}
u_{kl} \\
v_{kl}
\end{pmatrix} = \|f\|_{m}^2 < \infty.
\]

Now concerning the limit of the 2-D PD model as \(\delta \to 0\), we let \(u_{\delta} = u_{\delta}(x)\) be the solution of (4.9) in \(M_{\sigma}^{\infty}\) and \(u_0 = (u, v)\) be the solution of the Navier equation with a greased-wall boundary condition:

\[
\begin{aligned}
-\mu \Delta u(x) - (\mu + \lambda)\nabla \text{div} u(x) &= f(x), \quad x = (x_1, x_2) \in \Omega, \\
u &= \frac{\partial u}{\partial n} = 0 \quad \text{on} \ (0, \pi) \times \{x_2 = 0\} \text{ and } (0, \pi) \times \{x_2 = \pi\}, \\
v &= \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \{x_1 = 0\} \times (0, \pi) \text{ and } \{x_1 = \pi\} \times (0, \pi),
\end{aligned}
\]

where \(\mu = \lambda\) corresponding to the Poisson ratio \(\nu = 0.25\). Then, we have the following.

Theorem 4.4. Let \(\sigma = \sigma(|y|)\) satisfy (2.3). Then, as \(\delta \to 0\), solutions of (4.9) in \(M_{\sigma}^{\infty}\) converge to the solution of (4.12) in \(M_{\sigma}^{2\infty}\), provided that \(f \in L_{\sigma}^{2}\) and

\[
\tau_3 \to 8(\mu + \lambda)
\]

for the constant \(\tau_3\) defined in (2.3).

Proof. In the Fourier space, the coefficients of the solutions for (4.9) and (4.12) are, respectively,

\[
\begin{pmatrix}
u_{\delta,kl} \\
v_{\delta,kl}
\end{pmatrix} = M_{\delta,kl}^{-1} \begin{pmatrix}
f_{1,kl} \\
f_{2,kl}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
u_{0,kl} \\
v_{0,kl}
\end{pmatrix} = M_{0,kl}^{-1} \begin{pmatrix}
f_{1,kl} \\
f_{2,kl}
\end{pmatrix},
\]

where \(M_{0,kl}\) is as defined in (4.6). By simple calculation, we get

\[
\|u_{\delta} - u_0\|_{M_{\sigma}^{2\infty}}^2 = \sum_{k,l=1}^{\infty} (f_{1,kl}, f_{2,kl}) M_{0,kl}^{-1} (M_{0,kl} - M_{\delta,kl})^2 M_{0,kl}^{-1} \begin{pmatrix}
f_{1,kl} \\
f_{2,kl}
\end{pmatrix}.
\]

By Lemma 4.1, we have

\[
M_{0,kl}^{-1} (M_{0,kl} - M_{\delta,kl})^2 M_{0,kl}^{-1} \leq \left(1 + \frac{\tau_3}{2\mu}\right)^2 I.
\]
By Taylor expansion, we can see that

\[
M_{\delta,kl} = \frac{c_3}{2} \int_{B_0(0)} \left( \frac{k^2 \xi_1^4}{\sigma(|\xi|)} + 2k \xi_1^2 \xi_2^2 \frac{2kl \xi_1^2 \xi_2^2}{\sigma(|\xi|)} \right) d\xi + Z_{kl},
\]

where \(Z_{kl}\) contains higher order terms of the form

\[
Z_{kl} = \frac{c_3}{24} \int_{B_0(0)} \frac{(k \xi_1 + l \xi_2)^4}{\sigma(|\xi|)} \xi \otimes \xi \cos(\theta) d\xi
\]

for some \(\theta = \theta(\xi)\). We can see that for any \(k, l\),

\[
|Z_{kl}| \leq \frac{\tau_\delta}{24} (k^2 + l^2)^2 I.
\]

Hence \(Z_{kl}\) goes to the zero matrix as \(\delta \to 0\) for each given \(k, l\). Using the identity

\[
\int_{B_0(0)} \frac{\xi_1^4}{\sigma(|\xi|)} d\xi = \int_{B_0(0)} \frac{\xi_2^4}{\sigma(|\xi|)} d\xi = 3 \int_{B_0(0)} \frac{\xi_1^2 \xi_2^2}{\sigma(|\xi|)} d\xi
\]

and letting

\[
\mu = \lambda = \lim_{\delta \to 0} \frac{c_3}{2} \int_{B_0(0)} \frac{\xi_1^2 \xi_2^2}{\sigma(|\xi|)} d\xi = \lim_{\delta \to 0} \frac{\tau_\delta}{16},
\]

we may then get \(M_{\delta,kl} \to M_{0,kl}\), as \(\delta \to 0\), for any \(k, l\). To complete the proof of the lemma, we can use the dominant convergence theorem to get \(\|u_\delta - u_0\|^2_{H^2(\omega)} \to 0\).

**Remark 4.5.** As in the 1-D counterpart, the condition (4.13) is a natural assumption that leads to a consistent definition of the elastic constants of the material under consideration [14]. Convergence in conventional Sobolev space can also be shown using the space equivalence established earlier. Moreover, the analysis on finite-dimensional approximations including the error estimates and condition number estimates can also be carried out for the 2-D case. In fact, much of the conclusions given in the 1-D case can be extended to higher spatial dimensions without significant difficulties. For instance, one does not encounter the so-called locking phenomena [5] in the finite element approximations since the linear bond-based PD models correspond to elastic materials with a Poisson ratio 0.25.

**4.3. The time-dependent PD model.** The existence and uniqueness of the solutions of the time-dependent PD model (2.1) can also be derived.

**Theorem 4.6.** Let \(\sigma(\theta)\) satisfy (2.3). Then, for initial conditions \(u(x, 0) = g_0(x) \in M_{\sigma}^{\text{oc}}, u_t(x, 0) = g_1(x) \in L_0^2, \) and \(b(x, t) \in L^2(0, T; L_0^2), \) (2.1) has a unique solution \(u = u(x, t)\) with \(u \in C([0, T], M_{\sigma}^{\text{oc}}) \cap H^1(0, T; L_0^2)\).

Similar to the discussion given in the previous sections, we may also establish the convergence property, as \(\delta \to 0\), of the solutions of the time-dependent PD models to the time-dependent Navier equation (with \(\mu = \lambda\)), which is simply (4.12) with a term \(u_{tt}\) added to its left-hand side. Then, we have the following.

**Theorem 4.7.** Let \(\sigma(\theta)\) satisfy (2.3). Then, as \(\delta \to 0\), the solutions of (2.1) converge to the solution of the time-dependent Navier equation with the same initial conditions as that for (2.1), in the space \(L^2(0, T; M_{\sigma}^{\text{oc}}) \cap H^1(0, T; L_0^2)\), provided that (4.13) is satisfied with \(g_0 \in M_{\sigma}^{\text{oc}}, g_1 \in L_0^2, \) and \(b \in L^2(0, T; L_0^2). \) In addition, if \(\sigma(\theta)\) satisfies (4.8) with \(\alpha = \beta \in (0, 2)\), we also have the above convergence in \(L^2(0, T; H_0^\alpha)\).
4.4. Approximations of the time-dependent PD models. Similar to the 1-D stationary case, the convergence of finite-dimensional approximations can also be established for multidimensional time-dependent nonlocal problems as well. As an illustration, for the initial nonlocal BVP (2.1) defined with \( u \in C([0, T], M^w_\sigma) \) and \( u_t \in L^2(0, T; L^2_0) \), we consider a semidiscretization in spatial variables, that is, let \( \{V_n\} \) be a finite-dimensional subspace of \( M^w_\sigma \); that is, for any \( u \in M^w_\sigma \), there exists a sequence of \( \{u_n \in V_n\} \) such that \( \|u - u_n\|_{M^w_\sigma} \to 0 \) as \( n \to \infty \). For example, \( V_n \) can be subspaces of \( M^w_\sigma \) containing only a finite number (say, bounded by \( n \)) of Fourier modes.

We use \( \pi_n \) to denote the PD analogue of the elliptic projection operator on to \( V_n \). In our case here, more specifically, for any \( u \in M^w_\sigma \), \( \pi_n u \) is simply the best approximation to \( u \) in \( V_n \) corresponding to the norm in \( M^w_\sigma \), similar to the Galerkin–Ritz approximation we have described earlier for the 1-D case. For convenience, let us also introduce \( \tilde{\pi}_n \) as the standard \( L^2 \) projection onto \( V_n \) so that for any \( u \in L^2 \), we have \( \|u - \tilde{\pi}_n u\|_0 \to 0 \) as \( n \to \infty \). Meanwhile, as in the 1-D case, if \( \sigma = \sigma(|y|) \) satisfies (2.3), then as \( n \to \infty \), we have

\[
\|u - \pi_n u\|_{M^w_\sigma} \to 0 \quad \forall u \in M^w_\sigma.
\]

Taking the approximate initial conditions \( u_n(\cdot, 0) = \pi_n g_0, u_{n,t}(\cdot, 0) = \pi_n g_1 \), and using the projection operators, we have the equation for the numerical solution \( u_n \):

\[
u_n tt = \tilde{\pi}_n \mathcal{L}_\delta u_n + \tilde{\pi}_n b \quad \forall t \in (0, T).
\]

We define \( e_n = u - u_n \), \( \dot{e}_n = u - \pi_n u \), and \( \ddot{e}_n = \pi_n u - u_n \). To estimate \( e_n = u - u_n \), we may use the Duhamel principle to give an explicit expression \( u_n \) via the semigroup (or matrix function) generated by \( -\pi_n \mathcal{L}_\delta \) and then compare it directly with the solution \( u \). Alternatively, we can also follow a similar analysis of spatial discretizations for wave-type PDEs to control \( e_n \) in terms of \( \dot{e}_n \).

**Theorem 4.8.** If \( \sigma = \sigma(|y|) \) satisfies (2.3) and, in addition, the solution \( u \) of (2.1) is bounded in \( C([0, T], M^w_\sigma) \cap H^2(0, T; L^2_0) \), then there exists a generic constant \( c > 0 \), independent of \( n \), such that

\[
\|e_n(\cdot, t)\|^2 + \|e_{n,t}(\cdot, t)\|^2 \leq c \left\{ \|u(\cdot, t) - \pi_n u(\cdot, t)\|^2 + \|u_t(\cdot, t) - \pi_n u_t(\cdot, t)\|^2 \right. \\
+ \int_0^t \|u_{ss}(\cdot, s) - \pi_n u_{ss}(\cdot, s)\|^2 \, ds \right\}. \tag{4.16}
\]

Consequently, as \( n \to \infty \), we have the convergence of the sequence \( \{u_n\} \) to \( u \) in \( L^2(0, T; M^w_\sigma) \) and \( H^1(0, T; L^2_0) \).

**Proof.** Let us denote \( e_n = \pi_n u - u_n \), \( \dot{e}_n = u - \pi_n u \), and \( \ddot{e}_n = \pi_n u - u_n \). By the initial condition, we have \( e_n(\cdot, 0) = 0 \) and \( \dot{e}_n(\cdot, 0) = 0 \). We then subtract (2.1) and (4.15) and take the inner product with \( e_n \) to get

\[
\frac{1}{2} \frac{d}{dt} \left( \|e_n\|^2 + \|\dot{e}_n\|^2_{M^w_\sigma} \right) = (\pi_n u_{tt} - u_{tt}, e_n) \leq \frac{1}{4} \|\pi_n u_{tt} - u_{tt}\|^2 + \frac{1}{4} \|\dot{e}_n\|^2_{M^w_\sigma}.
\]

By Gronwall’s inequality, we get

\[
\|e_{n,t}(\cdot, t)\|^2 + \|e_n(\cdot, t)\|^2_{M^w_\sigma} \leq c \int_0^t \|u_{tt}(\cdot, t) - \pi_n u_{tt}(\cdot, t)\|^2 \, ds
\]

\[
\leq c \int_0^t \|u_{tt}(\cdot, t) - \pi_n u_{tt}(\cdot, t)\|^2_{M^w_\sigma} \, ds.
\]

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Using triangle inequality, we get (4.16). Then, using the convergence of the elliptic project \( \pi_n \), we get the convergence of \( u_n \) to \( u \) in the desired norms. \( \square \)

Note that if we replace \( M^\sigma \) by \( M^\nu \), we can get similar convergence for the approximations of the 1-D time-dependent PD models. Moreover, when the data are sufficiently smooth, we can obviously get the precise optimal order error estimates as those in (3.30) and (3.31); we omit the details. The estimates can also be generalized to fully discrete approximations. Moreover, the spectral bounds in (3.35) and (3.37) can also be used to determine the time-step size condition in the numerical stability analysis; we refer to [20, 24] for related discussions with some special choices of the micromodulus functions. The analytic framework developed here can again provide results in more general settings.

5. Conclusion. In this work, the analytic framework presented in [10] is extended to the nonlocal BVPs for bond-based linear PD models on a finite 1-D bar and a 2-D square. These problems are necessarily of nonlocal nature. Various analytical issues are established here under a unified framework, including basic well-posedness results and regularities of the solutions. In particular, it is shown that the solutions may or may not have improved smoothness over the given data, depending on the properties of the micromodulus function. These results are of interest when the PD models are used to study problems involving singularity formation and phase transition [9, 25]. Utilizing the analytic framework, we are able to establish the convergence of the weak solutions of the nonlocal PD models to that of the classical differential equations when the horizon parameter approaches to zero. Moreover, we are also able to show the convergence of finite-dimensional approximations to the nonlocal BVPs and to derive typical error estimates for both Fourier spectral and finite element methods. Important issues such as the dependence on the model parameter (PD horizon, for example) and the mesh parameter of the condition numbers of the stiffness matrices and the time-stability conditions can all be discussed within the framework. While the linear nonlocal boundary conditions considered here are special, we expect that similar results can be established for other nonlocal BVPs associated with the PD models, both bond- and state-based [22, 23], and problems in three- and higher-dimensional spatial domains as well as nonlinear problems. These studies, together with applications of PD models to practical problems, will be explored in the future.

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