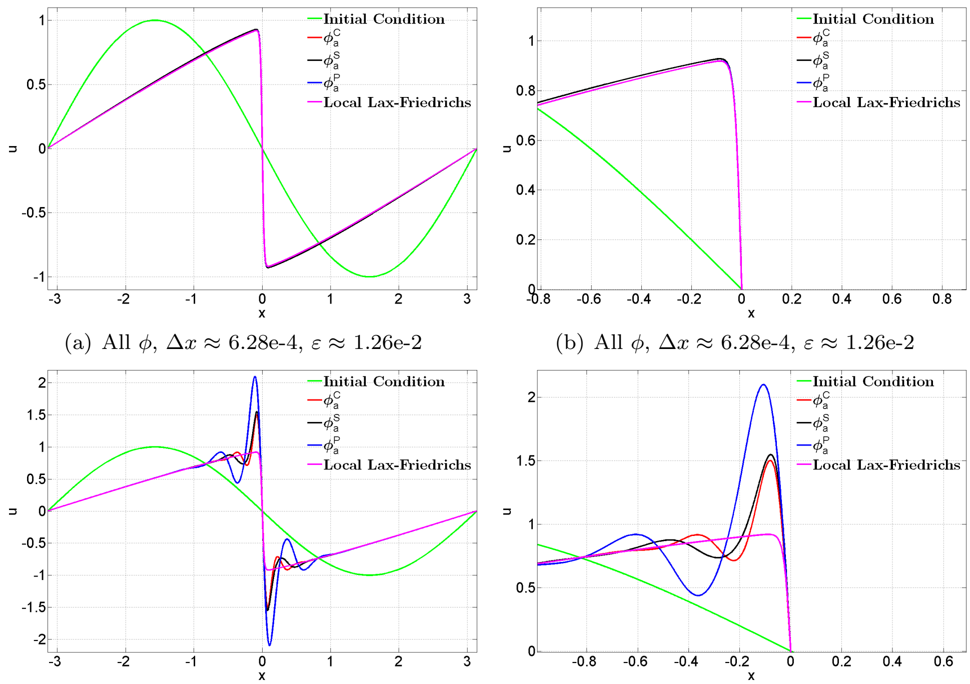
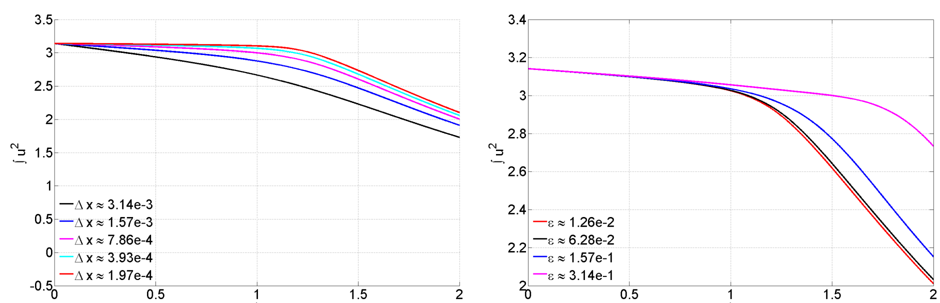


(a) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2 (b) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2

(c) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1 (d) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1

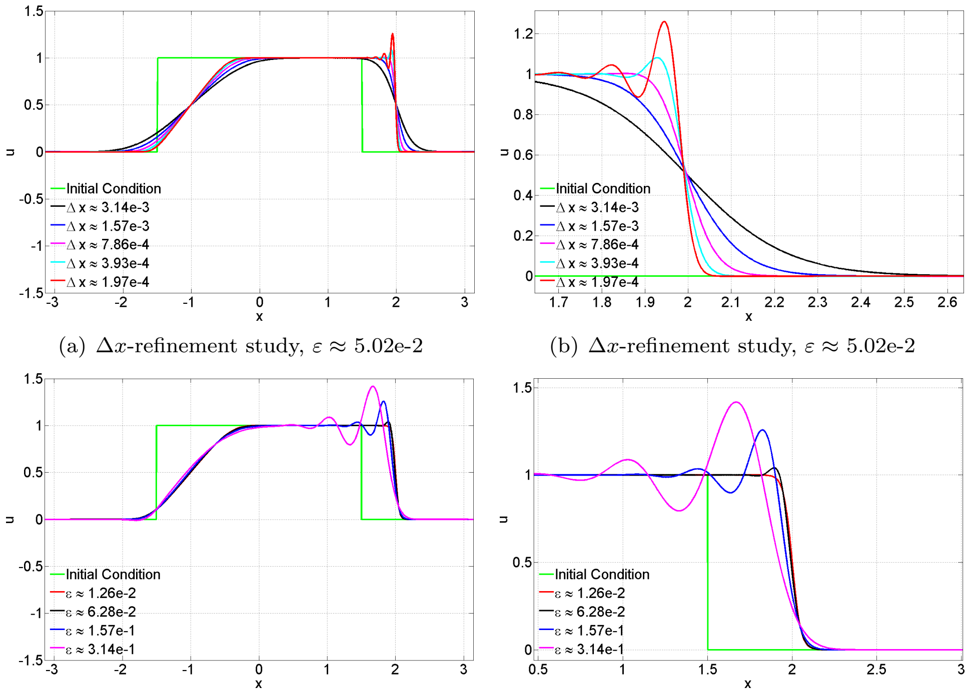
FIG. 6.2. *Numerical results for initial condition* (6.8)*, as described in* §*6.2. Results for all three kernel functions are shown. Figs. (a,b) show the smallest value of* ε *used, and Figs.(c,d) show the largest value of* ε *used. The solution produced by the classical Lax-Friedrichs method applied to the local inviscid Burgers equation is included for comparison. For small* ε*, all kernels display results close to the classical result, but this is not the case for larger values of* ε*.*



(a) Δx-refinement study, $\varepsilon \approx$ 5.02e-2 (b) ε-refinement study, $\Delta x \approx$ 6.28e-4

FIG. 6.3. *Plots of* $\int_{-\pi}^{\pi} u^2(x,t)\,dx$ *as a function of time, for the studies shown in Figure 6.1. Both plots confirm that the energy is strictly nonincreasing.*

prises the first steps toward formulation of a coherent mathematical approach for nonlocal advective phenomena consistent with existing peridynamics theory. Using integral operators, we proposed a nonlocal advection equation that we showed to be equivalent to the corresponding local advection equation in the sense of distributions, analyzed the specific case of nonlocal linear advection and demonstrated that the linear peridynamic equation can be written in terms of two nonlocal linear advective



(a) Δx-refinement study, $\varepsilon \approx$ 5.02e-2 (b) Δx-refinement study, $\varepsilon \approx$ 5.02e-2

(c) ε-refinement study, $\Delta x \approx$ 6.28e-4 (d) ε-refinement study, $\Delta x \approx$ 6.28e-4

FIG. 6.4. *Numerical results for initial condition* (6.9)*, as described in* §*6.3. The mesh refinement study of Figs. (a,b) and nonlocal horizon study of Figs. (c,d) are based on the kernel* (4.20).

equations. Moreover, we posited a nonlocal regularization that, likewise, reduces to the usual local case in the sense of distributions. Our analysis suggested a generalized concept of a flux that applies, in 1D, to disjoint open intervals on the line. We developed a simple conservative numerical method that was shown to be linearly stable under the usual von Neumann analysis. We performed basic computational experiments with this method on a nonlocal Burgers equation for initial conditions that correspond, in the local case, to shock formation and shock/rarefaction propagation. We also established the well-posedness of the nonlocal Burgers equation over finite time intervals when the antisymmetric kernel associated with the nonlocal advective operator is an element of L^1. Two important conclusions are that a shock cannot develop in finite time and the resulting nonlocal Burgers equation naturally incorporates regularization. The results of these calculations showed that dissipation on the coarser meshes considered significantly damped the solution structure. For the parameters considered, the effect of different nonlocal horizons, however, was slight, and, likewise, the influence of different kernel functions was minor.

Our numerical results are dependent not only on the parameters and kernel functions considered, but on the numerical scheme, as well. In future work, we intend to develop more sophisticated, less dissipative numerical methods with which to investigate solution behavior and go beyond L^1 kernels and seek exact solutions so that we may verify our numerical results.

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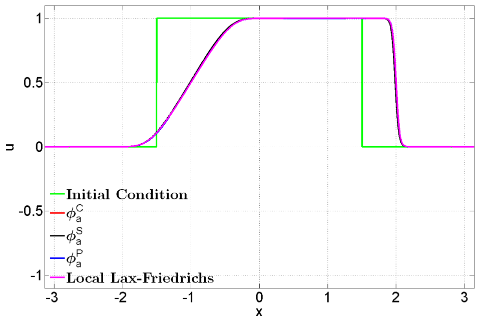
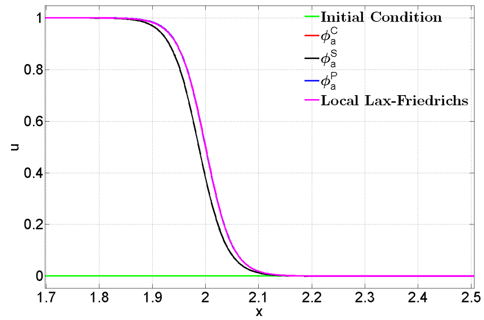
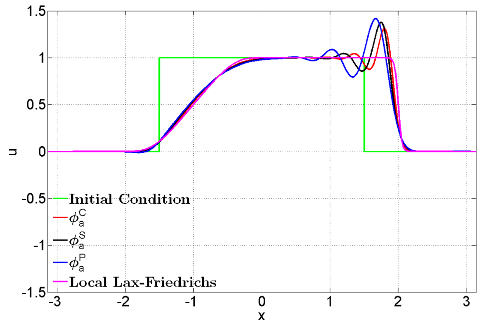


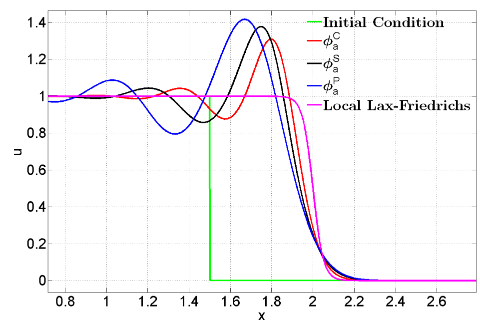
(a) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2



(b) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 1.26e-2



(c) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1



(d) All ϕ, $\Delta x \approx$ 6.28e-4, $\varepsilon \approx$ 3.14e-1

FIG. 6.5. *Numerical results for initial condition* (6.9)*, as described in* §*6.3. Results for all three kernel functions are shown. Figs. (a,b) show the smallest value of* ε *used, and Figs.(c,d) show the largest value of* ε *used. The solution produced by the classical Lax-Friedrichs method applied to the local inviscid Burgers equation is included for comparison. For small* ε*, all kernels display results close to the classical result, but this is not the case for larger values of* ε*.*

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