GINZBURG–LANDAU VORTICES: DYNAMICS, PINNING, AND HYSTERESIS

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Abstract. In this paper, we consider three problems related to the mathematical study of vortex phenomena in superconductivity based on the G-L models. First, we study the long-time behavior of the solutions of the time-dependent Ginzburg–Landau equations. Then we describe results concerning the pinning effect of thin regions in a variable thickness thin film. Finally, we prove the existence of vortex-like solutions to the steady state Ginzburg–Landau equations and study the hysteresis phenomenon near the lower critical field.

Key words. superconductivity, Ginzburg–Landau equations, vortex solution, long-time behavior, vortex pinning, hysteresis

AMS subject classifications. 35B40, 35A20, 35J65, 35K99, 82D55

1. Introduction. Below the critical temperature \( T_c \), the response of a superconducting material to an externally imposed magnetic field is most conveniently described by the diagram given in Figure 1, which shows the minimum energy state of the superconductor as a function of \( H_0 \), the applied magnetic field, and the dimensionless material parameter \( \kappa \) (known as the Ginzburg–Landau parameter). The parameter \( \kappa \) determines the type of superconducting material [10]. For \( \kappa < \frac{1}{\sqrt{2}} \), type-I superconductors, there is a critical magnetic field \( H_C \) below which the material will be the superconducting Meissner state but above which it will be in the normal state.

For \( \kappa > \frac{1}{\sqrt{2}} \), type-II superconductors, there is a third state known as the mixed (or vortex) state. The vortex state consists of many normal filaments embedded in a superconducting matrix. Each of these filaments carries with it a quantized amount of magnetic flux and is circled by a vortex of superconducting current. Thus, these filaments are often know as vortex lines. One of the most challenging problems to mathematicians working on the superconductivity models is to understand vortex phenomena in type-II superconductors, which include the recently discovered high-temperature superconductors.

The transition from the normal state to the vortex state takes place by a bifurcation as the magnetic field is lowered through some critical value \( H_{C_1} \). The critical field \( H_{C_1} \), on the other hand, is calculated so that at this field the energy of the wholly superconducting solution becomes equal to the energy of the single vortex filament solution for infinite superconductors.

The vortex structures have been studied extensively on the mezoscale by using the well-known Ginzburg–Landau (G–L) models of superconductivity [10, 13, 14]. The existence of vortex-like solutions for the full nonlinear G–L equations has been investigated by researchers using methods ranging from asymptotic analysis to numerical
simulations; however, it has not been justified by rigorous mathematical analysis. Much progress has been made in recent years [3, 20, 21] to establish a mathematical framework for a rigorous description of both the static and dynamic properties of the vortex solutions; in particular, as the coherence length tends to zero (κ goes to infinity), various results have been obtained. From a technological point of view, this is of interest since recently discovered high critical temperature superconductors are known to have large values of κ, say κ in excess of 50.

Vortex lines may move as a result of internal interactions between these filaments and external forces (due to applied fields or thermal fluctuations) acting on them. Unfortunately, such vortex motion in an applied magnetic field induces an effective electrical resistance in the material and, thus, a loss of superconductivity. Therefore, it is crucial to understand the dynamic of these vortex lines. At the same time, one is interested in studying mechanisms that can pin the vortices at a fixed location, i.e., prevent their motion. Various such mechanisms have been advanced by physicists, engineers, and material scientists. For example, normal (nonsuperconducting) impurities in an otherwise superconducting material sample are believed to provide sites at which vortices are pinned. Likewise, regions of the sample that are thin relative to other regions are also believed to provide pinning sites. These mechanisms have been introduced into the general G–L framework to derive various variants of the original G–L models of superconductivity. Numerical simulation clearly suggests the pinning effect.

In this paper, we deal with three independent problems, yet all of them are related to the study of vortex phenomena. First we study the long-time behavior of the solutions of the time-dependent G–L equations and the main result is Theorem 2.1. Then we describe results (Theorems 3.1–3.5) concerning the pinning effect of thin regions in a variable thickness thin film, based on the models developed in [6, 12], and finally, we prove the existence of vortex-like solutions to the steady state G–L equations (Lemma 4.1) and study the hysteresis phenomenon near the lower critical field.

Before addressing the above problems, we introduce the notation and the models that will be used in the paper. The starting point of our study is the phenomenological
model due to Ginzburg and Landau for superconductivity in isotropic, homogeneous material samples. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^3 \), occupied by the superconducting material. By ignoring the effect of the region exterior to the sample, the steady state model can be stated as a minimization problem of the free energy functional

\[
\mathcal{G}(\psi, A) = \int_{\Omega} \left\{ f_n + a|\psi|^2 + \frac{b}{2} |\psi|^4 + \frac{1}{2m_s} \left( i \hbar \nabla + \frac{e_s}{c} A \right) \psi^2 \right. \\
\left. + \frac{\mu_s}{8\pi} \mathbf{h} \cdot (\mathbf{h} - 2H_0) \right\} d\Omega,
\]

where \( f_n \) is the free energy density of the nonsuperconducting state in the absence of a magnetic field, \( \psi \) is the (complex-valued) superconducting order parameter, \( A \) is the magnetic vector potential, \( \mathbf{h} = (1/\mu_s) \text{curl} \mathbf{A} \) is the magnetic field, \( H_0 \) is the applied magnetic field, \( a \) and \( b \) are constants whose values depend on the temperature and such that \( b > 0 \), \( e_s \) is the mass of the superconducting charge carriers which is twice the electronic charge \( e \), \( c \) is the speed of light, \( \mu_s \) is the permeability, and \( 2\pi\hbar \) is Planck’s constant. It can be rewritten in nondimensionalized form:

\[
\mathcal{G}(\psi, A) = \int_{\Omega} \left( \frac{1}{2} (1 - |\psi|^2)^2 + \left( \frac{i}{\kappa} \nabla + A \right) \psi^2 + |\text{curl} \ A - H_0|^2 \right) d\mathbf{x},
\]

where \( \kappa \) is the so-called G–L parameter.

The functional \( \mathcal{G}(\psi, A) \) has an interesting gauge invariance property and the minimization of \( \mathcal{G} \) in appropriate functional spaces gives the following system of nonlinear differential equations that are named the G–L equations:

\[
\left( \frac{i}{\kappa} \nabla + A \right)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{in } \Omega,
\]

\[
\text{curl curl } A = \frac{i}{2\kappa} (\psi \nabla \psi^* - \psi^* \nabla \psi) - |\psi|^2 A \quad \text{in } \Omega,
\]

along with natural boundary conditions

\[
\text{curl } A \wedge \mathbf{n} = H_0 \wedge \mathbf{n} \quad \text{on } \partial \Omega
\]

and

\[
\left( \frac{i}{\kappa} \nabla \psi + A \psi \right) \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega,
\]

where \( \mathbf{n} \) is the exterior normal to the boundary \( \partial \Omega \). The G–L vortices are represented by the zeros of the complex order parameter \( \psi \). In section 4, we will prove the existence of vortex solutions to the above system when \( \kappa \) is large and the applied field is near the lower critical field.

Equations (3)–(4) are the steady state G–L equations. The time-dependent G–L model is often described by the Gorkov–Eliashberg evolution equation [17]:

\[
\begin{align*}
\eta \partial \psi \quad & + i \eta \kappa \Phi \psi + \left( \frac{i}{\kappa} \nabla + A \right)^2 \psi - \psi + |\psi|^2 \psi = 0, \\
\partial A \quad & + \nabla \Phi + \text{curl curl } A = -\frac{i}{2\kappa} (\psi^* \nabla \psi - \psi \nabla \psi^*) - A |\psi|^2.
\end{align*}
\]
Here, $\Phi$ denotes the (real) scalar electric potential, $\eta$ is a relaxation parameter, and $\psi^*$ denotes the complex conjugate of $\psi$. For simplicity, we take $\eta = 1$ in the rest of the paper.

The system is supplemented by the initial and boundary conditions

$$\psi(x, 0) = \psi_0(x), \quad A(x, 0) = A_0(x), \quad x \in \Omega; \quad (8)$$

$$\begin{cases}
\left( \frac{i}{\kappa} \nabla + A \right) \psi \cdot n = 0, \\
\text{curl} A \wedge n = H_0 \wedge n, \\
\left( \frac{\partial A}{\partial t} + \nabla \Phi \right) \cdot n = \vec{E} \cdot n = 0 \quad \text{on} \ \partial \Omega.
\end{cases} \quad (9)$$

Note that (7) and (9) are gauge invariant [11], in the sense that if $(\psi, A, \Phi)$ is a solution, then so is $(\psi_\chi, A_\chi, \Phi_\chi)$, where

$$\psi_\chi = \psi e^{i \kappa \chi}, \quad A_\chi = A + \nabla \chi, \quad \Phi_\chi = \Phi - \frac{\partial \chi}{\partial t}.$$ 

The dynamics of vortices can be determined from the solutions of the time-dependent equations (7)–(9). The long-time asymptotic behavior of solutions of equations (7)–(9) as $t \to \infty$ will be studied later in section 2.

For type-II superconductors, the minimizers of $\mathcal{G}$ are believed to exhibit vortex structures. Numerical experiments show that for large values of $\kappa$ and moderate field strength, the number of vortices could be exceedingly large even for a small sample size in actual physical scale. Thus, resolving the vortex phenomenon by using the full G–L equations remains computationally intensive.

Various simplifications have been made to reduce the complexity. For thin films of superconducting material, a two-dimensional model has been developed [6, 12] that can account for thickness variations through an averaging process. The model is given by the following minimization problem:

$$\mathcal{G}_\epsilon^a(\psi) = \int_\Omega a(x) \left( \left| (\nabla - i A_0)\psi \right|^2 + \frac{1}{2\epsilon^2} (1 - |\psi|^2)^2 \right) dx, \quad (10)$$

where $\Omega$ denotes the platform of the film, $a(x)$ measures the relative thickness of the film, and $A_0$ is a prescribed vector potential due to the normal (to film) component of the applied field. The role of the Ginzburg–Landau parameter $\kappa$ is assumed by the parameter $\epsilon(\propto 1/\kappa)$.

It was proved that for fixed $\epsilon$, the minimizers of the above problem, along with the prescribed vector potential $A_0$, provide the leading order approximation to the solution of the three-dimensional problem [6]. The creation and interaction of vortices based on the above model is connected to the prescribed magnetic potential. The number of vortices cannot be prescribed a priori, independently of $A_0$. To simplify the analysis further, a simpler problem, in which the number of vortices is prescribed and the magnetic potential is ignored, can be studied. By rescaling the spatial variables, one may consider the minimization of the functional

$$\mathcal{F}_\epsilon^a(\psi) = \int_\Omega a(x) \left( |\nabla \psi|^2 + \frac{1}{2\epsilon^2} (1 - |\psi|^2)^2 \right) dx. \quad (11)$$
with the boundary condition
\begin{equation}
\psi(x) = g(x) \quad \text{for } x \in \partial \Omega,
\end{equation}
where $g$ is smooth with $|g(x)| = 1$, $x \in \partial \Omega$. This can be viewed as a generalization of the problem studied in [3, 20] in which $a(x) \equiv 1$, i.e.,
\begin{equation}
\mathcal{F}_\epsilon(\psi) = \int_{\Omega} \left( |\nabla \psi|^2 + \frac{1}{2\epsilon^2} (1 - |\psi|^2)^2 \right) \, dx.
\end{equation}

The pinning effect of the variable thickness will be studied in section 3 by examining the properties of minimizers of (10) and (11).

The rest of the paper is devoted to the problems we have discussed above.

2. The uniqueness of the asymptotic limit. The global existence and uniqueness (up to gauge transformations) of classical solutions of (7)–(9) have been studied by various authors, e.g., [7, 11, 28]. The dynamics of vortices (including the simpler case which ignores the magnetic field) have been studied by [15, 16, 23, 24, 25] and recently proved in [22]. In [28], the long-time behavior, in particular the existence of the global attractor, is also investigated. Here, we shall sketch the proof of the asymptotic stability result, which shows that, as $t \to \infty$, (7)–(9) has a unique asymptotic limit up to gauge transformations.

As in [11], one may choose the so-called zero electric potential gauge for system (7)–(9). To do so, one must solve
\begin{equation}
\frac{\partial \chi}{\partial t} = \Phi
\end{equation}
and, at $t = 0$,
\begin{equation}
\Delta \chi = -\text{div} \, A \quad \text{in } \Omega \quad \text{with} \quad \nabla \chi \cdot n = -A \cdot n \quad \text{on } \partial \Omega.
\end{equation}

Thus, in this gauge, $\Phi \equiv 0$ and system (1) reduces to the gradient flow of the energy functional:
\begin{equation}
E(\psi, A) = \frac{1}{2} \int_{\Omega} \left[ \left( \frac{i}{\kappa} \nabla + A \right) \psi \right]^2 + \frac{1}{2} (|\psi|^2 - 1)^2 + |\text{curl} \, A - H_0|^2 \, dx.
\end{equation}

The initial condition satisfies $\text{div} \, A(0) = 0$ and $A(0) \cdot n = 0$. Moreover, we make a physically meaningful assumption that $\|\psi(0)\|_\infty \leq 1$. Let $v = (\psi, A)$. As shown in [11] and [28], the flow
\begin{equation}
\frac{dv}{dt} = -\text{grad} \, E(v), \quad v(0) = v_0,
\end{equation}
has a global classical solution. Let $V = \mathcal{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$, where $\mathcal{H}^2(\Omega)$ and $\mathbf{H}^2(\Omega)$ are spaces of functions whose components (or real and imaginary parts) are in the standard Sobolev space $H^2(\Omega)$.

Our main result concerning (17) now follows.

**Theorem 2.1.**
\begin{equation}
V_\infty = \lim_{t \to \infty} v(t) \quad \text{exists in } V.
\end{equation}
To describe the idea, we start with the ODE

\begin{equation}
\begin{aligned}
    \frac{dx}{dt} &= -\text{grad } f(x), \quad x \in \mathbb{R}^N \\
    x(0) &= x_0.
\end{aligned}
\end{equation}

We assume $f \in C^2(\mathbb{R}^N)$, $\nabla f(0) = 0$, and $x_0$ is close to 0.

**Case (i).** If $A = \nabla^2 f(0)$ is positive definite, then $x(t) \to 0$ (at an exponential rate) as $t \to +\infty$.

This result is standard. One can calculate

\[
\frac{d}{dt} |\dot{x}|^2 = -2(\nabla^2 f(x) \cdot \dot{x}, \dot{x}) \leq -2\lambda |\dot{x}|^2.
\]

Here we shall assume $|x(t)| \leq \delta_0$, and $(\nabla^2 f(x)) \geq \lambda I$ whenever $|x| \leq \delta_0$. Thus $|\dot{x}(t)| \leq |\dot{x}(0)| e^{-\lambda t}$ \forall $t \geq 0$ and $|x(t)| \leq |x_0| + \frac{1}{2}|\dot{x}(0)| = |x_0| + \frac{1}{2\lambda} |\nabla f(x_0)|$. We shall always assume $x_0$ is so close to the origin that $|x_0| + \frac{1}{2\lambda} |\nabla f(x_0)| < \delta_0$. Then the assumption $|x(t)| \leq \delta_0$ is true for all $t > 0$ and, thus, $x(t) \to 0$ at an exponential rate as $t \to +\infty$.

**Case (ii).** $\det (A) \neq 0$. Then one has that

\begin{align}
    \frac{d}{dt} (f(x) - f(0))^{1/2} &= \frac{1}{2} \frac{|\nabla f(x)| |\dot{x}|}{(f(x) - f(0))^{1/2}}, \quad \text{if } f(x) > f(0), \\
    \frac{d}{dt} (f(0) - f(x))^{1/2} &= \frac{1}{2} \frac{|\nabla f(x)| |\dot{x}|}{(f(x) - f(0))^{1/2}}, \quad \text{if } f(x) < f(0).
\end{align}

We obtain the following: either there is a $T \in (0, \infty)$ such that

\[
f(x)(T) \leq f(0) - \delta_0 \quad \text{for some } \delta_0 > 0
\]

or

\[
\lim_{t \to +\infty} x(t) = x_\infty \quad \text{exists.}
\]

Indeed, if $x$ is close to 0, then

\[
\frac{|\nabla f(x)|}{|f(x) - f(0)|^{1/2}} \geq \frac{\lambda_{\min}}{\sqrt{2\lambda_{\max}}} = C(A) > 0.
\]

Here $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum, respectively, eigenvalues of $(A^T A)^{1/2}$. Therefore,

\[
\int_0^\infty |\dot{x}(t)| dt \leq \frac{2\delta_0^{1/2}}{C(A)}
\]

by (19)–(20) whenever $f(x)(t) \geq f(0) - \delta_0$ for all $t \geq 0$. In such a case the conclusion $\lim_{t \to +\infty} x(t) = x_\infty$ follows (in fact $x_\infty = 0$) as for Case (i).

**Case (iii).** $\det (A) = 0$ and $f(x)$ is real analytic in $B_1$. We have the following well-known estimate (cf. [26, 27]). There are two positive constants $\theta_0, \sigma_0 \in (0, 1)$ depending on $f$ such that

\begin{equation}
|f(x) - f(0)|^{\theta_0} \leq |\text{grad } f(x)| \quad \text{whenever } x \in B_{\sigma_0}(0),
\end{equation}

\[
\text{and } \quad f(x) \leq f(0) - \delta_0 \quad \text{for some } \delta_0 > 0.
\]

\[
\lambda_{\min} \leq C(A) \lambda_{\max}.
\]
\[ \nabla f(0) = 0. \] Then, as for Case (ii), one has that \textit{either} there is a \( T \in (0, \infty) \) such that

\[ f(T) \leq f(0) - \delta_0 \]

\textit{or}

\[ \lim_{t \to \infty} x(t) = x_\infty \text{ exists.} \]

In [27], Simon considered the case

\[ E(u) = \int_M F(x, u, \nabla u) \, dx, \]

(22)

where \( M \) is a compact manifold without boundary and

\[ \begin{cases} \dot{u} = -\nabla E(u) =: \mathcal{M}(u), \\ u(0) = u_0 \simeq 0, \quad \mathcal{M}(0) = 0. \end{cases} \]

(23)

Here, \( F \) is assumed to be analytic in both \( u \) and \( \nabla u \) for \( u, \nabla u \) near 0.

Suppose \( L \) is elliptic, \( Lv = \frac{d}{ds} |_{s=0} \mathcal{M}(sv) \). Then \textit{either} there is a \( T \in (0, \infty) \) such that

\[ E(u(T)) \leq E(0) - \delta_0 \text{ for some } \delta_0 > 0 \]

\textit{or}

\[ u_\infty = \lim_{t \to \infty} u(t) \text{ exists.} \]

To apply the above idea to the time-dependent G–L equations, we need to use the following estimate given in [11] (also see [28] for similar results).

**Lemma 2.2.** Let \( v = (\psi, A) \) be the solution of the time-dependent G–L equations (17). Then

\[ \int_t^T \left[ \dot{\psi}(s), \dot{\psi}(s) + (\dot{A}(s), \dot{A}(s)) \right] \, ds + E(\psi(t), A(t)) = E(\psi(T), A(T)) \]

(24)

for any \( T > t > 0 \).

The following lemma will enable us to apply the above conclusion of Simon.

**Lemma 2.3.** Let \( (\psi, A) \) satisfy

\[ \begin{align*} \text{curl} A \land n &= H_0 \land n, \quad A \cdot n = 0, \quad \text{and} \quad \nabla \psi \cdot n = 0 \text{ on } \partial \Omega. \end{align*} \]

Let \( E(\psi, A) \) be defined by (16) and \( (\psi_*, A_*) \) be a steady state solution of the G–L equations in the gauge \( \text{div} A_* = 0 \) in \( \Omega \) and \( A_* \cdot n = 0 \) on \( \partial \Omega \). Then there exist constants \( \theta_0, \sigma_0 \) \( \in (0, 1) \) such that

\[ |E(\psi, A) - E(\psi_*, A_*)|^\theta_0 \leq \| \text{grad} E(\psi, A) \|_{L^2(\Omega)} \]

(25)

for any \( \| (\psi, A) - (\psi_*, A_*) \|_{C^2(\Omega)} \leq \sigma_0. \)

**Proof.** Let

\[ G(\psi, A) = E(\psi, A) + \frac{1}{2} \int_\Omega \| \text{div} A \|^2 \, dx. \]

(26)
By assumption, since \((\psi^*, A^*)\) is a steady state solution of the G–L equations in the
gauge \(\text{div} \ A = 0\) in \(\Omega\) and \(A^* \cdot \mathbf{n} = 0\) on \(\partial \Omega\), then

\[
G(\psi, A) = E(\psi, A)
\]

and \((\psi^*, A^*)\) remains a critical point of \(G\). By the ellipticity of \(\text{grad} \ G(\psi, A)\), see \([7, 11, 28]\), we may apply the result of \([27]\) to conclude that there exist constants \(\theta_0, \sigma_1 \in (0, 1)\) such that

\[
|G(\psi, A) - G(\psi^*, A^*)|^\theta_0 \leq \|\text{grad} \ G(\psi, A)\|_{L^2(\Omega)}
\]

for any \(\|(\psi, A) - (\psi^*, A^*)\|_{C^2(\Omega)} \leq \sigma_1\).

Let \(\chi\) be a gauge transformation function and \(\tilde{\psi} = e^{i\kappa \chi} \psi\) with \(\tilde{A} = A + \nabla \chi\). Let

us choose \(\chi\) such that \(\text{div} \ \tilde{A} = 0\). Simple calculation shows that

\[
\frac{\partial G}{\partial \tilde{\psi}} = e^{i\kappa \chi} \frac{\partial E}{\partial \psi} \quad \text{and} \quad \frac{\partial G}{\partial \tilde{A}} = \frac{\partial E}{\partial A}.
\]

So,

\[
\|\text{grad} \ E(\psi, A)\|_{L^2(\Omega)} = \|\text{grad} \ G(\tilde{\psi}, \tilde{A})\|_{L^2(\Omega)}.
\]

Since \(\text{div} \ A = 0\) for small enough \(\sigma_0\), if \(\|(\psi, A) - (\psi^*, A^*)\|_{C^2(\Omega)} \leq \sigma_0\), we have

\[
\|G(\tilde{\psi}, \tilde{A}) - G(\psi^*, A^*)\|^\theta_0 = \|G(\tilde{\psi}, \tilde{A}) - E(\psi^*, A^*)\|^\theta_0
\]

\[
\leq \|\text{grad} \ G(\tilde{\psi}, \tilde{A})\|_{L^2(\Omega)}
\]

\[
= \|\text{grad} \ E(\psi^*, A^*)\|_{L^2(\Omega)}.
\]

This proves the lemma.

We now turn to the proof of Theorem 2.1. It is important to observe that both
the energy functional and inequality in Lemma 2.3 are gauge invariant.

By Lemma 2.2, given any \(\epsilon > 0\), there exists a sufficiently large time \(t_n\) such that

\[
\|\dot{\psi}(t_n)\|_{L^2(\Omega)} < \epsilon,
\]

\[
\|\dot{A}(t_n)\|_{L^2(\Omega)} < \epsilon,
\]

and

\[
\int_{t_n}^T \left[\|\dot{\psi}(s)\|^2_{L^2(\Omega)} + \|\dot{A}(s)\|^2_{L^2(\Omega)}\right] ds < \epsilon \quad \forall \ T > t_n.
\]

By gauge invariance, one may define a gauge transformation function \(\tilde{\chi}(t_n)\) such that

\[
\tilde{\psi}(t_n) = e^{i\kappa \tilde{\chi}(t_n)} \psi(t_n) \quad \text{and} \quad \tilde{A}(t_n) = A(t_n) + \nabla \tilde{\chi}(t_n)
\]

\[
\text{div} \ \tilde{A}(t_n) = 0 \quad \text{in} \ \Omega,
\]

\[
\tilde{A}(t_n) \cdot \mathbf{n} = 0 \quad \text{on} \ \partial \Omega.
\]
We note that $\tilde{\psi}(t) = e^{i\chi(t_n)}\psi(t)$ and $\tilde{\mathbf{A}}(t) = \mathbf{A}(t) + \nabla \chi(t_n)$ are again solutions of the time-dependent G–L equations in the zero electric potential gauge with properly modified initial conditions and equation (24) still holds. Also,

$$\|\dot{\tilde{\psi}}(t_n)\|_{L^2(\Omega)} < \epsilon \quad \text{and} \quad \|\tilde{\mathbf{A}}(t_n)\|_{L^2(\Omega)} < \epsilon$$

and

$$\int_{t_n}^{T} \left[\|\dot{\tilde{\psi}}(s)\|_{L^2(\Omega)}^2 + \|\dot{\tilde{\mathbf{A}}}(s)\|_{L^2(\Omega)}^2\right] ds < \epsilon \quad \forall \, T > t_n$$

remain valid. To apply Lemma 2.3, we must construct solutions that are $C^2(\Omega)$ close to some steady state for some time $t_n \in [t_n, t_n + 1]$. To get the $C^2$ closeness, we use another gauge transformation $\tilde{\psi}(t) = e^{i\chi(t)}\tilde{\psi}(t)$ and $\tilde{\mathbf{A}}(t) = \tilde{\mathbf{A}}(t) + \nabla \chi(t)$, where $\chi$ is defined by

$$\frac{\partial \chi}{\partial t} - \Delta \chi = \text{div} \tilde{\mathbf{A}} \quad \text{in} \quad \Omega$$

with boundary condition

$$\frac{\partial \chi}{\partial n} = -\tilde{\mathbf{A}} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial \Omega$$

and initial condition at $t = t_n$: $\chi(t_n) = 0$.

Note that from the G–L equations (7)–(9), we may get [11]

$$\text{div} \dot{\tilde{\mathbf{A}}} = -\frac{i}{2} \kappa \left[\frac{\psi}{\psi^*} \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t}\right].$$

Thus, we have

$$\int_{t_n}^{t_n + 1} \left[\|\dot{\tilde{\mathbf{A}}}(s)\|_{L^2(\Omega)}^2\right] ds < c\epsilon$$

for some generic constant $c$. From this, we may get

$$\int_{t_n}^{t_n + 1} \left[\|
abla \chi(s)\|_{L^2(\Omega)}^2\right] ds < c\epsilon.$$ 

This further implies that

$$\int_{t_n}^{t_n + 1} \left[\|\Delta \chi(s)\|_{L^2(\Omega)}^2 + \|\dot{\chi}(s)\|_{L^2(\Omega)}^2\right] ds < c\epsilon.$$ 

It follows that $(\tilde{\psi}, \tilde{\mathbf{A}})$ also satisfies estimates similar to those in equations (29)–(31). Notice that $(\tilde{\psi}, \tilde{\mathbf{A}})$ is a solution of the time-dependent G–L equations in the gauge $\Phi = -\text{div} \mathbf{A}$ and it satisfies a parabolic system:

$$\begin{cases}
\frac{\partial \tilde{\psi}}{\partial t} - i \kappa \text{div} \tilde{\mathbf{A}} \tilde{\psi} + \left(\frac{i}{\kappa} \nabla + \tilde{\mathbf{A}}\right)^2 \tilde{\psi} - \tilde{\psi} + \tilde{\psi}^2 \tilde{\psi} = 0, \\
\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \Delta \tilde{\mathbf{A}} = -\frac{i}{2\kappa} (\tilde{\psi}^* \nabla \tilde{\psi} - \tilde{\psi} \nabla \tilde{\psi}^*) - \tilde{\mathbf{A}}|\tilde{\psi}|^2.
\end{cases}$$

(32)
with boundary conditions
\[
\text{curl } A \wedge n = H_0 \wedge n, \quad \vec{A} \cdot n = 0, \quad \text{and} \quad \nabla \vec{\psi} \cdot n = 0 \quad \text{on } \partial \Omega.
\]

Using standard parabolic regularity and estimates like (29)–(31) for \((\vec{\psi}, \vec{A})\), we may find a small \(\delta_0 > 0\) such that for all \(\delta_0 < \delta < 1\), we have
\[
\|\dot{\vec{\psi}}(t_n + \delta)\|_{C^2(\Omega)} < c\epsilon \quad \text{and} \quad \|\dot{\vec{A}}(t_n + \delta)\|_{C^2(\Omega)} < c\epsilon
\]
and
\[
\|\text{div } \vec{A}(t_n + \delta)\|_{C^1(\Omega)} < c\epsilon
\]
for some constant \(c\). Thus, we may conclude that in \(C^2(\Omega)\), \((\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta))\) is close to some solution \((\vec{\psi}_\infty, \vec{A}_\infty)\) of the steady state G–L equations with \(\text{div } \vec{A}_\infty = 0\).

Now, by Lemma 2.3, we get
\[
\left| E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \right|^{\theta_0} \\
\leq \|\text{grad } E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta))\|_{L^2(\Omega)}.
\]

Using the gauge invariance, however, this also implies that
\[
\left| E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \right|^{\theta_0} \\
\leq \|\text{grad } E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta))\|_{L^2(\Omega)}.
\]

Since \((\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) \in C^2\) close to \((\vec{\psi}_\infty, \vec{A}_\infty)\), we have
\[
\left| E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \right| \\
= \left| E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \right| \\
\leq c\epsilon.
\]

By the monotonicity of \(E(\vec{\psi}(t_n), \vec{A}(t_n))\), we have that for any \(T\), there exists \(t_n\) large enough such that
\[
E(\vec{\psi}(T), \vec{A}(T)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \\
\geq E(\vec{\psi}(t_n + \delta), \vec{A}(t_n + \delta)) - E(\vec{\psi}_\infty, \vec{A}_\infty) \\
\geq c\epsilon
\]
for some generic constant \(c\). Using the ideas given in Case (iii) for ODEs, we conclude that
\[
\int_T^{\infty} \left[ \|\dot{\vec{\psi}}(s)\|_{L^2(\Omega)} + \|\dot{\vec{A}}(s)\|_{L^2(\Omega)} \right] ds < c\epsilon.
\]

Thus, we must have that
\[
V_\infty = \lim_{t \to \infty} v(t) \quad \text{exists in } V.
\]

This concludes the proof of the Theorem 2.1.
3. The pinning effect of variable thickness in thin films. In the previous section, we were concerned with the dynamic properties of solutions to the time-dependent G–L equations. In this section, we focus our attention on the static case. In particular, we illustrate that the vortices may be pinned by inhomogeneities inside the material. The inhomogeneities we consider here are introduced due to the variation in thickness of the sample. During our writing, we also became aware of independent works [9, 1, 18] on the same subject; thus, we only give a brief outline here of the approach we have used.

3.1. The Dirichlet case. To present the main idea, we first ignore the magnetic potential and consider

$$\min\{F_{\epsilon}(\psi), \psi|_{\partial \Omega} = g\} = \min \left\{ \int_{\Omega} a(x) \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) dx, \psi|_{\partial \Omega} = g \right\}. \tag{33}$$

For convenience, we assume that $\Omega$ is a bounded, smooth (say Lipschitz) domain in $\mathbb{R}^2$. We may also view $\psi$ as a map from $\Omega$ to $\mathbb{R}^2$ and $g$ as a smooth map from $\partial \Omega$ to $S^1$ with $\deg(g, \partial \Omega) = d(\geq 0)$. The coefficient $a$ is a smooth (say Lipschitz or Hölder) function from $\Omega$ to $\mathbb{R}$ with $0 < m \leq a(x) \leq M$ for all $x \in \bar{\Omega}$. The minimizers of functional $F_{\epsilon}$ satisfy

$$\text{div} \left( a(x) \nabla \psi(x) \right) + \frac{a(x)}{\epsilon^2} (1 - |\psi(x)|^2) \psi(x) = 0 \text{ in } \Omega. \tag{34}$$

First of all, let us consider the case where, in $\Omega$, there are at least $d$ distinct points $b_1, \ldots, b_d, \ldots, b_k$ with $a(b_j) = m, j = 1, 2, \ldots, k$. Moreover, we assume that each $b_j$ is a strict minimum, that is, $a(x) > m$ for any $x \neq b_j$ in a small neighborhood of $b_j$. We define

$$\delta_0 = \frac{1}{2} \min\{|b_j - b_i|, \text{dist}(b_j, \partial \Omega); j \neq i, i, j = 1, \ldots, k\} > 0. \tag{35}$$

**Theorem 3.1.** Let $\psi_{\epsilon_n}, \epsilon_n \searrow 0$, be a sequence of minimizers of (33). Then

$$\psi_{\epsilon_n}(x) \rightarrow \psi^*(x) \text{ in } C^{1,\alpha}_{\text{loc}}(\Omega/\{b_1, \ldots, b_d\}), \tag{36}$$

where

$$\psi^*(x) = \prod_{j=1}^d x - b_j |x - b_j| e^{ih^*(x)} \text{ in } \Omega \tag{37}$$

and $b_1, \ldots, b_d$ are $d$ distinct points in $\Omega$ with

$$a(b_j) = m = \min_{x \in \Omega} a(x). \tag{38}$$

Moreover, if we write $\psi^*(x) = e^{i(\Theta(x) + h^*(x))}$, then $h^* \in H^1(\Omega) \cap C^{\alpha}(\bar{\Omega})$, $\psi^* = g$ on $\partial \Omega$ and

$$\text{div} \left[ a(x)(\nabla \Theta + \nabla h^*) \right] = 0 \text{ in } \Omega/\{b_1, \ldots, b_d\}. \tag{39}$$
The proof of the above theorem is based on a series of estimates as similarly presented in an earlier work [20]. One first has the energy upper bound
\[
\min \left\{ \mathcal{F}_\epsilon^a(\psi), \, \psi|_{\partial\Omega} = g \right\} \leq m\pi d \log \frac{1}{\epsilon} + C(a, g, \Omega)
\]
and energy lower bound
\[
\min \left\{ \int_B a(x) \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) \, dx, \, \psi|_{\partial B} = g, \, |g| \leq 1 \right\}
\geq m\pi d \log \frac{1}{\epsilon} - C(K),
\]
where \(m = \min_{x \in B} a(x)\) if \(\text{deg}(g, \partial B) \neq 0\). Now, the class \(S_g(\lambda, K)\) may be defined as in [20].

**Definition 3.2.** Let \(\Omega, g\) be given as before. We say a map \(u : \Omega \to \mathbb{R}^2\) belongs to the class \(S_g(\lambda, K)\) if

(i) \(\mathcal{F}_\epsilon(u) \leq m\pi d \log \frac{1}{\epsilon} + K\).

(ii) for any \(x \in \{x : x \in \Omega, |u(x)| \leq \frac{1}{2}\}\), \(B_{\lambda}(x) \cap \Omega \subset \{|u(x)| \leq \frac{3}{4}\}\).

Then one can show that if \(\psi\) minimizes (33), there exists some positive constants \(\lambda, K\) such that \(\psi \in S_g(\lambda, K)\).

One may then show that there are exactly \(d\) balls, say \(B_1, \ldots, B_d\), with \(x^*_1, \ldots, x^*_d\) their centers, such that the corresponding \(d_j = \text{deg}(u, \partial B_j)\) is 1 and
\[
\min \{ |x^*_j - x^*_k|, \text{dist}(x^*_j, \partial \Omega) ; j, k = 1, \ldots, d, j \neq k \} \geq \delta_1(\lambda, K) > 0.
\]
After extracting a subsequence, we have
\[
x^*_j \to x^*_j \quad \text{as} \quad \epsilon_n \to 0.
\]
The limits \(\{x^*_j\}\) are all different points; moreover, \(a(x^*_j) = m, \, j = 1, \ldots, d\). Combining with estimates away from vortices, the proof now follows similarly to that in [3, 20].

In the following, we briefly discuss the case when the number of minima is less than the degree of the boundary data. For simplicity, we focus on the case where \(\Omega\) is the unit disc \(B\) in \(\mathbb{R}^2\) and \(g(\theta) = e^{id\theta}\) for some positive integer \(d \geq 2\). Let \(a(x, y) = 1 + r^2, \, r^2 = x^2 + y^2\). We consider
\[
E_B = \min \left\{ \mathcal{F}_{\epsilon, B}(\psi) = \int_B \left( \frac{a}{2} |\nabla \psi|^2 + \frac{a}{4\epsilon^2} (1 - |\psi|^2)^2 \right) \, dx dy, \right\}
\]
\[
\psi|_{\partial \Omega} = g \right\}.
\]

**Theorem 3.3.** Let \(\psi_\epsilon\) be a sequence of minimizers of (41) as \(\epsilon \to 0\). Then all vortices of the \(\psi\) must be separated and of degree +1 for small enough \(\epsilon > 0\) and they go to the origin as \(\epsilon \to 0\).

**Proof.** Indeed, for any small parameter \(\delta > 0\) (say \(\delta < 1/d\)), one may choose \(\epsilon \leq \epsilon(\delta)\) and \(\psi\) with \(\psi|_{\partial \Omega} = g\) and vortices placed along \(\partial B_{\delta/2}\) such that
\[
E_B \leq \mathcal{F}_{\epsilon, B}(\psi) \leq \pi d (1 + \delta^2) \log \frac{\delta}{\epsilon} + \pi d^2 \log \frac{1}{\delta} + c_0 d^2
\]
\[
= \pi d (1 + \delta^2) \log \frac{1}{\epsilon} + \{\pi d^2 - \pi d (1 + \delta^2)\} \log \frac{1}{\delta} + c_0 d^2.
\]
On the other hand, if there are either vortices of degree no less than 2 or vortices outside \( B_{\sqrt{\delta}}(0) \), then by [3],
\[
F_{\epsilon,B}(\psi) \geq \pi(d + d\delta) \log \frac{1}{\epsilon} + c(\delta,d).
\]

For \( \delta > 0 \) and \( \epsilon \to 0 \), since \( \pi(d + d\delta) \log \frac{1}{\epsilon} + c(\delta,d) \), one sees that all vortices of the minimizer must go to the origin and be of degree +1.

The next question is for given \( d \) (say \( 2 \leq d \leq 5 \)) and \( \epsilon > 0 \) (but small): How close must these vortices be to the origin? We want to show that they cannot be too close.

Indeed, for a small generic constant \( \beta \), if all vortices of \( \psi \) are in the \( \beta \)-neighborhood of the origin, then
\[
\int_{B_{\beta}(0)} (1 + r^2) \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) \, dx\,dy \geq \pi d \log \frac{\beta}{\epsilon} - c_0 d^2
\]
and
\[
\int_{B \setminus B_{\beta}(0)} (1 + r^2) \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) \, dx\,dy \geq \pi d^2 \log \frac{1}{\beta}.
\]
So,
\[
F_{\epsilon,B}(\psi) \geq \pi d \log \frac{1}{\epsilon} + \pi d (d - 1) \log \frac{1}{\beta} - c_0 d^2.
\]

On the other hand, by placing vortices along \( \partial B_{\delta/2}(0) \), one may construct a map with energy
\[
F_{\epsilon,B}(\psi) \leq \pi d (1 + \delta^2) \log \frac{1}{\epsilon} + \pi d (d - 1 - \delta^2) \log \frac{1}{\delta} + c_0 d^2.
\]

Comparing the two bounds, we see that not all vortices are in the \( \delta \)-neighborhood of the origin if we have
\[
\pi d (d - 1) \log \frac{\delta}{\beta} \geq 2c_0 d^2 + \pi d \delta^2 \log \frac{1}{\epsilon}.
\]

By taking \( \beta \leq \delta^2 \), this means
\[
\pi d (d - 1) \log \frac{1}{\delta} - \pi d \delta^2 \log \frac{1}{\epsilon} \geq 2c_0 d^2.
\]

Again, letting \( \log \frac{1}{\delta} \geq 2c_0 \), it is sufficient to have
\[
\frac{d - 1}{2} \log \frac{1}{\delta} \geq \delta^2 \log \frac{1}{\epsilon},
\]
or
\[
\epsilon \geq \delta^{\frac{d-1}{2\delta^2}}.
\]

Thus, for small but positive \( \epsilon, \delta \) cannot be too small.

We have performed a series of numerical experiments to illustrate the pinning effect even for modest values of \( \epsilon \). The numerical methods used here are similar to
those discussed in [13, 14]. For computational convenience, the unit square $[0, 1]^2$ is taken to be our sample $\Omega$. In the first experiment, we choose the thickness function $a \equiv 1$. We solve for the minimizer of (33) by using the Dirichlet boundary conditions with $|\psi| = 1$ on the boundary. $\psi|_{\partial \Omega}$ has a winding number 4. The contour plots of the magnitude of the order parameter $\psi$ are given in Figure 2.

Next, we choose the thickness function $a = a(x, y)$ such that it has four minima at $(0.25, 0.25), (0.75, 0.35), (0.25, 0.65), (0.75, 0.75)$ with the same minimum value. The contour plots are given in Figure 3. We see that as $\epsilon$ gets smaller, the vortices get “pinned” at the minima of $a$. This is the case illustrated by Theorem 3.1.

Now, we continue the numerical experiments again using the Dirichlet boundary conditions with $|\psi| = 1$ on the boundary. However, we impose the boundary condition such that $\psi|_{\partial \Omega}$ has a winding number 3. We first choose the thickness function $a = a(x, y)$ such that it has four minima at $(0.25, 0.25), (0.25, 0.65), (0.75, 0.35), (0.75, 0.75)$ with the same minimum value. The contour plots are given in Figure 4. Since the number of minima is larger than the winding number, each vortex gets pinned to a different minimum point.

Then, we change the thickness function $a = a(x, y)$ such that it has two minima at $(0.25, 0.65), (0.75, 0.35)$ with the same minimum value. The contour plots are given in Figure 5. Since the number of minima is less than the winding number, one vortex gets pinned at one minimum but the other two get attracted to another minimum with some distance still between them.
Fig. 4. Contour plots of the magnitude of the order parameter for a model with variable thickness $a = a(x, y)$. The left-hand figure is for $\epsilon = .08$. The right-hand figure is for $\epsilon = .04$.

Fig. 5. Contour plots of the magnitude of the order parameter for a model with variable thickness $a = a(x, y)$. The left-hand figure is for $\epsilon = .08$. The right-hand figure is for $\epsilon = .04$.

For comparison purposes, we also give the pictures when choosing the thickness function $a \equiv 1$. The contour plots are given in Figure 6.

Finally, we present an experiment on the co-existence of vortex–antivortex solutions to the minimizers of (33) with constant thickness functions. This question has been rigorously studied in [22]. We again use the Dirichlet boundary conditions with $|\psi| = 1$ on the boundary. $\psi|_{\partial \Omega}$ has a winding number 0. Here, we start with a vortex state that has a vortex with winding number $+1$ on one side of the domain and a vortex with winding number $-1$ on the other side of the domain. We numerically follow the gradient flow of (33). The solution reaches steady state which again consists of two vortices with opposite winding numbers. The contour plots of the magnitude of the steady state solution $\psi$ are given in Figure 7.

3.2. The Neumann problem. We now study the minimization problem with Neumann-type boundary conditions. Let $a(x)$ have a strict local minimum at points $b_1, \ldots, b_d$ with $a(b_j) = m$, $j = 1, \ldots, d$. Recall that

$$\delta_0 = \frac{1}{2} \min \{|b_j - b_k|, \text{dist}(b_j, \partial \Omega); j \neq k, j, k = 1, \ldots, d\} > 0.$$ 

Let $r_0 < \delta_0$, $\Omega_0 = \Omega \setminus \bigcup_{i=1}^d B_{r_0}(b_i)$ and define

$$V = \left\{ u \in H^1(\Omega) \mid |u| \geq \frac{1}{2} \text{ in } \Omega_0, \right\}$$
Fig. 6. Contour plots of the magnitude of the order parameter for a model with constant thickness. The left-hand figure is for $\epsilon = .08$. The right-hand figure is for $\epsilon = .04$.

Fig. 7. Contour plots of the magnitude of the order parameter. The top figure is for $\epsilon = .08$. The bottom figure is for $\epsilon = .03$.

\begin{equation}
\deg \left( \frac{u}{|u|}, \partial B_{r_0}(b_k) \right) = 1, \ 1 \leq k \leq d.
\end{equation}

We consider

\begin{equation}
\min \left\{ \mathcal{F}_\epsilon(\psi), \ \psi \in V \right\} = \min \left\{ \int_\Omega a(x) \left( \frac{1}{2} |\nabla \psi|^2 + \frac{1}{4\epsilon^2}(1 - |\psi|^2)^2 \right) \, dx, \ \psi \in V \right\}.
\end{equation}

It is easy to see that $V$ is a weakly closed subset of $H^1(\Omega)$. In fact, it is also connected. For given $\epsilon > 0$, (43) has at least one minimizer, denoted by $\psi_\epsilon$.

**Theorem 3.4.** Let $\psi_{\epsilon_n}, \ \epsilon_n \downarrow 0$, be a sequence of minimizers of (43). Then,

$$\psi_{\epsilon_n}(x) \to \psi^*(x) \quad \text{in } C^{1,\alpha}_{\text{loc}} \left( \overline{\Omega}/\{b_1, \ldots, b_d\} \right),$$

where $\alpha$ is a constant depending on $\epsilon_n$.\[\]
where
\[ \psi^*(x) = \prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih^*(x)} \quad \text{in} \ \Omega. \]

Moreover, if we write \( \psi^*(x) = e^{i(\Theta(x) + h^*(x))} \), then \( h^* \in H^1(\Omega) \cap C^\alpha(\overline{\Omega}) \), \( \frac{\partial \phi^*}{\partial n} = 0 \) on \( \partial \Omega \) and
\[ \text{div} [a(x)(\nabla \Theta + \nabla h^*)] = 0 \quad \text{in} \ \Omega/\{b_1,\ldots,b_d\}. \]

The proof follows from constructions similar to those outlined in section 3.1.

3.3. Problems with prescribed magnetic potential. Unlike bulk material, in the thin film limit, superconductors of type-I and type-II display vortex-like structure. It has been shown that the G–L functional takes the special form
\[ F(\psi) = \int_{\Omega} a(x) \left( \frac{1}{2} |i \nabla \psi + A_0(x)\psi|^2 + (1 - |\psi|^2)^2 \right) dx, \]
where \( a(x) \) represents the relative thickness distribution of the thin film. Since, to leading order, the magnetic field penetrates the film uniformly, we get \( A_0 \) to be a given magnetic potential that can be prescribed by setting \( \text{curl} A_0(x) = H \).

Numerical simulation in [6, 14] suggests that the thickness variation provides a pinning mechanism for vortices. Using techniques similar to those in section 3.1, we now provide a rigorous analysis for such phenomena in the case where \( \kappa \) is large while the magnetic field is weak (\( H \approx \kappa^{-1} \)). By a proper scaling of the free energy, we get a functional of the form
\[ F^\epsilon_a(\psi) = \int_{\Omega} a(x) \left( \frac{1}{2} |\nabla \psi - i A_0(x)\psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) dx. \]

In the present form, \( \epsilon \) is a small parameter measuring the relative penetration depth and the sample size. Hence, we may consider problems similar to those in section 3.1 with the functional defined above. We first consider the Dirichlet problem
\[ \min \left\{ F^\epsilon_a(\psi), \psi|_{\partial \Omega} = g \right\} = \min \left\{ \int_{\Omega} a(x) \left( \frac{1}{2} |\nabla \psi - A_0(x)\psi|^2 \right. \right. \]
\[ + \left. \left. \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right) dx, \ \psi|_{\partial \Omega} = g \right\}. \]

For simplicity, we assume that \( |g| = 1 \) on \( \partial \Omega \), \( \text{deg}(g, \partial \Omega) = d \), and \( a(x) \) has \( d \) distinct strict minima at points \( b_1, b_2, \ldots, b_d \). With a proper choice of gauge, we may define
\[ A_0(x, y) = \frac{1}{2} (Hy, -Hx)^T, \]
where \( H \) is the scaled applied magnetic field. We then have the following theorem.

THEOREM 3.5. Let \( \psi_{\epsilon_n}, \epsilon_n \searrow 0 \), be a sequence of minimizers of (45). Then
\[ \psi_{\epsilon_n}(x) \rightharpoonup \psi^*(x) \quad \text{in} \ C^{1,\alpha}_{\text{loc}}(\overline{\Omega}/\{b_1, \ldots, b_d\}), \]
where
\[ \psi^*(x) = \prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih^*(x)} \quad \text{in } \Omega. \]

Moreover, if we write \( \psi^*(x) = e^{i(\Theta(x) + h^*(x))} \), then \( h^* \in H^1(\Omega) \cap C^\alpha(\Omega) \), \( \psi^*|_{\partial \Omega} = g \) on \( \partial \Omega \), and
\[ \text{div } [a(x)(\nabla \Theta + \nabla h^*)] = 0 \quad \text{in } \Omega/\{b_1, \ldots, b_d\}. \]

Next, we may also prove similar results for the Neumann-type problems. Let \( V \) be the space defined in (42). We consider
\[ \min \{ F^\alpha_c(\psi), \psi \in V \} = \min \left\{ \int_\Omega a(x) \left( \frac{1}{2} |\nabla \psi - A_0(x)\psi|^2 + \frac{1}{2} \frac{1}{4 \epsilon^2} (1 - |\psi|^2)^2 \right) dx, \psi \in V \right\}. \]

**Theorem 3.6.** Let \( \psi_{\epsilon_n}, \epsilon_n \searrow 0 \), be a sequence of minimizers of (46). Then
\[ \psi_{\epsilon_n}(x) \to \psi^*(x) \quad \text{in } C^{1,\alpha}_{\text{loc}}(\Omega/\{b_1, \ldots, b_d\}), \]
where
\[ \psi^*(x) = \prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih^*(x)} \quad \text{in } \Omega. \]

Moreover, if we write \( \psi^*(x) = e^{i(\Theta(x) + h^*(x))} = e^{i(\phi^*(x))} \), then \( h^* \in H^1(\Omega) \cap C^\alpha(\Omega) \), \( \frac{\partial \phi^*}{\partial n} = A_0 \cdot n \) on \( \partial \Omega \) (here, \( n \) is the outward normal of \( \partial \Omega \)), and
\[ \text{div } [a(x)(\nabla \Theta + \nabla h^*)] = 0 \quad \text{in } \Omega/\{b_1, \ldots, b_d\}. \]

Again, the proofs of Theorems 3.3 and 3.4 are omitted due to their similarities to our earlier discussions.

4. **The renormalized energy and the vortex solution of the full G–L model with applied magnetic field.** With proper scaling, we focus on the following form of the G–L functional:
\[ G(\psi, A) = \int_\Omega \left( \frac{1}{4 \epsilon^2} (1 - |\psi|^2)^2 + \frac{1}{2} |(\nabla - iA)\psi|^2 + \frac{1}{2} |\text{curl } A - H_0|^2 \right) dx. \]

Let \( \Omega \in \mathbb{R}^2 \) be a bounded Lipshitz domain and \( H_0 \) be a constant field. In this nondimensionalization, one may view \( \epsilon \) as proportional to \( \frac{1}{\kappa} \) and \( H_0 \) as proportional to \( \kappa \) times the (nondimensionalized) applied field. Studies of the densely packed vortex state when the nondimensionalized field is near the upper critical field may be found in [4]. In the low field case, variational problems concerning the G–L functional have been considered in [5] but with boundary conditions that are not completely physical. Our discussion here is valid for the natural (physically meaningful) boundary conditions and we will borrow many useful results from [5] and [20].
4.1. The renormalized energy. Following the discussions in [20, 21, 5], we now formulate the renormalized energy: Let

\[ \psi = e^{i\phi_b(x)} = \prod_{j=1}^{d} \frac{x - b_j}{|x - b_j|} e^{ih} \]

for some points \( b = (b_1, b_2, \ldots, b_d) \in \Omega^d \) and \( \frac{\partial \phi_b}{\partial n} = 0 \) on \( \partial \Omega \). Let

\[ B_\rho = \bigcup_{j=1}^{d} B_\rho(b_j) \].

Choose the gauge \( \text{div} A = 0 \) in \( \Omega \) and \( A \cdot n = 0 \) on \( \partial \Omega \). We may define \( \zeta \) such that

\[ A = \nabla^\perp \zeta \quad \text{in} \quad \Omega, \]

\[ \zeta = 0 \quad \text{on} \quad \partial \Omega. \]

Now, consider

\[ G_\rho = \int_{\Omega \setminus B_\rho} \left( \frac{1}{2} \left| \nabla \phi_b - \nabla^\perp \zeta \right|^2 + \left| \Delta \zeta - H_0 \right|^2 \right) d\Omega \]

\[ = \int_{\Omega \setminus B_\rho} \left( \frac{1}{2} \left| \nabla \phi_b \right|^2 + \frac{1}{2} \left| \nabla \zeta \right|^2 - \nabla \phi_b \cdot \nabla^\perp \zeta + \frac{1}{2} \left| \Delta \zeta - H_0 \right|^2 \right) d\Omega \]

\[ := d\pi \log \frac{1}{\rho} + W_\Omega(b, H_0) + O(\rho), \]

where the last equality may be taken as the definition of the renormalized energy \( W_\Omega(b, H_0) \). Note

\[ - \int_{\Omega \setminus B_\rho} \nabla \phi_b \cdot \nabla^\perp \zeta d\Omega = \int_{\Omega \setminus B_\rho} \text{div} (\zeta \cdot \nabla^\perp \phi_b) d\Omega \]

\[ = \sum_{j=1}^{d} 2\pi \zeta(b_j) + O(\rho). \]

So,

\[ W_\Omega(b, H_0) = \int_{\Omega \setminus B_\rho} \frac{1}{2} \left| \nabla \phi_b \right|^2 d\Omega - d\pi \log \frac{1}{\rho} + \sum_{j=1}^{d} 2\pi \zeta(b_j) \]

\[ + \int_{\Omega \setminus B_\rho} \frac{1}{2} \left( \left| \nabla \zeta \right|^2 + \left| \Delta \zeta - H_0 \right|^2 \right) d\Omega + O(\rho) \]

\[ = \int_{\Omega \setminus B_\rho} \frac{1}{2} \left| \nabla \phi_b \right|^2 d\Omega - d\pi \log \frac{1}{\rho} + 2\pi \sum_{j=1}^{d} \zeta(b_j) \]

\[ + \int_{\Omega \setminus B_\rho} \frac{1}{2} \left( \left| \nabla \zeta \right|^2 + \left| \Delta \zeta \right|^2 \right) d\Omega + \frac{1}{2} H_0^2 |\Omega| \]

\[ - \int_{\Omega \setminus B_\rho} H_0 \Delta \zeta d\Omega + O(\rho). \]
Minimizing the term involving $\phi_b$, we see that $\phi_b$ is a multivalued harmonic function on $\Omega \setminus \{b_1, b_2, \ldots, b_d\}$ and we may define the function $g_\Omega(b)$ by

$$\int_{\Omega \setminus B_\rho} \frac{1}{2} \left| \nabla \phi_b \right|^2 d\Omega - d\pi \log \frac{1}{\rho} := g_\Omega(b) + O(\rho).$$

Minimizing the terms involving $\zeta$, we can choose $\zeta$ to satisfy

$$-\Delta^2 \zeta + \Delta \zeta = 2\pi \sum_{j=1}^{d} \delta_{b_j} \quad \text{in } \Omega,$$

where $\delta_{b_j}$ is the Dirac–Delta measure with boundary conditions

$$\zeta = 0 \quad \text{on } \partial \Omega,$$

$$\Delta \zeta = H_0 \quad \text{on } \partial \Omega.$$

So,

$$2\pi \sum_{j=1}^{d} \zeta(b_j) = -\int_{\Omega} (|\nabla \zeta|^2 + |\Delta \zeta|^2) d\Omega + H_0 \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} d\Gamma.$$

Because $H_0$ is a constant, we get

$$\int_{\Omega \setminus B_\rho} H_0 \Delta \zeta d\Omega = H_0 \int_{\partial \Omega} \frac{\partial \zeta}{\partial n} d\Gamma + H_0 O(\rho).$$

Thus,

$$W_\Omega(b, H_0) = \frac{1}{2} H_0^2 |\Omega| - \frac{1}{2} \int_{\Omega} (|\nabla \zeta|^2 + |\Delta \zeta|^2) d\Omega + H_0 O(\rho) + g_\Omega(b).$$

Now, let us define $\zeta = \zeta_b + \zeta_{H_0}$, where

$$-\Delta^2 \zeta_b + \Delta \zeta_b = 2\pi \sum_{j=1}^{d} \delta_{b_j} \quad \text{in } \Omega,$$

$$\zeta_b = 0 \quad \text{on } \partial \Omega,$$

$$\Delta \zeta_b = 0 \quad \text{on } \partial \Omega,$$

and $\zeta_{H_0} = H_0 \zeta_1$ with

$$-\Delta^2 \zeta_1 + \Delta \zeta_1 = 0 \quad \text{in } \Omega,$$

$$\zeta_1 = 0 \quad \text{on } \partial \Omega,$$

$$\Delta \zeta_1 = 1 \quad \text{on } \partial \Omega.$$
Then
\[
\int_{\Omega} |\nabla \zeta|^2 d\Omega = H_0^2 \int_{\Omega} |\nabla \zeta_1|^2 d\Omega + \int_{\Omega} |\nabla \zeta_b|^2 d\Omega - 2H_0 \int_{\Omega} \Delta \zeta_1 \zeta_b d\Omega,
\]
and
\[
\int_{\Omega} |\Delta \zeta|^2 d\Omega = H_0^2 \int_{\Omega} |\Delta \zeta_1|^2 d\Omega + \int_{\Omega} |\Delta \zeta_b|^2 d\Omega + 2H_0 \int_{\Omega} \Delta \zeta_1 \Delta \zeta_b d\Omega,
\]
and
\[
\int_{\Omega} \Delta \zeta_1 (\Delta \zeta_b - \zeta_b) d\Omega = -2\pi \sum_{j=1}^{d} \zeta_1(b_j).
\]

So,
\[
\int_{\Omega} (|\nabla \zeta|^2 + |\Delta \zeta|^2) d\Omega = H_0^2 \int_{\Omega} (|\nabla \zeta_1|^2 + |\Delta \zeta_1|^2) d\Omega
+ \int_{\Omega} (|\nabla \zeta_b|^2 + |\Delta \zeta_b|^2) d\Omega - 2\pi \sum_{j=1}^{d} \zeta_1(b_j).
\]

Therefore,
\[
W_\Omega(b, H_0) = \frac{1}{2} H_0^2 C(\Omega) + 2\pi H_0 \sum_{j=1}^{d} \zeta_1(b_j) + \tilde{g}_\Omega(b) + O(\rho),
\]
where \(C(\Omega)\) is a constant and \(\tilde{g}_\Omega(b)\) has the property
\[
\tilde{g}_\Omega(b) = \begin{cases} +\infty, & b_i = b_j \text{ for some } i \neq j, \\ -\infty, & b \in \partial \Omega, \end{cases}
\]
otherwise it is a smooth function in \(\Omega^d\).

**Lemma 4.1.** \(W_\Omega(b, H_0)\) has a local minimum inside \(\Omega^d\) whenever \(H_0 \geq H_0(\Omega)\) for some constant \(H_0(\Omega)\).

**Proof.** Choose a small enough positive constant \(\delta_0\) and let
\[
\Omega_{\delta_0} = \{ x \in \Omega \mid \delta_0 \leq \text{dist}(x, \partial \Omega) \leq 2\delta_0 \}.
\]
If \(\text{dist}(b_j, \partial \Omega) \geq \delta_0\) for all \(j\), then \(\tilde{g}_\Omega(b) \geq -M(\delta_0)\), independently of \(H_0\). On the other hand, we can choose \(d\) distinct points \(b_1, b_2, \ldots, b_d \in B_R(x_0)\), where \(x_0\) satisfies \(\zeta_1(x_0) = \min_{x \in \Omega} \zeta_1(x) = -m_0 < 0\) such that \(B_R(x_0) \subset \{ x : \text{dist}(x, \partial \Omega) \geq 2\delta_0 \}\) and such that
\[
\tilde{g}_\Omega(b) \leq C(R)
\]
for some constant \(C(R)\) when \(R\) is small. Moreover,
\[
\sum_{j=1}^{d} \zeta_1(b_j) \leq -dm_0 + C_1 R
\]
for some constant $\tilde{C}_1$. So,

$$W_\Omega(b, H_0) \leq -2\pi H_0 dm_0 + C(R) + 2\pi H_0 \tilde{C}_1 R + \frac{1}{2} H_0^2 C(\Omega).$$

If, however, at least one $b_j$ satisfies $b_j \in \Omega_{\delta_0}$, then

$$W_\Omega(b, H_0) \geq -M(\delta_0) - 2\pi H_0 \tilde{C}_2 \delta_0 - 2\pi H_0 (d - 1)m_0 + \frac{1}{2} H_0^2 C(\Omega)$$

$$\geq -2\pi H_0 dm_0 + C(R) + 2\pi H_0 \tilde{C}_1 R + \frac{1}{2} H_0^2 C(\Omega)$$

for some constant $\tilde{C}_2$ if $R, \delta_0$ are small enough and $H_0$ is large enough. Thus, $W_\Omega(b, H_0)$ has a local minimum inside $\Omega^d$ whenever $H_0 > H_0(\Omega)$.

### 4.2. The existence of vortex solutions.

Using the renormalized energy, we now study the solutions of the G–L equations (3)–(6).

**Theorem 4.2.** If $H_0 \geq H_0(\Omega)$ and $W_\Omega(b, H_0)$ has a nondegenerate local minimum for some $b \in \Omega^d$, then, for small $\epsilon$, there are solutions $(\psi^\epsilon, A^\epsilon)$ to the full steady state G–L equations with the gauge choice $A^\epsilon = \nabla \perp \xi^\epsilon$ in $\Omega$, $A^\epsilon \cdot n = 0$ on $\partial \Omega$ such that

$$\psi(\xi) \to \psi^*(\xi) \quad \text{in } C^{1,\alpha}_{\text{loc}} \left(\frac{\Omega}{\{b_1, b_2, \ldots, b_d\}}\right),$$

$$\zeta(\xi) \to \zeta^*(\xi) \quad \text{in } H^2(\Omega),$$

where

$$-\Delta^2 \zeta^* + \Delta \zeta^* = 2\pi \sum_{j=1}^d \delta(b_j) \quad \text{in } \Omega$$

with

$$\zeta^* = 0 \quad \text{on } \partial \Omega,$$

$$\Delta \zeta^* = H_0 \quad \text{on } \partial \Omega,$$

and

$$\psi^*(\xi) = \prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{ih^*(\xi)} \quad \text{in } \Omega.$$  

Moreover, if we write $\psi^*(\xi) = e^{i\theta_b(\xi) + ih^*(\xi)} = e^{i\phi_b}$, then $\phi_b$ is a multivalued harmonic function on $\Omega \setminus \{b_1, b_2, \ldots, b_d\}$ with $\frac{\partial \phi_b}{\partial n} = 0$ on $\partial \Omega$.

To complete the proof, we use the approach in [21]. We consider the solution of the following system:

$$\frac{\partial \psi}{\partial t} - (\nabla - iA)^2 \psi - \frac{1}{\epsilon^2} \psi(1 - |\psi|^2) = 0 \quad \text{in } \Omega,$$

$$\frac{\partial A}{\partial t} - \Delta A + \frac{i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2 A = 0 \quad \text{in } \Omega,$$
and
\begin{align}
(50) & \quad \text{curl} A = H_0 \quad \text{on } \partial \Omega, \\
(51) & \quad (\nabla - i A) \psi \cdot n = 0 \quad \text{on } \partial \Omega, \\
(52) & \quad A \cdot n = 0 \quad \text{on } \partial \Omega
\end{align}

with properly defined initial conditions.

Let
\[ E_\epsilon(\psi, A) = \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 + \frac{1}{2} |(\nabla - i A) \psi|^2 + \frac{1}{2} |\text{curl} A - H_0|^2 \]

and
\[ \tilde{E}_\epsilon(\psi, A) = E_\epsilon(\psi, A) + \frac{1}{2} |\text{div} A|^2. \]

It is easy to see that (see [11], for example)
\[ \int_\Omega \tilde{E}_\epsilon(\psi(t), A(t)) \, dx \leq \int_\Omega \tilde{E}_\epsilon(\psi(0), A(0)) \, dx. \]

Assuming that the initial condition satisfies
\[ \text{div } A(0) = 0, \]
this in turn implies
\[ \int_\Omega E_\epsilon(\psi(t), A(t)) \, dx \leq \int_\Omega E_\epsilon(\psi(0), A(0)) \, dx. \]

Let \(\rho_0\) be small enough such that
\[ \delta_0 = \min \{ \text{dist}(\partial \Omega, \partial B_{2\rho_0}(b_j)) \}, |b_j - b_i|, i \neq j, i, j = 1, 2, \ldots, d \} > 0 \]

and
\[ \min_{x \in \partial B_{\rho_0}(b)} W(x) \geq C(\rho_0) + W(b). \]

Let us construct initial conditions. Let \(A(0) = \nabla^\perp \zeta_b \) in \(\Omega\) with \(\zeta_b\) defined as before. For small \(\rho\), let \(B_\rho = \bigcup_{j=1}^d B_\rho(b_j)\), and let \(\psi(0) = e^{i\phi_b}\) in \(\Omega \setminus B_\rho\) and extend \(\psi(0)\) inside each ball \(B_\rho(b_j)\) by the minimizer of
\[ I_\rho(\psi, A) = \min \left\{ \int_{B_\rho(b_j)} \left( \frac{1}{2} |\nabla \psi - i A \psi|^2 + \frac{1}{4\epsilon^2} (1 - |\psi|^2)^2 \right. \right. \]
\[ + \left. \left. \frac{1}{2} |\text{curl} A - H_0|^2 \right) dx \mid \text{div } A = 0 \text{ in } \Omega, \quad A \cdot n = 0 \text{ on } \partial \Omega, \right. \]
\[ \left. |\psi| \big|_{\partial B_\rho(b_j)} = 1, \quad (i \psi, (\nabla - i A) \psi \cdot \tau) = g \text{ on } \partial B_\rho(b_j), \right. \]
\[ \text{and } \deg(\psi, \partial B_\rho(b_j)) = 1, \quad j = 1, 2, \ldots, d \right\}. \]
where $\tau$ is the unit tangent vector and $g = \nabla(\theta_b - \zeta_b) \cdot \tau$ on $\partial B_p(b_j)$.

Using the argument in [5] and the gauge invariance, one may show that

$$I_p(\psi, A) = \pi d \log \left( \frac{p}{\epsilon} \right) + \gamma d + o(1) + O(\rho).$$

Thus, we may assume

$$E_\epsilon(\psi(0), A(0)) \leq d \pi \log \frac{1}{\epsilon} + W_\Omega(b, H_0) + \gamma d + o(1).$$

**Lemma 4.3.** The solution of (48)–(52) has the property, for all sufficiently small $\epsilon$ and for all $t \geq 0$, that the set $\{x \in \Omega, |\psi_\epsilon(x)| \leq \frac{1}{2}\}$ is contained in a union of disjoint discs $B_j, j = 1, 2, \ldots, d$, where

(i) $B_j = B_\alpha(x_j^\epsilon), x_j^\epsilon \in B_{\rho_0}(b_j)$, for some small $\alpha$.

(ii) $\epsilon^a \int_{\partial B_j} \mathcal{E}_\epsilon(\psi_\epsilon, A_\epsilon)d\mathbf{x} \leq C(K_0).

Thus, $d_j = \text{deg}(\partial B_j, \psi)$ is well defined and $d_j = 1, j = 1, 2, \ldots, d$.

**Proof.** Let us define

$$P = \{t \geq 0 \mid (\psi_\epsilon, A_\epsilon) \text{ s.t. (i) -- (ii)}\}.$$ 

Clearly, $P$ is a closed set and by choosing the initial conditions properly, we have $0 \in P$. We now show that $P$ is open; thus $P = [0, \infty)$. First, we have the following claim.

**Claim.** For any $t \in P$, there exists a $\rho^* \in (0, \rho_0)$, independent of sufficiently small $\epsilon$, such that $\min\{\text{dist}(x^\epsilon, \partial B_{\rho_0}(b))\} \geq \rho^*$.

Assume that the claim is true. Then for any $t' > t$ but sufficiently close to $t$,

$$\text{deg}(\partial B_{\rho^*}(x_j^\epsilon), \psi_\epsilon(\cdot, t')) = 1.$$

We may then follow the ideas in [21] and [5] to construct, for $\psi_\epsilon(\cdot, t')$, a new disc $\hat{B}_j$ which satisfies (i)–(ii). Again, the center of the disc can be shown to satisfy the above claim. Thus, the lemma is proved.

Now we verify the claim. Suppose the claim is not true, there exists a sequence $\epsilon_n \searrow 0, t_n \nearrow t^* \in P$, such that $x_i^{\epsilon_n} \to b_i \in B_{\rho_0}(b_i)$ for $i = 1, 2, \ldots, d$, and $b_j \in \partial B_{\rho_0}(b_j)$ for some $j$.

We will follow constructions similar to those in [21] and [5] to show that one may replace $(\psi_\epsilon, A_\epsilon)$ by $(\bar{\psi}_\epsilon, \bar{A}_\epsilon)$ to satisfy

$$E(\bar{\psi}_\epsilon, \bar{A}_\epsilon) \geq \pi d \log \frac{1}{\epsilon} + W_\Omega(b, H_0) + \gamma d + o(1).$$

First, let us do gauge transformation such that $\bar{A}_\epsilon = \nabla^\perp \zeta_\epsilon$ with $\zeta_\epsilon = 0$ on $\partial \Omega$. Let us consider

$$\min \left\{ \int_{\Omega} \mathcal{E}_\epsilon(\bar{\psi}, \bar{A})d\mathbf{x} \mid \bar{A} = \nabla^\perp \zeta \text{ in } \Omega, \ deg(\bar{\psi}, \partial B_{\alpha}(x_j^\epsilon)) = 1, \right.$$ 

$$|\bar{\psi}|_{\partial B_{\alpha}(x_j)} = |\psi_\epsilon|_{\partial B_{\alpha}(x_j^\epsilon)}, \ (i\bar{\psi}, (\nabla - i\bar{A})\bar{\psi} \cdot \tau) = g_\epsilon \text{ on } \partial B_{\alpha}(x_j^\epsilon) \right\},$$

where $g_\epsilon = (i\psi_\epsilon, (\nabla - iA_\epsilon)\psi_\epsilon \cdot \tau)$. Note that the boundary conditions are the same gauge invariant boundary conditions considered in [5] and $\alpha$ satisfies property (ii) listed above.
Let \((\bar{\psi}_\epsilon, \bar{A}_\epsilon)\) denote a minimizer. Let \(\tilde{\psi}_\epsilon = \bar{\rho}_\epsilon e^{i\bar{\phi}_\epsilon}\) and \(\psi_\epsilon = \rho_\epsilon e^{i\phi_\epsilon}\) on the boundary of \(\bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)\). To extend it inside the \(\epsilon^\alpha\) balls, we define a gauge transformation by

\[\Delta^2 \chi = 0 \quad \text{in} \quad \bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)\]

with boundary conditions

\[\kappa \chi = \tilde{\phi}_\epsilon - \phi_\epsilon \quad \text{and} \quad \frac{\partial \chi}{\partial n} = 0 \quad \text{on} \quad \bigcup_{j=1}^d \partial B_{\epsilon^\alpha}(x_j^\epsilon).\]

Note that \(\chi\) is well defined on \(\bigcup_{j=1}^d \partial B_{\epsilon^\alpha}(x_j^\epsilon)\). Thus, inside \(\bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)\), we define \(\bar{\psi}_\epsilon = \psi_\epsilon e^{i\chi}\) and \(\bar{A}_\epsilon = A_\epsilon + \nabla \chi\).

By the gauge invariance and the choice of the small \(\alpha\), we have the energy lower bound inside the \(\epsilon^\alpha\) balls:

\[
\int_{\bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)} E_\epsilon(\bar{\psi}_\epsilon, \bar{A}_\epsilon) \, dx = \int_{\bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)} E_\epsilon(\psi_\epsilon, A_\epsilon) \, dx \\
\geq \pi d \log \left( \frac{\epsilon^\alpha}{\epsilon} \right) - C
\]

for small enough \(\epsilon\). So, we have the energy upper bound outside:

\[
\int_{\Omega \setminus \bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)} E_\epsilon(\bar{\psi}_\epsilon, \bar{A}_\epsilon) \, dx \leq \pi d \alpha \log \left( \frac{1}{\epsilon} \right) + C.
\]

Using arguments similar to those in [21] and [5], we have

\[|\bar{\psi}_\epsilon| \geq \frac{1}{2} \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon).\]

Moreover, we may modify the arguments of [21] and [5] to show that we have strong convergence of

\[\bar{\psi}_\epsilon \to e^{i\phi_\delta}\]

and

\[\bar{A}_\epsilon \to \nabla^\perp \zeta_\delta\]

outside any small neighborhood of \(B_j, j = 1, 2, \ldots, d\). Indeed, using the fact that \(H = \text{curl} \, A_\epsilon\) satisfies the equation

\[
\text{div} \left( \frac{1}{|\psi_\epsilon|^2} \nabla H \right) = H \quad \text{in} \quad \Omega \setminus \bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon),
\]

we note that \(\rho = |\bar{\psi}_\epsilon| > 1/2\) in \(\Omega \setminus \bigcup_{j=1}^d B_{\epsilon^\alpha}(x_j^\epsilon)\) (in fact, \(\rho\) is arbitrarily close to 1 if we allow \(\epsilon\) small). Therefore, we obtain from elliptic estimates that the \(W^{1,p}\) norm
and the $C^\gamma$ norm of $H$ are bounded locally for some $p > 2$ and $\gamma > 0$. On the other hand, from Lemma 4.1 in [21] and arguments in [21] and [5], one deduces the local strong convergence of $\bar{\psi}_\epsilon$. Then the assertion on the convergence of $\bar{A}_\epsilon$ follows. We omit the details.

Next, for small $\delta$, by strong convergence,

$$
\int_{\Omega} \bigcup_{j=1}^d \mathcal{E}_\epsilon(\bar{\psi}_\epsilon, \bar{A}_\epsilon) \, dx 
\geq \pi d \log \left( \frac{1}{\delta} \right) + W_{\Omega}(\bar{b}, H_0) + o(1) + O(\delta).
$$

(53)

We can also get $\text{deg}(\bar{\psi}_\epsilon, \partial B_\delta(\bar{b}_j)) = 1$ for any $j$.

Let us write $\bar{\psi}_\epsilon = |\bar{\psi}_\epsilon| e^{i\phi_\epsilon} = \rho e^{i\theta} + ih$, where $\theta$ is the angle function inside each $B_\delta(x_j^\epsilon)$ so that $e^{i\theta(x)} = \frac{x - \bar{b}_j}{|x - \bar{b}_j|}$ in $B_\delta(x_j^\epsilon) \forall j$.

We have, by gauge invariance, the energy upper bound outside the $\epsilon^\alpha$ balls:

$$
\int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) \left( |\nabla \rho|^2 + \rho^2 |\nabla \theta - \nabla^\perp \zeta|^2 + |\Delta \zeta|^2 \right) \, dx \leq C \log \left( \frac{1}{\epsilon} \right) + K,
$$

(54)

where $\nabla^\perp \zeta = \bar{A}_\epsilon$ is the magnetic potential after a gauge transformation.

Thus, we have

$$
\int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) \left( 1 - \rho^2 \right) |\nabla \theta - \nabla^\perp \zeta|^2 \, dx
\leq \int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) \left[ 2(1 - \rho^2) |\nabla^\perp \zeta|^2 + 2(1 - \rho^2) |\nabla \theta|^2 \right] \, dx
\leq \int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) \left( \frac{1}{\epsilon} (1 - \rho^2)^2 \, dx + \epsilon |\nabla^\perp \zeta|^4 \right) \, dx + o(1)
\leq C \epsilon \log \left( \frac{1}{\epsilon} \right) + \epsilon \left( \int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) |\Delta \zeta|^2 \, dx \right)^2 + o(1)
\leq C \epsilon \log \left( \frac{1}{\epsilon} \right) + C \epsilon \left( \log \left( \frac{1}{\epsilon} \right) \right)^2 + o(1)
\leq o(1),
$$

where $C$ is some generic constant.

This implies in particular that

$$
\int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) \mathcal{E}(\bar{\psi}_\epsilon, \bar{A}_\epsilon) \, dx
\geq \int \bigcup_{j=1}^d B_\delta(x_j^\epsilon) \setminus B_{\epsilon^\alpha}(x_j^\epsilon) |\nabla \theta - \nabla^\perp \zeta|^2 \, dx + o(1)
\geq \pi d \log \left( \frac{\delta}{\epsilon^\alpha} \right) + o(1).
$$
Therefore, by arguments in [5] and the convergence of \((\tilde{\psi}_\epsilon, \tilde{A}_\epsilon)\), we have

\[
\sum_{j=1}^{d} \int_{B_j(\delta)} E_e(\tilde{\psi}_\epsilon, \tilde{A}_\epsilon) \, dx \geq \pi d \log \left( \frac{\delta}{\epsilon} \right) + \gamma d + o(1) + O(\delta).
\]

Thus, for small \(\epsilon\) and small \(\delta\),

\[
\int_{\Omega} E_e(\tilde{\psi}_\epsilon, \tilde{A}_\epsilon) \, dx \geq \pi d \log \left( \frac{1}{\epsilon} \right) + W_{\Omega}(\bar{b}, H_0) + \gamma d + o(1).
\]

This leads to a contradiction of the energy upper bound obtained from the energy dissipation from the energy of the initial condition since \(b\) is the nondegenerate local minimum of \(W(b, H_0)\). Hence, the claim is true.

To complete the proof of the theorem, let us use the uniform bound on the solution \((\psi^\epsilon, A^\epsilon)\) as \(t \to \infty\). We get a subsequence \(t_n\) such that \((\psi^\epsilon(t_n), A^\epsilon(t_n)) \to (\psi^\epsilon, A^\epsilon)\) as \(t \to \infty\). One may then easily check that \((\psi^\epsilon, A^\epsilon)\) is the critical point of the G–L functional. This proves the theorem.

Remark.

(1) One may prove, using arguments given in [5], that as \(\epsilon \to 0\), the zeros of the \(\psi^\epsilon\) go to the local minimum of the renormalized energy.

(2) By a more careful analysis, one may replace the assumption that the renormalized energy has a nondegenerate local minimum by simply the existence of a local minimum.

(3) One can also prove, via [19], the existence of general critical points (saddle points) of the G–L functional under similar conditions.

(4) According to the nondimensionalization used here, our theorem describes the phenomenon that the G–L system has a vortex solution for an applied field of strength on the order of \(\frac{1}{\kappa}\). Recall that the standard estimates for the lower critical field are \(\log(\kappa) / \kappa\). That is, we can prove the existence of a stable vortex state below \(H_{c1}\).

4.3. The weak hysteresis near the lower critical field \(H_{c1}\) for type-II superconductors. We have just shown that when we decrease the applied magnetic field, there may be stable vortex states (even local energy minimizing states) even when the field strength is below the lower critical field \(H_{c1}(\approx \log(\kappa) / \kappa\), say, \(c_0 / \kappa\) for some constant \(c_0\). On the other hand, it is rather straightforward to check from the definition of \(W_{\Omega}(b, H_0)\) that there is no local minimum of the function \(W_{\Omega}(b, H_0)\) when \(H_0\) is sufficiently small. That is, we have shown the existence of a subcooling field \(0 < H_{sc} < H_{c1}\).

Let us consider another case where we gradually increase the applied field. We start, say, with a perfect superconducting state (or the Meissner state). It is quite possible that the Meissner state exists even when the applied field is much larger than the lower critical field \(H_{c1}\). Indeed, if one looks for a solution to the full steady state G–L equations \((\psi, A)\) with \(|\psi| \neq 0\), then one may write \(\psi = f e^{i\chi}\) for some \(f \neq 0\).

Using gauge invariance, we define \(Q = A - \frac{1}{\kappa} \nabla \chi\), then \((f, Q)\) remains a solution of the steady state G–L equations which can now be written in the following form:

\[
\begin{align*}
\frac{1}{\kappa} \Delta f &= f^3 - f + f|Q|^2 \\
-\text{curl}^\Omega Q &= f^2 Q
\end{align*}
\]

in \(\Omega\).
with boundary conditions

\begin{equation}
\begin{aligned}
\begin{cases}
\frac{\partial f}{\partial n} = 0 \\
\mathbf{Q} \cdot \mathbf{n} = 0 \\
H = \text{curl} \mathbf{Q} = H_0
\end{cases}
\end{aligned}
\end{equation}

From the equation for \( \mathbf{Q} \), we obtain

\begin{equation}
\begin{aligned}
\begin{cases}
- \frac{\partial H}{\partial x_2} = f^2 Q_1 \\
\frac{\partial H}{\partial x_1} = f^2 Q_2
\end{cases}
\end{aligned}
\end{equation}

or, equivalently,

\[ \text{div} \left( \frac{\nabla H}{f^2} \right) = H \quad \text{in } \Omega. \]

As in [2], we let \( \kappa \to \infty \) to obtain

\[
f = 1 - |\mathbf{Q}|^2.
\]

So, \( |\nabla H|^2 = f^2(1 - f^2) \). Let \( f = \rho(|\nabla H|) \). Then the equation for \( H \) becomes

\begin{equation}
\begin{aligned}
\begin{cases}
\text{div} \left( \rho(\nabla H) \nabla H \right) = H \\
H = H_0
\end{cases}
\end{aligned}
\end{equation}

For \( H_0 > 0 \) small enough (but obviously larger than \( \lim_{\kappa \to \infty} \frac{\log \kappa}{\kappa} = 0 \)), one can show the existence of a solution to the above equation. Moreover, such a solution is a linearly stable wholly superconducting state (Meissner state) for \( 0 < H_0 \leq H_0^* \). The new field strength \( H_0^* \) is called the superheating field; see [2] for details.

Combining this latter result with ours given in section 4.2, a weak-hysteresis diagram of type-II superconductors near the lower critical field \( H_{c1} \) is completed (see Figure 8).
REFERENCES