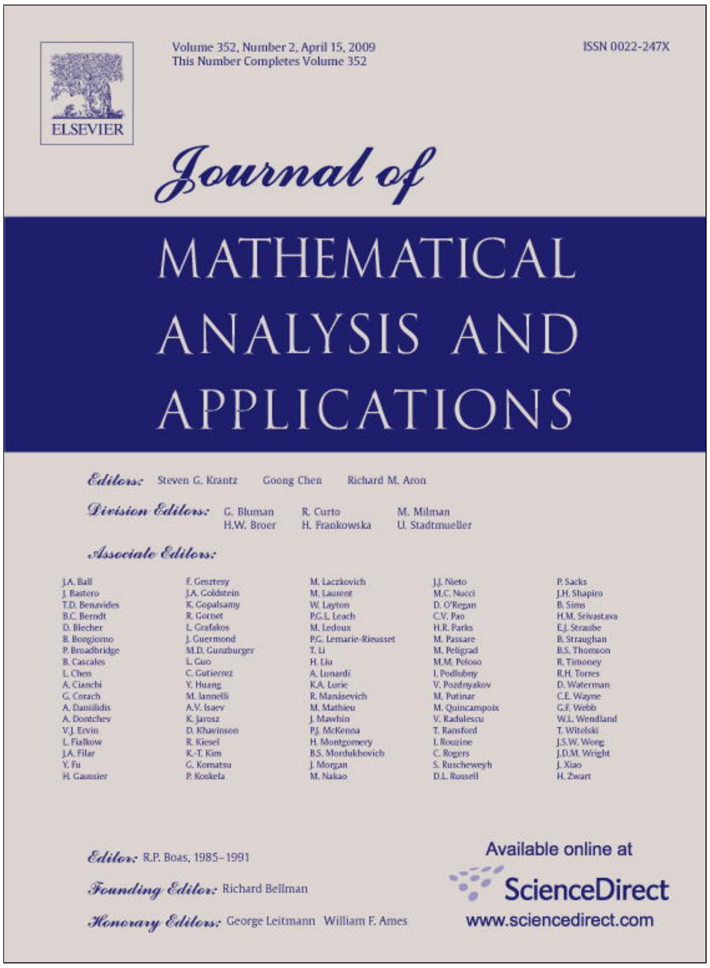
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A finite volume method on general surfaces and its error estimates

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ABSTRACT

In this paper, we study a finite volume method and its error estimates for the numerical solution of some model second order elliptic partial differential equations defined on a smooth surface. The discretization is defined via a surface mesh consisting of piecewise planar triangles and piecewise polygons. The optimal error estimates of the approximate solution are proved in both the H^1 and L^2 norms which are of first order and second order respectively under mesh regularity assumptions. Some numerical tests are also carried out to experimentally verify our theoretical analysis.

1. Introduction

Numerical solutions of partial differential equations on arbitrary surfaces or two dimensional Riemannian manifolds are needed in diverse applications such as fluid dynamics, weather forecast and climate modeling, chemical coating, cell membrane modeling and image processing [5,9,18,25,34–36,45,47]. Many discretization techniques developed for these type of problems are based on finite element methods or finite difference methods, including direct discretizations on surface meshes [4,23] or discretizations via level set techniques for implicitly defined surfaces [5,30,43]. Meanwhile, finite volume methods (also called finite volume element methods or co-volume methods) for the numerical solution of partial differential equations have also been gaining popularity in recent decades, see for instance, the barycenter-based method [2,6,7,11,39,41], the circumcenter-based method [40,42], the unified approach [8,28,29,37,38,46], the discretizations on the sphere [15,16,33–36,44,45] and references cited therein. Finite volume methods can be applied to general unstructured meshes, and their advantages include the natural preservation of conservation properties and the easy extension to upwinding and high-order fluxes to ensure stability and solution monotonicity at the discrete level. It is thus interesting to consider the application of finite volume methods to solve PDEs on general surfaces. Yet, the theoretical analysis of such approximations remains limited in comparison with the analysis of finite element approximations for which a priori estimates for general surfaces meshes (not just simplices), pointwise estimates and a posteriori estimates have all been developed recently [12,13].

The objective of this paper is to analyze a finite volume method based on the primal-dual meshes for the numerical solution of some linear second order elliptic equations defined on smooth surfaces. We choose to work directly with a surface discretization, in the form of a piecewise linear complex representation, rather than using an implicitly defined surface approach. The latter often avoids the difficulty of dealing with complex (and perhaps evolving) surfaces at the expense of solving equations in a higher space dimension. The former approach, on the other hand, relies its success more on a good

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geometric representation of the underlying surface. Naturally, another alternative is to use the surface parameterization to map the problem to a planar domain entirely and then make it tractable via conventional discretization methods in $\mathbb{R}^2$. A comprehensive discussion on the pros and cons of these different approaches is beyond the scope of this paper. The focus here is rather on some theoretical issues related to the direct discrete approximations, in the situation where a good piecewise (locally defined) representation of the surface is available or can be efficiently constructed [14,21]. The main results of this paper are the rigorous analysis of a finite volume method for some model elliptic equations (diffusion–reaction problem) based on primal-dual surface meshes. To our knowledge, there has not been any rigorous error estimate for the finite volume methods on general surfaces of the type presented here. Unlike the error estimates of conventional finite volume methods for problems defined on planar domains, the key contribution of this work is to take into account of the errors in the approximate representation of the surface while proving some optimal error estimates for the approximate solutions. By carefully analyzing the finite volume discretization of the differential equation, together with the elegant analysis of the discrete mesh approximation of the surface done in [23], we show that the errors of our finite volume approximation in the discrete H^1 norm and the L^2 norm are of first and second order respectively in the mesh parameter under the smoothness assumptions on the manifold and the solutions to the PDEs, and some mesh regularity assumptions. We note the results are similar to that established for planar problems.

The paper is organized as follows: we first introduce a model equation defined on general surfaces in Section 2. Then in Section 3, we present the finite volume discretization schemes. A short summary of notations used in the paper is given in the beginning of Section 3 to serve as references. In Section 4, the existence of the discrete solution and stability estimates are discussed. The rigorous H^1 and L^2 error estimates are given in Sections 5 and 6 respectively. We discuss the generalization to diffusion–convection problems in Section 7. In Section 8, some numerical experiments are performed to demonstrate the optimal convergence rates. Finally, brief discussions on the surface mesh regularity and concluding remarks are given in Section 9.

2. Model problem and weak solution

For a bounded $C^{k,\alpha}$-hypersurface $\mathbf{S}$ ($k \in \mathbb{N} \cup \{0\}$ and $0 \leqslant \alpha < 1$) in $\mathbb{R}^3$ [31,32] with boundary $\partial\mathbf{S}$, it may be represented globally by some oriented distance function (level set function) $d = d(\mathbf{x})$ defined in some open subset Ω of $\mathbb{R}^3$ such that $\mathbf{S} = \{\mathbf{x} \in \Omega \mid d(\mathbf{x}) = 0\}$ with $d \in C^{k,\alpha}$ and $\nabla d \neq 0$. The unit outward normal to $\mathbf{S}$ (with increasing d) at $\mathbf{x}$ is given by $\vec{\mathbf{n}}(\mathbf{x}) = \nabla d(\mathbf{x})/|\nabla d(\mathbf{x})|$, where $|\cdot|$ denotes the Euclidean norm and ∇ denotes the standard gradient operator in $\mathbb{R}^3$. Without loss of generality, we assume that $|\nabla d| \equiv 1$ in Ω.

Let $\nabla_s = (\nabla_{s,1}, \nabla_{s,2}, \nabla_{s,3}) = \nabla - \vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \nabla)$ and $\Delta_s = \nabla_s \cdot \nabla_s$ be the tangential (surface) gradient operator and the classical Laplace–Beltrami operator on $\mathbf{S}$ respectively [32]. We use the standard notation for Sobolev spaces $L^p(\mathbf{S})$, $W^{m,p}(\mathbf{S})$, and $H^m(\mathbf{S}) = W^{m,2}(\mathbf{S})$ on $\mathbf{S}$. To make the space $H^m(\mathbf{S})$ well defined, it is customary to assume $k + \alpha \geqslant \max\{1, m\}$, see [32]. To avoid technical complexities, we further assume that $\partial\mathbf{S} \neq \emptyset$ and $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth (say, of class C^3 for the rest of the paper unless stated otherwise.

In this paper, we consider the following model equation on $\mathbf{S}$ (diffusion–reaction problem):

$$-\nabla_s \cdot \big(a(\mathbf{x})\nabla_s u(\mathbf{x})\big) + b(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \text{for } \mathbf{x} \in \mathbf{S}, \tag{2.1}$$

with the homogeneous Dirichlet boundary condition,

$$u(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \partial\mathbf{S}. \tag{2.2}$$

While our discussion here can be extended to more general cases such as having $a = a(\mathbf{x})$ being a symmetric positive definite tensor, we focus on the above problems since they have direct applications in areas such as texture synthesis and images inpainting on surfaces [9] and provide simple illustrations of the analytical issues. For simplicity, we assume that the data in (2.1) satisfy:

Assumption 1. *$f \in L^2(\mathbf{S})$, a is uniformly continuous in $\mathbf{S}$, $b \in L^\infty(\mathbf{S})$, $a(\mathbf{x}) \geqslant \alpha_1 > 0$ and $b(\mathbf{x}) \geqslant \alpha_2 \geqslant 0$, $\forall \mathbf{x} \in \mathbf{S}$.*

For any $u, v \in H_0^1(\mathbf{S})$, define the bilinear functional $\mathcal{A}$ by

$$\mathcal{A}(u, v) = \int_{\mathbf{S}} a(\mathbf{x})\big(\nabla_s u(\mathbf{x}) \cdot \nabla_s v(\mathbf{x})\big)\, ds + \int_{\mathbf{S}} b(\mathbf{x})u(\mathbf{x})v(\mathbf{x})\, ds, \tag{2.3}$$

then we have (for some constants $c > 0$ and $\alpha_0 > 0$)

$$\mathcal{A}(u, v) \leqslant c\|u\|_{H^1(\mathbf{S})}\|v\|_{H^1(\mathbf{S})}, \tag{2.4}$$

$$\mathcal{A}(u, u) \geqslant \alpha_0\|u\|^2_{H^1(\mathbf{S})}. \tag{2.5}$$

We say that $u \in H_0^1(\mathbf{S})$ is a weak solution of Eq. (2.1) if and only if

$$\mathcal{A}(u, v) = (f, v), \quad \forall v \in H_0^1(\mathbf{S}) \tag{2.6}$$

where

$$(f, v) = \int_{\mathbf{S}} f(\mathbf{x}) v(\mathbf{x})\, ds.$$

The existence of the weak solution under the given assumptions follows from the standard elliptic theory:

Theorem 1. *Under the Assumption* 1, *for any* $f \in L^2(\mathbf{S})$ *and* $a \in W^{1,\infty}(\mathbf{S})$, *there exists a unique weak solution* $u \in H_0^1(\mathbf{S})$ *of* (2.1). *Moreover,* $u \in H^2(\mathbf{S})$ *and satisfies, for some constant* $C > 0$, *that*

$$\|u\|_{H^2(\mathbf{S})} \leqslant C\|f\|_{L^2(\mathbf{S})}. \tag{2.7}$$

We note that if $\partial\mathbf{S} = \emptyset$, then for any $f \in L^2(\mathbf{S})$, one can also show that, if $\alpha_2 > 0$, there exists a unique weak solution $u \in H^1(\mathbf{S})$ of (2.1).

3. Finite volume discretization

We now present a finite volume discretization of Eq. (2.1). The discrete solution is determined by Eq. (3.4) given later, but first, to make it easier for the readers to follow the discussion, we briefly summarize some glossaries used later. For example, $\mathcal{T} = \{T_i\}_1^n$ and $\mathcal{T}^h = \{T_i^h\}_1^n$ are used to denote the curved triangulation of the surface $\mathbf{S}$ and the planar triangulation of its piecewise polygonal approximation $\mathbf{S}^h$, these triangulations are related to each other by the lift map $\mathcal{L}$ from $\mathbf{S}^h$ to $\mathbf{S}$ as defined in (5.1); $\mathcal{K}$ and $\mathcal{K}^h$ are the corresponding dual tessellations of $\mathbf{S}$ and $\mathbf{S}^h$; $\mathcal{U}$ and $\mathcal{V}$ denote piecewise linear and piecewise constant function spaces defined on the triangulation $\mathcal{K}^h$ of $\mathbf{S}^h$; Π_u and Π_v are interpolation operators into $\mathcal{U}$ and $\mathcal{V}$, while π_u and π_v, defined by (5.3), are the counterparts onto the pair of spaces induced by $\mathcal{U}$ and $\mathcal{V}$ on $\mathbf{S}$ through the lift $\mathcal{L}$; $\mathbf{P}_h$ and $\mathbf{P}$ are projection operators defined by (5.2); $\mathcal{A}$, $\mathcal{A}_G^h$, $\mathcal{A}_*^h$, $\mathcal{A}_G$ and $\mathcal{A}^h$ are bilinear forms defined by (2.3), (3.3), (3.6), (5.10) and (6.1) respectively (the subscript G refers to the use of Green's formula in the definition).

We now present detailed discussions. For the smooth surface $\mathbf{S}$, we may assume that there is a strip (*band*)

$$\mathbf{U} = \{\mathbf{x} \in \Omega \mid \operatorname{dist}(\mathbf{x}, \mathbf{S}) < \delta\}, \quad \text{for some } \delta > 0,$$

around $\mathbf{S}$ such that there is a unique decomposition for any $\mathbf{x} \in \mathbf{U}$,

$$\mathbf{x} = \mathbf{p}(\mathbf{x}) + d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x}),$$

where $\mathbf{p}(\mathbf{x}) \in \mathbf{S}$, $d(\mathbf{x})$ is the signed distance to $\mathbf{S}$, and $\vec{\mathbf{n}}(\mathbf{x})$ denotes the unit outward normal of $\mathbf{S}$ at $\mathbf{p}(\mathbf{x})$. The parameter δ can be determined by the surface curvatures if $\mathbf{S}$ is sufficiently smooth. Then, a function u defined on $\mathbf{S}$ can be extended unambiguously in the strip by

$$U(\mathbf{x}) = u(\mathbf{p}(\mathbf{x})) = u(\mathbf{x} - d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x})), \quad \forall \mathbf{x} \in \mathbf{U}.$$

Let $\mathbf{S}$ be approximated by a sequence of continuous piecewise linear complex $\{\mathbf{S}^h \subset \mathbf{U}\}$, consisting of a sequence of regular triangulations $\{\mathcal{T}^h = \{T_i^h\}_{i=1}^m\}$ with $h \to 0$ being the mesh parameter. Each T^h contains vertices $\{\mathbf{x}_i\}_{i=1}^n$ on $\mathbf{S}$ (i.e., $\{\mathbf{x}_i\}_{i=1}^n \in \mathbf{S} \cap \mathbf{S}^h$), see Fig. 1 (left). Clearly, $\mathbf{S}^h$ is globally of class $C^{0,1}$. In order to avoid global double covering, we further assume that for each point $\mathbf{y} \in \mathbf{S}$ there is at most one point $\mathbf{x} \in \mathbf{S}^h$ such that $\mathbf{p}(\mathbf{x}) = \mathbf{y}$ as suggested in [24]. We use $m(\cdot)$ to denote the area for planar regions or the length for arcs and segments.

In addition, we assume that $\mathcal{T}^h$ satisfies the following mesh regularity condition:

$$c_1 h^2 \leqslant m(T_i^h) \leqslant c_2 h^2, \tag{3.1}$$

where h is the mesh parameter (size) for $\mathcal{T}^h$, c_1 and c_2 are positive constants independent of h. Comments on meshes satisfying such regularity conditions are given at the last section of this paper.

By the uniqueness of the vector decomposition discussed above, we define $T_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in T_i^h\}$ and let $\mathcal{T} = \{T_i\}_{i=1}^m$, then $\mathbf{S} = \bigcup_{i=1}^m T_i$. Note that this implies in particular that $\mathbf{p}(\partial\mathbf{S}^h) = \partial\mathbf{S}$.

Let the tangential gradient operator ∇_{s_h} on $\mathbf{S}^h$ be given by:

$$\nabla_{s_h} = (\nabla_{s_h,1}, \nabla_{s_h,2}, \nabla_{s_h,3}) = \nabla - \vec{\mathbf{n}}_h(\vec{\mathbf{n}}_h \cdot \nabla),$$

where $\vec{\mathbf{n}}_h(\mathbf{x}) = (n_{h1}(\mathbf{x}), n_{h2}(\mathbf{x}), n_{h3}(\mathbf{x}))$ is the unit normal vector to $\mathbf{S}^h$. Since $\vec{\mathbf{n}}_h$ is constant on each triangle T_i^h, ∇_{s_h} only needs to be locally defined as a two dimensional gradient operator on the plane formed by T_i^h, and the Sobolev space $W^{m,p}(\mathbf{S}^h)$ is well-defined for $m \leqslant 1$.

We follow a strategy adopted in [23] to numerically solve the equation on $\mathbf{S}^h$ instead of $\mathbf{S}$. But, we consider a finite volume method [8,38] (also named a finite volume element method, see for instance, [7,26,27,46]), instead of the standard Galerkin finite element methods.

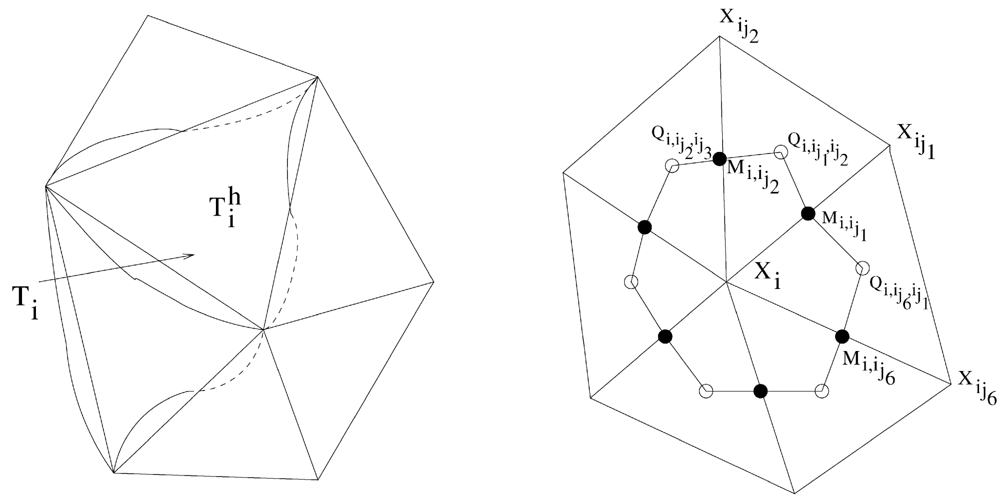

Fig. 1. Approximate mesh surface and the control volume.

We now discuss the discretization scheme. First, we project the coefficients and the data a, b and f in (2.1) from $\mathbf{S}$ onto $\mathbf{S}^h$ such that for any $\mathbf{x} \in \mathbf{S}^h$, $A(\mathbf{x}) = a(\mathbf{p}(\mathbf{x}))$, $B(\mathbf{x}) = b(\mathbf{p}(\mathbf{x}))$, and $F(\mathbf{x}) = f(\mathbf{p}(\mathbf{x}))$. It is easy to verify that $A, B \in W^{1,\infty}(T_i^h)$ for any $T_i^h \in \mathcal{T}^h$, and $F \in L^2(\mathbf{S}^h)$.

Denote by $\mathcal{U}$ the space of continuous piecewise linear polynomials on $\mathbf{S}^h$ with respect to $\mathcal{T}^h$, that is,

$$\mathcal{U} = \{U^h \in C^0(\mathbf{S}^h) \mid U^h|_{\partial \mathbf{S}^h} = 0,\ U^h|_{T_i^h} \in \mathbb{P}_1(T_i^h)\}, \tag{3.2}$$

where $\mathbb{P}_k(D)$ denote the space of polynomials of degree no larger than k on any planar domain D. It is easy to see that $U^h \in H_0^1(\mathbf{S}^h)$ and $\nabla_{s_h} U^h$ is constant on each triangle $T_i^h \in \mathcal{T}^h$.

We now construct the dual tessellation of $\mathcal{T}^h$ on $\mathbf{S}^h$, see Fig. 1 (right). For each vertex $\mathbf{x}_i$, let $\chi_i = \{i_s\}_{s=1}^{m_i}$ be the set of indices of its neighbors, $Q_{i,i_j,i_{j+1}}$ (where $i_{s+1} = i_1$ if $s = m_i$) be the centroid of the triangle $T_{i_j}^h = \Delta \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}}$ and M_{i,i_j} be the midpoint of $\overline{\mathbf{x}_i \mathbf{x}_{i_j}}$ for $i_j \in \chi_i$. Let $K_i^h = \bigcup_{i_j \in \chi_i} \Omega_{i,i_j,i_{j+1}}$ where $\Omega_{i,i_j,i_{j+1}}$ denotes the polygonal region bounded by $\mathbf{x}_i$, M_{i,i_j}, $Q_{i,i_j,i_{j+1}}$ and $M_{i,i_{j+1}}$. K_i^h is in general only piecewise planar and we define its projection on $\mathbf{S}$ by $K_i = \{\mathbf{p}(\mathbf{x}) \in \mathbf{S} \mid \mathbf{x} \in K_i^h\}$. In the remaining part of this paper, for simplicity, we let i_j mean $i_{(j-1) \bmod (m_i)+1}$ if $j > m_i$ when $i_j \in \chi_i$ ($\mathbf{x}_{i_j}$ is a neighbor vertex of $\mathbf{x}_i$), otherwise i_j means $i_{(j-1) \bmod (3)+1}$ if $j > 3$ when $\mathbf{x}_{i_j}$ is a vertex of $T_i^h = \Delta \mathbf{x}_{i_1} \mathbf{x}_{i_2} \mathbf{x}_{i_3}$.

Now, denote by σ the set of indices of the interior vertices of $\mathcal{T}^h$, then, $\mathcal{K} = \{K_i\}_{i \in \sigma}$ and $\mathcal{K}^h = \{K_i^h\}_{i \in \sigma}$ may be viewed as dual tessellations of $\mathbf{S} = \bigcup_{i=1}^m T_i$ and $\mathbf{S}^h = \bigcup_{i=1}^m T_i^h$. Denote by $\mathcal{V}$ the space of grid functions on S^h with respect to $\mathcal{K}^h$:

$$\mathcal{V} = \{V^h \mid V^h|_{\partial \mathbf{S}^h} = 0,\ V^h|_{K_i^h} \in \mathbb{P}_0(K_i^h)\}.$$

A set of basis functions $\{\Psi_i^h\}_{i \in \sigma}$ of $\mathcal{V}$ is given by

$$\Psi_i^h(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in K_i^h, \\ 0, & \mathbf{x} \in \mathbf{S}^h - K_i^h. \end{cases}$$

For $V^h \in \mathcal{V}$ and $U \in H^1(S^h)$ with $U|_{T_i^h} \in H^2(T_i^h)$ for any $T_i^h \in \mathcal{T}^h$, define the bilinear functionals $\mathcal{A}_G^h$ such that

$$\mathcal{A}_G^h(U, V^h) = \sum_{i \in \sigma} V_i^h \mathcal{A}_G^h(U, \Psi_i^h), \tag{3.3}$$

where $V_i^h = V^h(\mathbf{x}_i)$ and

$$\begin{aligned} \mathcal{A}_G^h(U, \Psi_i^h) &= -\int_{\partial K_i^h} A(\mathbf{x}) \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \int_{K_i^h} B(\mathbf{x}) U(\mathbf{x})\, ds_h \\ &= -\sum_{i_j \in \chi_i} \int_{\Gamma_{i,i_j,i_{j+1}}} A(\mathbf{x}) \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \int_{K_i^h} B(\mathbf{x}) U(\mathbf{x})\, ds_h \end{aligned}$$

with $\Gamma_{i,i_j,i_{j+1}} = \partial K_i^h \cap \Delta \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}} = \overline{M_{i,i_j} Q_{i,i_j,i_{j+1}} M_{i,i_{j+1}}}$ and $\vec{\mathbf{n}}_{K_i^h}$ denoting the outward unit normal of ∂K_i^h.

For any $V^h \in \mathcal{V}$, define

$$\left(F, V^h\right)_{s_h} = \int_{\mathbf{S}^h} F(\mathbf{x}) V^h(\mathbf{x})\, ds_h.$$

Then the *discrete finite volume method* is given by: find $U^h \in \mathcal{U}$ such that

$$\mathcal{A}_G^h\left(U^h, V^h\right) = \left(F, V^h\right)_{s_h}, \quad \forall V^h \in \mathcal{V}. \tag{3.4}$$

Notice that the linear system resulting from (3.4) may not be symmetric.

3.1. A mass-lumped scheme

In practical implementation, noticing that U^h is piecewise linear on $\mathbf{S}^h$ with respect to $\mathcal{T}^h$, $\nabla_{s_h} U^h$ is constant on each triangle $T_{i_j}^h = \Delta \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}}$, and defining

$$B_i = \frac{1}{m(K_i^h)} \int_{K_i^h} B(\mathbf{x})\, ds_h, \qquad F_i = \frac{1}{m(K_i^h)} \int_{K_i^h} F(\mathbf{x})\, ds_h$$

as averages over K_i^h, we may use the approximations:

$$\left(F, V^h\right)_{s_h} \approx \sum_{i \in \sigma} m\left(K_i^h\right) V_i^h F_i, \tag{3.5}$$

$$\mathcal{A}_*^h\left(U^h, V^h\right) \approx \sum_{i \in \sigma} V_i^h \mathcal{A}_*^h\left(U^h, \Psi_i^h\right), \tag{3.6}$$

where

$$\begin{aligned} \mathcal{A}_*^h\left(U^h, \Psi_i^h\right) &= -\sum_{i_j \in \chi_i} A_{i,i_j,i_{j+1}} \left[q_{i,i_j,i_{j+1}}^1 \left(U_{i_j}^h - U_i^h\right) + q_{i,i_j,i_{j+1}}^2 \left(U_{i_{j+1}}^h - U_i^h\right)\right] + m\left(K_i^h\right) B_i U_i^h \\ &= -\sum_{i_j \in \chi_i} p_{i,i_j} \left(U_{i_j}^h - U_i^h\right) + m\left(K_i^h\right) B_i U_i^h \end{aligned} \tag{3.7}$$

with

$$U_i^h = U^h(\mathbf{x}_i), \qquad A_{i,i_j,i_{j+1}} = A(Q_{i,i_j,i_{j+1}}), \qquad p_{i,i_j} = A_{i,i_j,i_{j+1}} q_{i,i_j,i_{j+1}}^1 + A_{i,i_{j-1},i_j} q_{i,i_{j-1},i_j}^2,$$

$$q_{i,i_j,i_{j+1}}^k = \frac{1}{8m(\Delta \mathbf{x}_i \mathbf{x}_{i_j} \mathbf{x}_{i_{j+1}})} \left((-1)^{k-1} \|\mathbf{x}_{i_{j+1}} - \mathbf{x}_i\|^2 + (-1)^k \|\mathbf{x}_{i_j} - \mathbf{x}_i\|^2 + \|\mathbf{x}_{i_j} - \mathbf{x}_{i_{j+1}}\|^2\right), \quad k = 1, 2.$$

With the above numerical integration, we may transform (3.4) to the following problem in the practical implementation: find $U^h \in \mathcal{U}$ such that

$$\mathcal{A}_*^h\left(U^h, V^h\right) = \left(F, V^h\right)_{s_h}, \quad \forall V^h \in \mathcal{V}. \tag{3.8}$$

Rewriting (3.8) in a form of a discrete system, we then get the following system of linear equations:

$$-\sum_{i_j \in \chi_i} p_{i,i_j} \left(U_{i_j}^h - U_i^h\right) + m\left(K_i^h\right) B_i U_i^h = m\left(K_i^h\right) F_i, \quad \text{for } i \in \sigma. \tag{3.9}$$

This corresponding coefficient matrix is a symmetric, positive definite M-matrix.

Remark 1. It is clear that the above system (3.9) satisfies the discrete conservation law since

$$\sum_{i \in \sigma} \sum_{i_j \in \chi_i} p_{i,i_j} \left(U_{i_j}^h - U_i^h\right) = 0. \tag{3.10}$$

Moreover, by the properties of the M-matrix, we see that the solution of the system (3.9) also satisfies the maximum principle, and in particular, if $F_i \leqslant 0$, we have $U_i^h \leqslant 0$ for all i.

Remark 2. Although a global triangulation for $\mathbf{S}$ is provided for the description of the algorithm, we note that the finite volume discretization may be constructed locally using the geometry of a locally defined triangular mesh and the corresponding dual cells as seen from Eq. (3.9).

In this paper, we only analyze the error of the finite volume approximation (3.4). The bilinear form $\mathcal{A}_*^h$ given above turns out to be useful in the derivation of the coercivity of $\mathcal{A}_G^h$. The analysis can be generalized to (3.9) but more stringent regularity assumptions on the data and the exact solution would be required.

4. Existence and stability estimates

The analysis below resembles closely the similar framework used in [23,37,38] and also [8,16]. For given functions $U^h, V^h \in \mathcal{U}$, we define, similar to [16,29,37,38], the following discrete inner products and norms associated with $\mathcal{T}^h$ and a particular triangle $T_i^h = \Delta \mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3}$:

$$\begin{cases} (U^h, V^h)_{0,T_i^h} = \frac{1}{3} m(T_i^h)\left(\sum_{j=1}^{3} U^h(\mathbf{x}_{i_j}) V^h(\mathbf{x}_{i_j})\right), \\ \|U^h\|^2_{0,T_i^h} = (U^h, U^h)_{0,T_i^h}, \qquad |U^h|^2_{1,T_i^h} = m(T_i^h)\left|\nabla_{s_h} U^h|_{T_i^h}\right|^2 \end{cases}$$

and $\|U^h\|^2_{0,\mathcal{T}^h} = (U^h, U^h)_{0,\mathcal{T}^h}$, $\|U^h\|^2_{1,\mathcal{T}^h} = \|U^h\|^2_{0,\mathcal{T}^h} + |U^h|^2_{1,\mathcal{T}^h}$ where

$$\begin{cases} (U^h, V^h)_{0,\mathcal{T}^h} = \sum_{T_i^h \in \mathcal{T}^h} (U^h, V^h)_{0,T_i^h}, \\ |U^h|^2_{1,\mathcal{T}^h} = \sum_{T_i^h \in \mathcal{T}^h} |U^h|^2_{1,T_i^h}. \end{cases}$$

As the norms are defined locally with piecewise planar triangles, the following technical lemma is a trivial generalization of the same result given in [38] and the Poincare inequality.

Lemma 1. *There exist some constants $c_1, c_2 > 0$ such that for any $U^h \in \mathcal{U}$,*

$$\begin{aligned} &c_1\|U^h\|_{0,\mathcal{T}^h} \leqslant \|U^h\|_{L^2(\mathbf{S}^h)} \leqslant c_2\|U^h\|_{0,\mathcal{T}^h}, \\ &c_1|U^h|_{1,\mathcal{T}^h} \leqslant \|U^h\|_{H^1(\mathbf{S}^h)} \leqslant c_2|U^h|_{1,\mathcal{T}^h}. \end{aligned} \tag{4.1}$$

Similarly, for any $U \in C^0(\mathbf{S}^h)$, denote by $\Pi_u(U)$ the standard interpolant of U onto $\mathcal{U}$ and by $\Pi_v(U)$ the standard interpolant onto $\mathcal{V}$ with respect to $\mathbf{S}^h$, then we have

Lemma 2. *For $T_i^h = \Delta \mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3} \in \mathcal{T}^h$, there exists some $c_1 > 0$ such that*

$$\left\|U - \Pi_u(U)\right\|_{L^2(T_i^h)} + h\left\|U - \Pi_u(U)\right\|_{H^1(T_i^h)} \leqslant c_1 h^2 \|U\|_{H^2(T_i^h)}, \tag{4.2}$$

for any $U \in H^2(T_i^h)$. Moreover, for any $U \in W^{1,p}(T_i^h)$ with $p > 2$, we have

$$\left\|U - \Pi_v(U)\right\|_{L^2(T_i^h)} \leqslant c_2 h \|U\|_{W^{1,p}(T_i^h)}, \tag{4.3}$$

for some $c_2 > 0$. In addition, for any $U^h \in \mathcal{U}$, we have

$$\int_{T_i^h} \left(\Pi_v(U^h) - U^h\right) ds_h = 0, \tag{4.4}$$

and for any edge $\mathbf{x}_{i_j}\mathbf{x}_{i_k}$ of T_i^h, we also have

$$\int_{\mathbf{x}_{i_j}\mathbf{x}_{i_k}} \left(\Pi_v(U^h) - U^h\right) d\gamma_h = 0. \tag{4.5}$$

The estimates in (4.2) and (4.3) are classical approximation results. The $W^{1,p}$ regularity for $p > 2$ is to insure the validity of pointwise interpolation. Eq. (4.4) follows from the equi-area property $m(K_{i_j}^h \cap T_i^h) = m(T_i^h)/3$ for $j = 1, 2, 3$, which is a consequence of Q_{i_1,i_2,i_3} being the barycenter of T_i^h. Eq. (4.5) follows from the symmetry property. Since Q_i is the centroid of T_i^h, it is easy to find that for any $U^h \in \mathcal{U}$,

$$\begin{aligned} \left\|\Pi_v(U^h)\right\|_{L^2(\mathbf{S}^h)} &= \left(\sum_{T_i^h \in \mathcal{T}^h} \sum_{j=1}^{3} (U_{i_j}^h)^2 m(K_{i_j}^h \cap T_i^h)\right)^{1/2} \\ &= \left(\sum_{T_i^h \in \mathcal{T}^h} \frac{1}{3}\left(\sum_{j=1}^{3} (U_{i_j}^h)^2 m(T_i^h)\right)\right)^{1/2} = \|U^h\|_{0,\mathcal{T}^h}, \end{aligned} \tag{4.6}$$

which gives us the norm equivalence. Note that if the pointwise interpolation is replaced by Clement type interpolation [10], then standard H^1 regularity is enough, but Eqs. (4.4) and (4.5) may not hold simultaneously.

We now derive the coercivity of the operator $\mathcal{A}_G^h$.

Proposition 1. *Assume that a is uniformly continuous in* **S**. *There exists a constant $c > 0$ such that when h is sufficiently small,*

$$\mathcal{A}_G^h\big(U^h, \Pi_v\big(U^h\big)\big) \geqslant c\big\|U^h\big\|^2_{H^1(\mathbf{S}^h)} \tag{4.7}$$

for any $U^h \in \mathcal{U}$.

Proof. First we have

$$\mathcal{A}_G^h\big(U^h, \Pi_v\big(U^h\big)\big) = \big[\mathcal{A}_G^h\big(U^h, \Pi_v\big(U^h\big)\big) - \mathcal{A}_*^h\big(U^h, \Pi_v\big(U^h\big)\big)\big] + \mathcal{A}_*^h\big(U^h, \Pi_v\big(U^h\big)\big). \tag{4.8}$$

From (3.6), we get

$$\begin{aligned}
\mathcal{A}_*^h\big(U^h, \Pi_v\big(U^h\big)\big) &= \sum_{i\in\sigma} U_i^h \mathcal{A}_*^h\big(U^h, \Psi_i^h\big) \\
&= \sum_{i\in\sigma}\Bigg(-\sum_{i_j\in\chi_i} A_{i,i_j,i_{j+1}} U_i^h \int\limits_{\Gamma_{i,i_j,i_{j+1}}} \nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + m\big(K_i^h\big) B_i \big(U_i^h\big)^2\Bigg) \\
&\geqslant -\sum_{i\in\sigma}\sum_{i_j\in\chi_i} A(Q_{i,i_j,i_{j+1}}) U_i^h \int\limits_{\Gamma_{i,i_j,i_{j+1}}} \nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h \\
&= \sum_{T_i^h\in\mathcal{T}^h} A(Q_i)\Bigg(-\sum_{j=1}^3 U_{i_j}^h \int\limits_{\partial K_{i_j}^h\cap T_i^h} \nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_{i_j}^h}\, d\gamma_h\Bigg),
\end{aligned}$$

where $Q_i = Q_{i_1,i_2,i_3}$ be the centroid of $T_i^h = \Delta \mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3} \in \mathcal{T}^h$.

Note that each T_i^h can be regarded as a triangle in the xy-plane with some suitable affine mapping and ∇_{s_h} as the standard two-dimensional gradient operator, then using the result from [38, Theorem 3.2.1, p. 126], we immediately have that

$$-\sum_{j=1}^3 U_{i_j}^h \int\limits_{\partial K_{i_j}^h\cap T_i^h} \nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_{i_j}^h}\, d\gamma_h = m\big(T_i^h\big)\big|\nabla_{s_h} U^h\big|_{T_i^h}\big|^2.$$

By Lemma 1, we then have

$$\mathcal{A}_*^h\big(U^h, \Pi_v\big(U^h\big)\big) \geqslant \sum_{T_i^h\in\mathcal{T}^h} A(Q_i) m\big(T_i^h\big)\big|\nabla_{s_h} U^h\big|_{T_i^h}\big|^2 \geqslant c\big|U^h\big|^2_{1,\mathcal{T}^h} \geqslant c\big\|U^h\big\|^2_{H^1(\mathbf{S}^h)}. \tag{4.9}$$

On the other hand, we have

$$\big|\mathcal{A}_G^h\big(U^h, \Pi_v\big(U^h\big)\big) - \mathcal{A}_*^h\big(U^h, \Pi_v\big(U^h\big)\big)\big| \leqslant I_1 + I_2$$

where

$$\begin{aligned}
I_1 &= \Bigg|-\sum_{T_i^h\in\mathcal{T}^h}\sum_{j=1}^3 U_{i_j}^h \int\limits_{\partial K_{i_j}^h\cap T_i^h} \big[A(\mathbf{x}) - A(Q_i)\big]\nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_{i_j}^h}\, d\gamma_h\Bigg|, \\
I_2 &= \Bigg|\sum_{i\in\sigma}\int\limits_{k_i^h} B(\mathbf{x})\big(U^h(\mathbf{x}) - U_i^h\big) U_i^h\, ds_h\Bigg|.
\end{aligned}$$

Rearranging I_1, we get

$$I_1 = \Bigg|\sum_{T_i^h\in\mathcal{T}^h}\Bigg(\sum_{j=1}^3 \big(U_{i_{j+2}}^h - U_{i_{j+1}}^h\big) \int\limits_{M_{i_{j+1},i_{j+2}}Q_i} \big[A(\mathbf{x}) - A(Q_i)\big]\nabla_{s_h} U^h(\mathbf{x})\cdot \vec{\mathbf{n}}_{K_{i_{j+1}}^h}\, d\gamma_h\Bigg)\Bigg|.$$

Since in each triangle T_i^h, we have

$$\begin{aligned}
&\left|U_{i_{j+2}}^h - U_{i_{j+1}}^h\right| \leqslant ch\left|\nabla_{s_h} U^h|_{T_i^h}\right|,\\
&\left|A(\mathbf{x}) - A(Q_i)\right| \leqslant \epsilon,\\
&\left|\nabla_{s_h} U^h(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_{i_{j+1}}^h}\right| \leqslant \left|\nabla_{s_h} U^h|_{T_i^h}\right|,\\
&m(M_{i_{j+1},i_{j+2}} Q_i) \leqslant ch,
\end{aligned}$$

where ϵ is a constant that can be arbitrarily small for sufficiently small h, as implied by the uniform continuity of a.

With the mesh regularity condition (3.1) and Lemma 1, we get

$$\begin{aligned}
I_1 &\leqslant \sum_{T_i^h \in \mathcal{T}^h} c\epsilon h^2 \left|\nabla_{s_h} U^h|_{T_i^h}\right|^2\\
&\leqslant c\epsilon \sum_{T_i^h \in \mathcal{T}^h} \left|\nabla_{s_h} U^h|_{T_i^h}\right|^2 m(T_i^h)\\
&= c\epsilon \left\|U^h\right\|_{1,\mathcal{T}^h}^2 \leqslant c\epsilon \left\|U^h\right\|_{H^1(\mathbf{S}^h)}^2.
\end{aligned} \tag{4.10}$$

As for I_2, with Lemma 2 and (4.6), we have

$$\begin{aligned}
I_2 &\leqslant c\|B\|_{L^\infty(\mathbf{S}^h)} \left\|U^h - \Pi_v(U^h)\right\|_{L^2(\mathbf{S}^h)} \left\|\Pi_v(U^h)\right\|_{L^2(\mathbf{S}^h)}\\
&\leqslant ch\left\|U^h\right\|_{H^1(\mathbf{S}^h)} \left\|\Pi_v(U^h)\right\|_{L^2(\mathbf{S}^h)}\\
&\leqslant ch\left\|U^h\right\|_{L^2(\mathbf{S}^h)} \left\|U^h\right\|_{H^1(\mathbf{S}^h)}.
\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we know

$$\left|\mathcal{A}_G^h\big(U^h, \Pi_v(U^h)\big) - \mathcal{A}_*^h\big(U^h, \Pi_v(U^h)\big)\right| \leqslant c(\epsilon + h)\left\|U^h\right\|_{H^1(\mathbf{S}^h)}^2. \tag{4.12}$$

Using (4.8), (4.9), (4.12), and the Poincare inequality in $H_0^1(\mathbf{S}^h)$, we finally obtain (4.7) when h is sufficiently small. □

It is also easy to see

$$\begin{aligned}
\left|\big(F, \Pi_v(U^h)\big)_{S^h}\right| &\leqslant \|F\|_{L^2(\mathbf{S}^h)} \left\|\Pi_v(U^h)\right\|_{L^2(\mathbf{S}^h)}\\
&= c\|F\|_{L^2(\mathbf{S}^h)} \left\|U^h\right\|_{0,\mathcal{T}^h} \leqslant c\|F\|_{L^2(\mathbf{S}^h)} \left\|U^h\right\|_{H^1(\mathbf{S}^h)}.
\end{aligned} \tag{4.13}$$

By Proposition 1 and (4.13), we have the following results:

Theorem 2. *The discrete problem* (3.4) *has a unique solution $U^h \in \mathcal{U}$ when h is sufficiently small.*

Remark 3. If a is a constant function or piecewise constant function with respect to $\mathcal{T}^h$, then both conditions that h must be sufficiently small and the mesh is quasi-uniform can be removed.

5. H^1 error estimate

When h is small enough, it is easy to find that $|d(\mathbf{x})| \leqslant ch^2$ for any $\mathbf{x} \in \mathbf{S}^h$ (see [23]). To compare the discrete solution on $\mathbf{S}^h$ with the continuous solution on $\mathbf{S}$, we lift a function U defined on $\mathbf{S}^h$ to $\mathbf{S}$ by

$$\mathcal{L}: U \to u = \mathcal{L}(U) \quad \text{where } u(\mathbf{y}) = U\big(\mathbf{p}^{-1}(\mathbf{y})\big),\ \forall \mathbf{y} \in \mathbf{S}, \tag{5.1}$$

that is, $U(\mathbf{x}) = u(\mathbf{p}(\mathbf{x})) = u(\mathbf{x} - d(\mathbf{x})\vec{\mathbf{n}}(\mathbf{x}))$ for $\mathbf{x} \in \mathbf{S}^h$. Let $\mathbf{y} = \mathbf{p}(\mathbf{x})$ and

$$\mu_h(\mathbf{x}) = \frac{ds(\mathbf{x})}{ds_h(\mathbf{p}(\mathbf{x}))}, \qquad \xi_h(\mathbf{x}) = \frac{d\gamma(\mathbf{x})}{d\gamma_h(\mathbf{p}(\mathbf{x}))}.$$

Since $\mathbf{S}$ and $\partial\mathbf{S}$ are sufficiently smooth, we have

$$\left|1 - \mu_h(\mathbf{x})\right| \leqslant ch^2, \qquad \left|1 - \xi_h(\mathbf{x})\right| \leqslant ch^2, \qquad \left|\vec{\mathbf{n}}(\mathbf{y}) - \vec{\mathbf{n}}_h(\mathbf{x})\right| < ch.$$

For the relations between ∇_s and ∇_{s_h}, we have

$$\begin{aligned}
&\nabla_{s_h} U(\mathbf{x}) = \mathbf{P}_h \nabla U(\mathbf{x}), \qquad \nabla_s u(\mathbf{y}) = \mathbf{P}\nabla u(\mathbf{y}),\\
&\nabla U(\mathbf{x}) = (\mathbf{P} - d\mathbf{H})\nabla u(\mathbf{y})
\end{aligned}$$

where

$$\mathbf{P}_h = (\delta_{i,j} - n_{h_i i} n_{h_j}), \qquad \mathbf{P} = (\delta_{i,j} - n_i n_j),$$
$$\mathbf{H} = (d_{x_i,x_j}) = \big((n_i)_{x_j}\big) = \big((n_j)_{x_i}\big). \tag{5.2}$$

Since $\mathbf{P}$ is in fact a projection, we can easily find that

$$\mathbf{PP} = \mathbf{P}, \qquad \mathbf{PH} = \mathbf{HP} = \mathbf{H},$$

and consequently,

$$\nabla_{s_h} U(\mathbf{x}) = \mathbf{P}_h(\mathbf{I} - d\mathbf{H})\nabla_s u(\mathbf{y}).$$

Lemma 3. *For any $q \geqslant 2$, there exists some constants $c_1, c_2, c_3, c_4, c > 0$ such that*

$$\begin{cases} c_1\|U\|_{L^q(T_i^h)} \leqslant \|u\|_{L^q(T_i)} \leqslant c_2\|U\|_{L^q(T_i^h)}, \\ c_3\|U\|_{W^{1,q}(T_i^h)} \leqslant \|u\|_{W^{1,q}(T_i)} \leqslant c_4\|U\|_{W^{1,q}(T_i^h)}, \\ |U|_{H^2(T_i^h)} \leqslant c\big[|u|_{H^2(T_i)} + h|u|_{H^1(T_i)}\big], \\ \|U\|_{H^3(T_i^h)} \leqslant c\|u\|_{H^3(T_i)} \end{cases}$$

for any $T_i^h \in \mathcal{T}^h$.

Proof. The first three inequalities are proved in [23]. The last inequality is a consequence of the first three. □

For any $u \in C^0(\mathbf{S})$, we define the interpolants $\pi_u(u)$ and $\pi_v(u)$ by

$$\pi_u(u) = \mathcal{L}\big(\Pi_u\big(\mathcal{L}^{-1}(u)\big)\big), \qquad \pi_v(u) = \mathcal{L}\big(\Pi_v\big(\mathcal{L}^{-1}(u)\big)\big). \tag{5.3}$$

Then we have the following results (see [23]):

Lemma 4. *There exist some $c_1, c_2 > 0$ such that*

$$\big\|u - \pi_u(u)\big\|_{L^2(\mathbf{S})} + h\big\|u - \pi_u(u)\big\|_{H^1(\mathbf{S})} \leqslant c_1 h^2 \|u\|_{H^2(\mathbf{S})}, \tag{5.4}$$

for $u \in H^2(\mathbf{S})$, and

$$\big\|u - \pi_v(u)\big\|_{L^2(\mathbf{S})} \leqslant c_2 h \|u\|_{W^{1,p}(\mathbf{S})}, \tag{5.5}$$

for $u \in W^{1,p}(\mathbf{S})$ with $p > 2$.

Furthermore, it is straightforward to establish:

Lemma 5. *There exist some $\{c_i > 0\}_{i=1}^4$ such that for any $W \in W^{1,p}(\mathbf{S}^h)$ with $p > 2$,*

$$\big\|\Pi_u(W) - W\big\|_{L^2(\mathbf{S}^h)} \leqslant c_1 h \|W\|_{W^{1,p}(\mathbf{S}^h)}, \tag{5.6}$$
$$\big\|\Pi_u(W)\big\|_{H^1(\mathbf{S}^h)} \leqslant c_2 \|W\|_{W^{1,p}(\mathbf{S}^h)}, \tag{5.7}$$

and for any $W^h \in \mathcal{U}$,

$$\big\|\Pi_v\big(W^h\big) - W^h\big\|_{L^2(\mathbf{S}^h)} \leqslant c_3 h \|W\|_{H^1(\mathbf{S}^h)}, \tag{5.8}$$
$$\big\|\Pi_v\big(W^h\big)\big\|_{L^2(\mathbf{S}^h)} \leqslant c_4 \|W\|_{L^2(\mathbf{S}^h)}. \tag{5.9}$$

For any $U^h \in \mathcal{U}$ and $V^h \in \mathcal{V}$, we lift them onto $\mathbf{S}$ by $u^h = \mathcal{L}(U^h)$ and $v^h = \mathcal{L}(V^h)$, and let

$$\psi_i^h(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in K_i, \\ 0, & \mathbf{x} \in \mathbf{S} - K_i. \end{cases}$$

Let $\vec{\mathbf{n}}_{K_i}$ denote the outward normal of ∂K_i. For $v^h \in \mathcal{L}(\mathcal{V})$ and $u \in H^1(S)$ with $u|_{T_i} \in H^2(T_i)$ for any $T_i \in \mathcal{T}$, we then define the bilinear functional $\mathcal{A}_G$ such as

$$\mathcal{A}_G\big(u, v^h\big) = \sum_{i \in \sigma} v_i^h \mathcal{A}_G\big(u, \psi_i^h\big), \tag{5.10}$$

where $v_i^h = v^h(\mathbf{x}_i)$ and

$$\mathcal{A}_G\big(u, \psi_i^h\big) = -\int_{\partial K_i} a(\mathbf{x})\nabla_s u(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}\, d\gamma + \int_{K_i} b(\mathbf{x})u(\mathbf{x})\, ds.$$

To avoid excessively long formulae, we assume $a(\mathbf{x}) \equiv 1$, so that $A(\mathbf{x}) \equiv 1$ in the remaining parts of this paper. In addition, we assume that $b \in W^{1,\infty}(\mathbf{S})$. We note that the results hold in fact for more general coefficients. The next two lemmas describe the consistency of bilinear forms with Lemma 6 measuring the difference between a function defined on $\mathbf{S}$ and its interpolant on $\mathbf{S}^h$ and Lemma 7 measuring the difference between the bilinear forms on the surface $\mathbf{S}$ and $\mathbf{S}^h$ respectively. While the derivation of the former result is slightly more involved than the planar case as a lifting is needed for defining suitable interpolants, the need for deriving the latter is completely due to the extra surface approximation which can be avoided in the planar case.

Lemma 6. *For any $u \in H^2(\mathbf{S})$ and $W^h \in \mathcal{U}$, there exists a constant $c > 0$ such that*

$$\big|\mathcal{A}_G^h\big(U, \Pi_v\big(W^h\big)\big) - \mathcal{A}_G^h\big(\Pi_u(U), \Pi_v\big(W^h\big)\big)\big| \leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}, \tag{5.11}$$

where $U = \mathcal{L}^{-1}(u)$.

Proof. It is easy to see that $U \in H^2(T_i^h)$ for each $T_i^h \in \mathcal{T}^h$ and $w^h \in H^1(\mathbf{S})$. We know

$$\mathcal{A}_G^h\big(U, \Pi_v\big(W^h\big)\big) - \mathcal{A}_G^h\big(\Pi_u(U), \Pi_v\big(W^h\big)\big) = I_1 + I_2,$$

where

$$I_1 = \sum_{i\in\sigma} -W^h(\mathbf{x}_i)\int_{\partial K_i^h} \nabla_{s_h}\big(U - \Pi_u(U)\big)\cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h,$$

$$I_2 = \sum_{i\in\sigma} W^h(\mathbf{x}_i)\int_{K_i^h} B\big(U - \Pi_u(U)\big)\, ds_h.$$

Let $W_i^h = W^h(\mathbf{x}_i)$ and $T_i^h = \Delta \mathbf{x}_{i_1}\mathbf{x}_{i_2}\mathbf{x}_{i_3}$, then we get

$$\begin{aligned} I_1 &= \sum_{T_i^h\in\mathcal{T}^h}\left(-\sum_{j=1}^{3} W_{i_j}^h \int_{\partial K_{i_j}^h\cap T_i^h} \nabla_{s_h}\big(U - \Pi_u(U)\big)\cdot\vec{\mathbf{n}}_{K_{i_j}^h}\, d\gamma_h\right) \\ &= \sum_{T_i^h\in\mathcal{T}^h}\left(\sum_{j=1}^{3}\big(W_{i_{j+2}}^h - W_{i_{j+1}}^h\big)\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \nabla_{s_h}\big(U - \Pi_u(U)\big)\cdot\vec{\mathbf{n}}_{K_{i_{j+1}}^h}\, d\gamma_h\right). \end{aligned}$$

In each triangle T_i^h, we have

$$\big|W_{i_{j+2}}^h - W_{i_{j+1}}^h\big| \leqslant h\big|\nabla_{s_h}W^h|_{T_j^h}\big| \leqslant c\big\|W^h\big\|_{1,T_i^h}.$$

Using the trace theorem and Lemma 2, we get

$$\begin{aligned} \left|\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \nabla_{s_h}\big(U - \Pi_u(U)\big)\cdot\vec{\mathbf{n}}_{K_{i_{j+1}}^h}\, d\gamma_h\right| &\leqslant ch^{1/2}\left(\int_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \big|\nabla_{s_h}\big(U - \Pi_u(U)\big)\big|^2\, d\gamma_h\right)^{1/2} \\ &\leqslant ch\big(h^{-1}\big|\nabla_{s_h}\big(U - \Pi_u(U)\big)\big|_{L^2(T_i^h)} + \big|\nabla_{s_h}\big(U - \Pi_u(U)\big)\big|_{H^1(T_i^h)}\big) \\ &\leqslant ch\|U\|_{H^2(T_i^h)}. \end{aligned}$$

By Lemmas 1 and 3, we then obtain

$$\begin{aligned} |I_1| &\leqslant \sum_{T_i^h\in\mathcal{T}^h} ch\|U\|_{H^2(T_i^h)}\big\|W^h\big\|_{1,T_i^h} \\ &\leqslant ch\sum_{T_i^h\in\mathcal{T}^h}\|u\|_{H^2(T_i)}\big\|W^h\big\|_{H^1(T_i^h)} \\ &\leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \end{aligned} \tag{5.12}$$

Also by Lemmas 2 and 3, we achieve

$$\begin{aligned}
|I_2| &= \left| \sum_{i\in\sigma} \int_{K_i^h} B\Pi_v\left(W^h\right)\left(U - \Pi_u(U)\right) ds_h \right| \\
&\leqslant \|B\|_{L^\infty(\mathbf{S}^h)} \int_{\mathbf{S}^h} \left|\Pi_v\left(W^h\right)\right| \cdot \left|U - \Pi_u(U)\right| ds_h \\
&\leqslant c\left\|\Pi_u\left(W^h\right)\right\|_{L^2(\mathbf{S}^h)} \left\|U - \Pi_u(U)\right\|_{L^2(\mathbf{S}^h)} \\
&\leqslant ch\|U\|_{W^{1,p}(\mathbf{S})} \left\|W^h\right\|_{L^2(\mathbf{S}^h)} \\
&\leqslant ch\|u\|_{W^{1,p}(\mathbf{S})} \left\|W^h\right\|_{L^2(\mathbf{S}^h)}.
\end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13), we get (5.11) and complete the proof. □

Lemma 7. *For any $u \in H^2(\mathbf{S})$ and $W^h \in \mathcal{U}$, there exists a constant $c > 0$ such that*

$$\left|\mathcal{A}_G\left(u, \pi_v\left(w^h\right)\right) - \mathcal{A}_G^h\left(U, \Pi_v\left(W^h\right)\right)\right| \leqslant ch\|u\|_{H^2(\mathbf{S})} \left\|W^h\right\|_{H^1(\mathbf{S}^h)}, \tag{5.14}$$

where $U = \mathcal{L}^{-1}(u)$ and $w^h = \mathcal{L}(W^h)$.

Proof. We know

$$\mathcal{A}_G\left(u, \pi_v\left(w^h\right)\right) - \mathcal{A}_G^h\left(U, \Pi_v\left(W^h\right)\right) = I_1 + I_2 + I_3$$

where

$$I_1 = \sum_{i\in\sigma} -W_i^h\left(\int_{\partial K_i} \nabla_s u(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}(\mathbf{x})\, d\gamma - \int_{\partial K_i^h} \nabla_{s_h} U(\mathbf{x}) \cdot \vec{\mathbf{n}}_{K_i}\left(\mathbf{p}(\mathbf{x})\right) d\gamma_h\right),$$

$$I_2 = \sum_{i\in\sigma} -W_i^h\left(\int_{\partial K_i^h} \nabla_{s_h} U(\mathbf{x}) \cdot \left[\vec{\mathbf{n}}_{K_i}\left(\mathbf{p}(\mathbf{x})\right) - \vec{\mathbf{n}}_{K_i^h}(\mathbf{x})\right] d\gamma_h\right),$$

$$I_3 = \sum_{i\in\sigma} W_i^h\left(\int_{K_i} bu\, ds - \int_{K_i^h} BU\, ds_h\right)$$

with $W_i^h = W^h(\mathbf{x}_i)$.

As for I_1, we have

$$\begin{aligned}
I_1 &= \sum_{i\in\sigma} -W_i^h \int_{\partial K_i^h} \left[\xi_h \nabla_s u\left(\mathbf{p}(\mathbf{x})\right) - \nabla_{s_h} U(\mathbf{x})\right] \cdot \vec{\mathbf{n}}_{K_i}\left(\mathbf{p}(\mathbf{x})\right) d\gamma_h \\
&= \sum_{i\in\sigma} -W_i^h \int_{\partial K_i^h} \left(\left[\xi_h \mathbf{I} - \mathbf{P}_h(\mathbf{I} - d\mathbf{H})\right]\nabla_s u\left(\mathbf{p}(\mathbf{x})\right)\right) \cdot \vec{\mathbf{n}}_{K_i}\left(\mathbf{p}(\mathbf{x})\right) d\gamma_h \\
&= \sum_{i\in\sigma} -W_i^h \int_{\partial K_i^h} \left(\xi_h\left[\mathbf{P} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right]\nabla_s u\left(\mathbf{p}(\mathbf{x})\right)\right) \cdot \vec{\mathbf{n}}_{K_i}\left(\mathbf{p}(\mathbf{x})\right) d\gamma_h \\
&= \sum_{T_i^h \in \mathcal{T}^h} \left(\sum_{j=1}^3 \left(W_{i_{j+2}}^h - W_{i_{j+1}}^h\right) \int_{\overline{M_{i_{j+1},i_{j+2}} Q_i}} \left(\xi_h\left[\mathbf{P} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right]\nabla_s u\left(\mathbf{p}(\mathbf{x})\right)\right) \cdot \vec{\mathbf{n}}_{K_{i_{j+1}}}\left(\mathbf{p}(\mathbf{x})\right) d\gamma_h\right)
\end{aligned}$$

where the second last step is due to the fact that $\mathbf{P}$ is the tangential projection onto $\mathbf{S}$. Thus

$$|I_1| \leqslant \left|\xi_h\left(\mathbf{P} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right)\right| \sum_{T_i^h \in \mathcal{T}^h} \left(\sum_{j=1}^3 \left|W_{i_{j+2}}^h - W_{i_{j+1}}^h\right| \int_{\overline{M_{i_{j+1},i_{j+2}} Q_i}} \left|\nabla_s u\left(\mathbf{p}(\mathbf{x})\right)\right| d\gamma_h\right).$$

Since $|1-\xi_h| < ch^2$, it is easy to see that

$$\begin{aligned}\left|\mathbf{P} - \frac{1}{\xi_h}\mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P}\right| &\leqslant \left|\mathbf{P} - \mathbf{P}_h(\mathbf{I} - d\mathbf{H})\mathbf{P})\right| + ch^2 \\ &\leqslant |\mathbf{P} - \mathbf{P}_h\mathbf{P}| + ch^2 \\ &\leqslant ch.\end{aligned}$$

Consequently, using similar analysis as used in Lemma 6, we obtain

$$\begin{aligned}|I_1| &\leqslant ch \sum_{T_i^h \in \mathcal{T}^h} h\big(h^{-1}|\nabla_s u|_{L^2(T_i)} + |\nabla_s u|_{H^1(T_i)}\big)\big\|W^h\big\|_{H^1(T_i^h)} \\ &\leqslant ch \sum_{T_i^h \in \mathcal{T}^h} \|\nabla_s u\|_{H^1(T_i)}\big\|W^h\big\|_{H^1(T_i^h)} \\ &\leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}.\end{aligned} \tag{5.15}$$

As for I_2, we have

$$I_2 = \sum_{T_i^h \in \mathcal{T}^h}\left(\sum_{j=1}^{3}\big(W^h_{i_{j+2}} - W^h_{i_{j+1}}\big)\int\limits_{\overline{M_{i_{j+1},i_{j+2}}Q_i}} \nabla_{s_h}U(\mathbf{x})\cdot\big[\vec{\mathbf{n}}_{K_{i_{j+1}}}\big(\mathbf{p}(\mathbf{x})\big) - \vec{\mathbf{n}}_{K^h_{i_{j+1}}}(\mathbf{x})\big]\,d\gamma_h\right).$$

Since $|\vec{\mathbf{n}}_{K_{i_{j+1}}}(\mathbf{p}(\mathbf{x})) - \vec{\mathbf{n}}_{K^h_{i_{j+1}}}(\mathbf{x})| < ch$, we again get

$$\begin{aligned}|I_2| &\leqslant ch\|\nabla_{s_h}U\|_{L^2(\mathbf{S}^h)}\big\|W^h\big\|_{H^1(\mathbf{S}^h)} \\ &\leqslant ch\|u\|_{H^1(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}.\end{aligned} \tag{5.16}$$

As for I_3, we also can get

$$\begin{aligned}|I_3| &\leqslant \|b\|_{L^\infty(\mathbf{S})}\sum_{i\in\sigma}\big|W_i^h\big|\int\limits_{K_i^h}|\mu_h - 1|\,|U|\,ds_h \\ &\leqslant ch^2\|U\|_{L^2(\mathbf{S}^h)}\big\|W^h\big\|_{L^2(\mathbf{S}^h)} \\ &\leqslant ch^2\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{L^2(\mathbf{S}^h)}.\end{aligned} \tag{5.17}$$

Combining (5.15)–(5.17), we finally obtain (6.16). □

Theorem 3. *Suppose that u is the weak solution of the problem* (2.1) *with $u|_{\partial\mathbf{S}} = 0$, $U^h \in \mathcal{U}$ is the solution of discrete problem* (3.4) *and $u^h = \mathcal{L}(U^h)$. If $u \in H^2(\mathbf{S})$, then it holds that*

$$\big\|u - u^h\big\|_{H^1(\mathbf{S})} \leqslant ch\|u\|_{H^2(\mathbf{S})}, \tag{5.18}$$

for some constant $c > 0$.

Proof. Let us extend u onto $\mathbf{S}^h$ by $U = \mathcal{L}^{-1}(u)$. By Proposition 1, we have

$$\big\|U^h - \Pi_u(U)\big\|^2_{H^1(\mathbf{S}^h)} \leqslant c\mathcal{A}^h_G\big(U^h - \Pi_u(U), \Pi_v\big(U^h - \Pi_u(U)\big)\big). \tag{5.19}$$

For any $W^h \in \mathcal{U}$, let $w^h = \mathcal{L}(W^h)$, then we get

$$\begin{aligned}\mathcal{A}^h_G\big(U^h - \Pi_u(U), \Pi_v\big(W^h\big)\big) &= \big[\mathcal{A}^h_G\big(U^h, \Pi_v\big(W^h\big)\big) - \mathcal{A}_G\big(u, \pi_v\big(w^h\big)\big)\big] \\ &\quad + \mathcal{A}^h_G\big(U - \Pi_u(U), \Pi_v\big(W^h\big)\big) \\ &\quad + \big[\mathcal{A}_G\big(u, \pi_v\big(w^h\big)\big) - \mathcal{A}^h_G\big(U, \Pi_v\big(W^h\big)\big)\big].\end{aligned} \tag{5.20}$$

According to the weak form and Green's theorem, we have

$$\mathcal{A}^h_G\big(U^h, \Pi_v\big(W^h\big)\big) = \big(F, \Pi_v\big(W^h\big)\big)_{S_h}, \qquad \mathcal{A}_G\big(u, \pi_v\big(w^h\big)\big) = \big(f, \pi_v\big(w^h\big)\big).$$

So by Lemmas 1–3, we get

$$\begin{aligned}\left|\mathcal{A}_G^h\big(U^h,\Pi_v\big(W^h\big)\big)-\mathcal{A}_G\big(u,\pi_v\big(w^h\big)\big)\right| &= \left|\big(F,\Pi_v\big(W^h\big)\big)_{s_h}-\big(f,\pi_v\big(w^h\big)\big)\right| \\ &= \left|\int_{\mathbf{S}^h} F\Pi_v\big(W^h\big)\,ds_h-\int_{\mathbf{S}} f\pi_v\big(w^h\big)\,ds\right| \\ &= \left|\int_{\mathbf{S}^h} F\Pi_v\big(W^h\big)\,ds_h-\int_{\mathbf{S}^h} F\Pi_v\big(W^h\big)\mu_h\,ds_h\right| \\ &= \left|\int_{\mathbf{S}^h}(1-\mu_h)F\Pi_v\big(W^h\big)\,ds_h\right| \\ &\leqslant ch^2\|F\|_{L^2(\mathbf{S}^h)}\big\|\Pi_v\big(W^h\big)\big\|_{L^2(\mathbf{S}^h)} \\ &\leqslant ch^2\|f\|_{L^2(\mathbf{S})}\big\|W^h\big\|_{L^2(\mathbf{S}^h)} \\ &\leqslant ch^2\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{L^2(\mathbf{S}^h)}.\end{aligned} \tag{5.21}$$

By Lemma 6, we have

$$\left|\mathcal{A}_G^h\big(U-\Pi_u(U),\Pi_v\big(W^h\big)\big)\right|\leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \tag{5.22}$$

By Lemma 7, we get

$$\left|\mathcal{A}_G\big(u,\pi_v\big(w^h\big)\big)-\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)\right|\leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \tag{5.23}$$

Using (5.19)–(5.23) and setting $W^h=U^h-\Pi_u(U)$, we then obtain

$$\big\|U^h-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)}^2\leqslant ch\|u\|_{H^2(\mathbf{S})}\big\|U^h-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)},$$

that is,

$$\big\|U^h-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)}\leqslant ch\|u\|_{H^2(\mathbf{S})}. \tag{5.24}$$

Additionally, by Lemma 4, we have

$$\big\|U-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)}\leqslant\big\|u-\pi_u(u)\big\|_{H^1(\mathbf{S})}\leqslant ch\|u\|_{H^2(\mathbf{S})}. \tag{5.25}$$

Combining (5.24) and (5.25), we finally have

$$\begin{aligned}\big\|u-u^h\big\|_{H^1(\mathbf{S})} &\leqslant c\big\|U-U^h\big\|_{H^1(\mathbf{S}^h)} \\ &\leqslant c\big(\big\|U^h-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)}+\big\|U-\Pi_u(U)\big\|_{H^1(\mathbf{S}^h)}\big) \\ &\leqslant ch\|u\|_{H^2(\mathbf{S})},\end{aligned}$$

which completes the proof. □

The optimal error estimate presented in Theorem 3 is similar to that obtained by the finite element method, see [23].

6. L^2 error estimate

Before presenting the main result for L^2 error estimate, let us first prove additional estimates on the bilinear forms. Lemma 8 is used to measure the differences of bilinear forms on $\mathbf{S}$ and $\mathbf{S}^h$, while Lemmas 9 and 10 measure the differences of the bilinear forms due to the integration by parts on $\mathbf{S}^h$ and $\mathbf{S}$ respectively. Moreover, Lemma 11 is a strengthened version of the Lemma 7 under extra regularity, and Lemma 12 may be seen as the *dual* version of the Lemma 11. The presence of the surface approximation and the requirement on the higher regularity again demand the derivation of these separate estimates.

Now let us define a bilinear operator as follows: for any $U,V\in H^1(\mathbf{S}^h)$,

$$\mathcal{A}^h(U,V)=\int_{\mathbf{S}^h}\nabla_{s_h}U\cdot\nabla_{s_h}V\,ds_h+\int_{\mathbf{S}^h}BUV\,ds_h. \tag{6.1}$$

Lemma 8. *For any $u,v\in H^1(\mathbf{S})$, there exists a constant $c>0$ such that*

$$\left|\mathcal{A}(u,v)-\mathcal{A}^h(U,V)\right|\leqslant ch^2\|u\|_{H^1(\mathbf{S})}\|v\|_{H^1(\mathbf{S})}, \tag{6.2}$$

where $U=\mathcal{L}^{-1}(u)$ and $V=\mathcal{L}^{-1}(v)$.

Proof. We know

$$\mathcal{A}(u,v)-\mathcal{A}^h(U,V)=I_1+I_2 \tag{6.3}$$

where

$$I_1=\int_{\mathbf{S}}\nabla_s u\cdot\nabla_s v\,ds-\int_{\mathbf{S}^h}\nabla_{s_h}U\cdot\nabla_{s_h}V\,ds_h,$$

$$I_2=\int_{\mathbf{S}} buv\,ds-\int_{\mathbf{S}^h} BUV\,ds_h.$$

It is clear that

$$\begin{aligned}|I_1|&=\Bigg|\sum_{T_i\in\mathcal{T}}\int_{T_i}\nabla_s u\cdot\nabla_s v\,ds-\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\nabla_{s_h}U\cdot\nabla_{s_h}V\,ds_h\Bigg|\\&=\Bigg|\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\mu_h\big(\nabla_s u\big(\mathbf{p}(\mathbf{x})\big)\cdot\nabla_s v\big(\mathbf{p}(\mathbf{x})\big)\big)-\nabla_s U(\mathbf{x})\cdot\nabla_s V(\mathbf{x})\,ds_h\Bigg|\\&=\Bigg|\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\big[\mu_h\mathbf{I}-\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\big]\big(\nabla_s u\big(\mathbf{p}(\mathbf{x})\big)\cdot\nabla_s v\big(\mathbf{p}(\mathbf{x})\big)\big)\,ds_h\Bigg|.\end{aligned}$$

Using the result proved in [23, p. 148] that

$$\big|\mu_h\mathbf{I}-\mathbf{P}(\mathbf{I}-d\mathbf{H})\mathbf{P}_h(\mathbf{I}-d\mathbf{H})\mathbf{P}\big|\leqslant ch^2,$$

we immediately obtain

$$|I_1|\leqslant ch^2\|u\|_{H^1(\mathbf{S})}\|v\|_{H^1(\mathbf{S})}. \tag{6.4}$$

It is also easy to see that

$$\begin{aligned}|I_2|&=\Bigg|\int_{\mathbf{S}^h}(\mu_h-1)BUV\,ds_h\Bigg|\\&\leqslant ch^2\|U\|_{L^2(\mathbf{S}^h)}\|V\|_{L^2(\mathbf{S}^h)}\\&\leqslant ch^2\|u\|_{L^2(\mathbf{S})}\|v\|_{L^2(\mathbf{S})}.\end{aligned} \tag{6.5}$$

Combination of (6.3)–(6.5) deduces (6.2). □

Lemma 9. *For any $u\in H^3(\mathbf{S})$ and $W^h\in\mathcal{U}$, there exists a constant $c>0$ such that*

$$\big|\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)-\mathcal{A}^h\big(U,W^h\big)\big|\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}, \tag{6.6}$$

where $U=\mathcal{L}^{-1}(u)$ and $w^h=\mathcal{L}(W^h)$.

Proof. It is easy to see

$$\begin{aligned}\sum_{i\in\sigma}-W^h(\mathbf{x}_i)\int_{\partial K_i^h}\nabla_{s_h}U(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_i^h}(\mathbf{x})\,d\gamma_h&=\sum_{T_i^h\in\mathcal{T}^h}\Bigg(-\sum_{j=1}^{3}\int_{\partial K_{i_j}^h\cap T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{K_{i_j}^h}\big)\Pi_v\big(W^h\big)\,d\gamma_h\Bigg)\\&=\sum_{T_i^h\in\mathcal{T}^h}\Bigg(-\int_{T_i^h}\Delta_{s_h}U\Pi_v\big(W^h\big)\,ds_h+\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)\Pi_v\big(W^h\big)\,d\gamma_h\Bigg),\end{aligned}$$

and

$$\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\nabla_{s_h}U\cdot\nabla_{s_h}W^h\,ds_h=\sum_{T_i^h\in\mathcal{T}^h}\Bigg(-\int_{T_i^h}\Delta_{s_h}UW^h\,ds_h+\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)W^h\,d\gamma_h\Bigg).$$

Then we get

$$\begin{aligned}\left|\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)-\mathcal{A}^h\big(U,W^h\big)\right| \leqslant{}& \left|\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\Delta_{s_h}U\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h\right| \\ &+\left|\sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h\right| \\ &+\left|\int_{\mathbf{S}^h}BU\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h\right|. \end{aligned} \tag{6.7}$$

Let $\overline{\Delta_{s_h}U^i}$ denote the average of $\Delta_{s_h}U$ over $\mathcal{T}_i^h$, then by (4.5),

$$\int_{T_i^h}\Delta_{s_h}U\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h=\int_{T_i^h}\big(\Delta_{s_h}U-\overline{\Delta_{s_h}U^i}\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h.$$

Now we can obtain

$$\begin{aligned}\left|\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\Delta_{s_h}U\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h\right| &\leqslant \sum_{T_i^h\in\mathcal{T}^h}\big\|\Delta_{s_h}U-\overline{\Delta_{s_h}U^i}\big\|_{L^2(T_i^h)}\big\|\Pi_v\big(W^h\big)-W^h\big\|_{L^2(T_i^h)} \\ &\leqslant ch^2\|U\|_{H^3(\mathbf{S}^h)}\big\|W^h\big\|_{H^1(\mathbf{S}^h)} \\ &\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \end{aligned} \tag{6.8}$$

Using (4.5), we get

$$\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h=\sum_{j=1}^{3}\int_{\mathbf{x}_{i_j}\mathbf{x}_{i_{j+1}}}\big[\big(\nabla_{s_h}U-\nabla_{s_h}\Pi_u(U)\big)\cdot\vec{\mathbf{n}}_{T_i^h}\big]\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h.$$

Since $W^h=0$ on the boundary $\partial\mathbf{S}^h$, we only need to consider the set of interior edges denoted by $\mathcal{E}^h=\{e_i^h\}$. Suppose that e_i^h is shared by two triangles $T_{i_1}^h$ and $T_{i_2}^h$, then we have

$$\sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h=\sum_{e_i^h\in\mathcal{E}^h}\int_{e_i^h}\big[\big(\nabla_{s_h}U-\nabla_{s_h}\Pi_u(U)\big)\cdot\big(\vec{\mathbf{n}}_{T_{i_1}^h,e_i^h}+\vec{\mathbf{n}}_{T_{i_2}^h,e_i^h}\big)\big]\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h.$$

Notice that $|\vec{\mathbf{n}}_{T_{i_1}^h,e_i^h}+\vec{\mathbf{n}}_{T_{i_2}^h,e_i^h}|<ch$, then using the trace theorem we can get

$$\begin{aligned}\left|\sum_{T_i^h\in\mathcal{T}^h}\int_{\partial T_i^h}\big(\nabla_{s_h}U\cdot\vec{\mathbf{n}}_{T_i^h}\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,d\gamma_h\right| &\leqslant ch\sum_{T_i^h\in\mathcal{T}^h}ch^{1/2}\big(h^{-1}\big|\nabla_{s_h}\big(U-\Pi_u(U)\big)\big|_{L^2(\mathcal{T}_i^h)}+\big|\nabla_{s_h}\big(U-\Pi_u(U)\big)\big|_{H^1(\mathcal{T}_i^h)}\big) \\ &\quad\times h^{1/2}\big(h^{-1}\big|\Pi_v\big(W^h\big)-W^h\big|_{L^2(\mathcal{T}_i^h)}+\big|\Pi_v\big(W^h\big)-W^h\big|_{H^1(\mathcal{T}_i^h)}\big) \\ &\leqslant ch^2\|U\|_{H^2(\mathbf{S}^h)}\big\|W^h\big\|_{H^1(\mathbf{S}^h)} \\ &\leqslant ch^2\|u\|_{H^2(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \end{aligned} \tag{6.9}$$

It is also trivial to verify

$$\begin{aligned}\int_{\mathbf{S}^h}BU\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h={}&\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\big(B-\Pi_v(B)\big)U\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h \\ &+\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\Pi_v(B)\big(U-\Pi_v(U)\big)\big(\Pi_v\big(W^h\big)-W^h\big)\,ds_h. \end{aligned}$$

Then we get

$$\left| \int_{\mathbf{S}^h} BU\big(\Pi_v(W^h) - W^h\big)\, ds_h \right| \leqslant ch^2 \|U\|_{H^1(\mathbf{S}^h)} \big\|W^h\big\|_{H^1(\mathbf{S}^h)}$$
$$\leqslant ch^2 \|u\|_{H^1(\mathbf{S})} \big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \tag{6.10}$$

Combination of (6.7)–(6.10) gives us (6.6). □

Lemma 10. *For any $u \in H^3(\mathbf{S})$ and $W^h \in \mathcal{U}$, there exists a constant $c > 0$ such that*

$$\big|\mathcal{A}_G\big(u, \pi_v(w^h)\big) - \mathcal{A}\big(u, w^h\big)\big| \leqslant ch^2 \|u\|_{H^3(\mathbf{S})} \big\|W^h\big\|_{H^1(\mathbf{S}^h)}, \tag{6.11}$$

where $w^h = \mathcal{L}(W^h)$.

Proof. By Green's theorem, we can easily find that

$$\mathcal{A}_G\big(u, \pi_v(w^h)\big) = \sum_{T_i \in \mathcal{T}} \left(-\int_{T_i} \Delta_s u \pi_v(w^h)\, ds + \int_{\partial T_i} (\nabla_s u \cdot \vec{\mathbf{n}}_{T_i}) \pi_v(w^h)\, d\gamma \right) + \int_{\mathbf{S}} bu\pi_v(w^h)\, ds,$$

and

$$\mathcal{A}\big(u, w^h\big) = \sum_{T_i \in \mathcal{T}} \left(-\int_{T_i} \Delta_s u w^h\, ds + \int_{\partial T_i} (\nabla_s u \cdot \vec{\mathbf{n}}_{T_i}) w^h\, d\gamma \right) + \int_{\mathbf{S}} buw^h\, ds.$$

So that

$$\big|\mathcal{A}_G\big(u, \pi_v(w^h)\big) - \mathcal{A}\big(u, w^h\big)\big| \leqslant \left| \sum_{T_i \in \mathcal{T}} \int_{T_i} \Delta_s u\big(\pi_v(w^h) - w^h\big)\, ds \right|$$
$$+ \left| \sum_{T_i \in \mathcal{T}} \int_{\partial T_i} (\nabla_s u \cdot \vec{\mathbf{n}}_{T_i})\big(\pi_v(w^h) - w^h\big)\, d\gamma \right|$$
$$+ \left| \int_{\mathbf{S}} bu\big(\pi_v(w^h) - w^h\big)\, ds \right|. \tag{6.12}$$

Notice that

$$\left| \int_{T_i} \pi_v(w^h) - w^h\, ds \right| = \left| \int_{T_i} \pi_v(w^h) - w^h\, ds - \int_{T_i^h} \Pi_v(W^h) - W^h\, ds_h \right|$$
$$= \left| \int_{T_i^h} (\mu_h - 1)\big(\Pi_v(W^h) - W^h\big)\, ds_h \right|$$
$$\leqslant ch^3 \big\|W^h\big\|_{H^1(T_i^h)}.$$

Let $\overline{\Delta_s u^i}$ denote the average of $\Delta_s u$ over T_i, then we have

$$\left| \sum_{T_i \in \mathcal{T}} \int_{T_i} \Delta_s u\big(\pi_v(w^h) - w^h\big)\, ds_h \right| \leqslant \left| \sum_{T_i \in \mathcal{T}} \int_{T_i} \big(\Delta_s u - \overline{\Delta_s u^i}\big)\big(\pi_v(w^h) - w^h\big)\, ds_h \right|$$
$$+ \left| \sum_{T_i \in \mathcal{T}} \int_{T_i} \overline{\Delta_s u^i}\big(\pi_v(w^h) - w^h\big)\, ds_h \right|$$
$$\leqslant ch^2 \|u\|_{H^3(\mathbf{S})} \big\|w^h\big\|_{H^1(\mathbf{S})} + ch^3 \|u\|_{H^3(\mathbf{S})} \big\|W^h\big\|_{H^1(\mathbf{S}^h)}$$
$$\leqslant ch^2 \|u\|_{H^3(\mathbf{S})} \big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \tag{6.13}$$

Let $\mathcal{E} = \{e_i\}$ denote the set of interior edges of $\mathcal{T}$. Since $w^h = 0$ on $\partial\mathbf{S}$, then we find by the fact $\vec{\mathbf{n}}_{T_{i_1}, e_i} = -\vec{\mathbf{n}}_{T_{i_2}, e_i}$ and the continuity of $\nabla_s u$ that

$$\sum_{T_i\in\mathcal{T}}\int_{\partial T_i}(\nabla_s u\cdot\vec{\mathbf{n}}_{T_i})\big(\pi_v\big(w^h\big)-w^h\big)\,d\gamma=\sum_{e_i\in\mathcal{E}^h}\int_{e_i^h}\big[\nabla_s u\cdot(\vec{\mathbf{n}}_{T_{i_1},e_i}+\vec{\mathbf{n}}_{T_{i_2},e_i})\big]\big(\pi_v\big(w^h\big)-w^h\big)\,d\gamma=0. \tag{6.14}$$

Similar to derivation of (6.10), it is also easy to see

$$\left|\int_{\mathbf{S}} bu\big(\pi_v\big(w^h\big)-w^h\big)\,ds\right|\leqslant ch^2\|u\|_{H^1(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}. \tag{6.15}$$

Finally, the combination of (6.12)–(6.15) gives us (6.11). □

Lemma 11. *For any $u\in H^3(\mathbf{S})$ and $w\in W^{1,p}(\mathbf{S})\cap H_0^1(\mathbf{S})$ with $p>2$, there exists a constant $c>0$ such that*

$$\big|\mathcal{A}_G\big(u,\pi_v(w)\big)-\mathcal{A}_G^h\big(U,\Pi_v(W)\big)\big|\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}, \tag{6.16}$$

where $u=\mathcal{L}(U)$ and $W=\mathcal{L}^{-1}(w)$.

Proof. Let $W^h=\Pi_u(W)$, notice that $\Pi_v(W^h)=\Pi_v(W)$ and $\pi_v(w^h)=\pi_v(w^h)$, we know that

$$\begin{aligned}\big|\mathcal{A}_G^h\big(U,\Pi_v(W)\big)-\mathcal{A}_G\big(u,\pi_v(w)\big)\big|&=\big|\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)-\mathcal{A}_G\big(u,\pi_v\big(w^h\big)\big)\big|\\&\leqslant\big|\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)-\mathcal{A}^h\big(U,W^h\big)\big|\\&\quad+\big|\mathcal{A}_G\big(u,\pi_v\big(w^h\big)\big)-\mathcal{A}\big(u,w^h\big)\big|\\&\quad+\big|\mathcal{A}\big(u,w^h\big)-\mathcal{A}^h\big(U,W^h\big)\big|.\end{aligned} \tag{6.17}$$

Combining (6.17) with Lemmas 8–10, we get

$$\begin{aligned}\big|\mathcal{A}_G^h\big(U,\Pi_v\big(W^h\big)\big)-\mathcal{A}_G\big(u,\pi_v\big(w^h\big)\big)\big|&\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\big\|W^h\big\|_{H^1(\mathbf{S}^h)}\\&\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}.\end{aligned}$$

The proof is complete. □

Lemma 12. *Suppose that u is the weak solution of the problem (2.1) with $u|_{\partial\mathbf{S}}=0$, and $U^h\in\mathcal{U}$ is the solution of discrete problem (3.4). If $u\in H^3(\mathbf{S})$, then for any $w\in W^{1,p}(\mathbf{S})\cap H_0^1(\mathbf{S})$ for $p>2$, there exists a constant $c>0$ such that*

$$\big|\mathcal{A}^h\big(U-U^h,\Pi_u(W)\big)-\mathcal{A}_G^h\big(U-U^h,\Pi_v(W)\big)\big|\leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}, \tag{6.18}$$

where $W=\mathcal{L}^{-1}(w)$.

Proof. We first have that

$$\begin{aligned}\mathcal{A}^h\big(U-U^h,\Pi_u(W)\big)&=\int_{\mathbf{S}^h}\nabla_{s_h}\big(U-U^h\big)\cdot\nabla_{s_h}\Pi_u(W)\,ds_h+\int_{\mathbf{S}^h}B\big(U-U^h\big)\Pi_u(W)\,ds_h\\&=\sum_{T_i^h\in\mathcal{T}^h}\int_{T_i^h}\nabla_{s_h}\big(U-U^h\big)\cdot\nabla_{s_h}\Pi_u(W)\,ds_h+\int_{\mathbf{S}^h}B\big(U-U^h\big)\Pi_u(W)\,ds\\&=\sum_{T_i^h\in\mathcal{T}^h}\left(-\int_{T_i^h}\Delta_{s_h}U\;\Pi_u(W)\,ds_h+\int_{\partial T_i^h}\big(\nabla_{s_h}\big(U-U^h\big)\cdot\vec{\mathbf{n}}_{T_i^h}\big)\Pi_u(W)\,d\gamma_h\right)\\&\quad+\int_{\mathbf{S}^h}B\big(U-U^h\big)\Pi_u(W)\,ds_h\end{aligned}$$

and by Green's theorem

$$\begin{aligned}\mathcal{A}_G^h\big(U-U^h,\Pi_v(W)\big)&=\sum_{i\in\sigma}\left(\int_{\partial K_i^h}-\big(\nabla_{s_h}\big(U-U^h\big)\cdot\vec{\mathbf{n}}_{K_i^h}\big)\Pi_v(W)\,d\gamma_h+\int_{K_i^h}B\big(U-U^h\big)\Pi_v(W)\,ds_h\right)\\&=\sum_{T_i^h\in\mathcal{T}^h}\sum_{j=1}^{3}\int_{\partial K_{i_j}^h\cap T_i^h}-\big(\nabla_{s_h}\big(U-U^h\big)\cdot\vec{\mathbf{n}}_{K_i^h}\big)\Pi_v(W)\,d\gamma_h+\int_{\mathbf{S}^h}B\big(U-U^h\big)\Pi_v(W)\,ds_h\end{aligned}$$

$$= \sum_{T_i^h \in \mathcal{T}^h} \left(-\int_{T_i^h} \Delta_{s_h} U \, \Pi_v(W)\, ds_h + \int_{\partial T_i^h} \left(\nabla_{s_h}\left(U - U^h\right) \cdot \vec{\mathbf{n}}_{T_i^h}\right) \Pi_v(W)\, d\gamma_h \right)$$
$$+ \int_{\mathbf{S}^h} B\left(U - U^h\right) \Pi_v(W)\, ds_h.$$

So we obtain

$$\mathcal{A}^h\left(U - U^h, \Pi_u(W)\right) - \mathcal{A}_G^h\left(U - U^h, \Pi_v(W)\right) = I_1 + I_2 + I_3$$

where

$$I_1 = \sum_{T_i^h \in \mathcal{T}^h} \int_{T_i^h} -\Delta_{s_h} U\left(\Pi_u(W) - \Pi_v(W)\right) ds_h,$$
$$I_2 = \sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} \left[\nabla_{s_h}\left(U - U^h\right) \cdot \vec{\mathbf{n}}_{T_i^h}\right]\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h,$$
$$I_3 = \int_{\mathbf{S}^h} B\left(U - U^h\right)\left(\Pi_u(W) - \Pi_v(W)\right) ds_h.$$

Let $\overline{\Delta_{s_h} U^i}$ denote the average of $\Delta_{s_h} U$ over T_i^h. Using (4.4), we have

$$\int_{T_i^h} \Delta_{s_h} U\left(\Pi_u(W) - \Pi_v(W)\right) ds_h = \int_{T_i^h} \left(\Delta_{s_h} U - \overline{\Delta_{s_h} U^i}\right)\left(\Pi_u(W) - \Pi_v(W)\right) ds_h.$$

Therefore, for $p > 2$,

$$\begin{aligned} |I_1| &\leqslant \sum_{T_i^h \in \mathcal{T}^h} \left\| \Delta_{s_h} U - \overline{\Delta_{s_h} U^i} \right\|_{L^2(T_i^h)} \left\| \Pi_u(W) - \Pi_v(W) \right\|_{L^2(T_i^h)} \\ &\leqslant ch^2 \|U\|_{H^3(\mathbf{S}^h)} \left\|\Pi_u(W)\right\|_{H^1(\mathbf{S}^h)} \\ &\leqslant ch^2 \|u\|_{H^3(\mathbf{S})} \|w\|_{W^{1,p}(\mathbf{S})}. \end{aligned} \tag{6.19}$$

Similarly, by (4.5), we have

$$\begin{aligned} \int_{\partial T_i^h} \left(\nabla_{s_h}\left(U - U^h\right) \cdot \vec{\mathbf{n}}_{T_i^h}\right)\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h &= \sum_{j=1}^{3} \int_{\mathbf{x}_{i_j}\mathbf{x}_{i_{j+1}}} \left(\nabla_{s_h} U \cdot \vec{\mathbf{n}}_{T_i^h}\right)\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h \\ &= \sum_{j=1}^{3} \int_{\mathbf{x}_{i_j}\mathbf{x}_{i_{j+1}}} \left[\left(\nabla_{s_h} U - \nabla_{s_h} \Pi_u\left(U^h\right)\right) \cdot \vec{\mathbf{n}}_{T_i^h}\right]\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h. \end{aligned}$$

Since $W = 0$ on $\partial\mathbf{S}^h$, we have

$$\begin{aligned} &\sum_{T_i^h \in \mathcal{T}^h} \int_{\partial T_i^h} \left(\nabla_{s_h} U \cdot \vec{\mathbf{n}}_{T_i^h}\right)\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h \\ &\quad = \sum_{e_i^h \in \mathcal{E}^h} \int_{e_i^h} \left[\left(\nabla_{s_h} U - \nabla_{s_h} \Pi_u(U)\right) \cdot (\vec{\mathbf{n}}_{T_{i_1}^h, e_i^h} + \vec{\mathbf{n}}_{T_{i_2}^h, e_i^h})\right]\left(\Pi_u(W) - \Pi_v(W)\right) d\gamma_h. \end{aligned}$$

Notice that $|\vec{\mathbf{n}}_{T_{i_1}^h, e_i^h} + \vec{\mathbf{n}}_{T_{i_2}^h, e_i^h}| \leqslant ch$. Then by the similar analysis used for (6.9), we can get

$$|I_2| \leqslant ch^2 \|u\|_{H^2(\mathbf{S})} \|w\|_{W^{1,p}(\mathbf{S})}. \tag{6.20}$$

About I_3, we have

$$\begin{aligned} |I_3| &\leqslant c\|B\|_{L^\infty(\mathbf{S}^h)} \left\|U - U^h\right\|_{L^2(\mathbf{S}^h)} \left\|\Pi_u(W) - \Pi_v(W)\right\|_{L^2(\mathbf{S}^h)} \\ &\leqslant ch^2 \|u\|_{H^2(\mathbf{S})} \|w\|_{W^{1,p}(\mathbf{S})}. \end{aligned} \tag{6.21}$$

Combining (6.19)–(6.21), we immediately obtain (6.18). □

Theorem 4. *Suppose that u is the weak solution of the problem (2.1) with $u|_{\partial\mathbf{S}} = 0$, $U^h \in \mathcal{U}$ is the solution of discrete problem (3.4) and $u^h = \mathcal{L}(U^h)$. If $u \in H^3(\mathbf{S})$, then it holds that*

$$\|u - u^h\|_{L^2(\mathbf{S})} \leqslant ch^2 \|u\|_{H^3(\mathbf{S})} \tag{6.22}$$

for some constant $c > 0$.

Proof. Since $u - u^h \in H_0^1(\mathbf{S})$, according to Theorem 1, we know that there exists a weak solution $w \in H^2(\mathbf{S}) \cap H_0^1(\mathbf{S})$ satisfying

$$\mathcal{A}(w, v) = (u - u^h, v), \quad \forall v \in H_0^1(\mathbf{S}).$$

Put $v = u - u^h$ in the above equality, then we get

$$\|u - u^h\|_{L^2(\mathbf{S})}^2 = (u - u^h, u - u^h) = \mathcal{A}(u - u^h, w).$$

Furthermore, it also holds

$$\|w\|_{H^2(\mathbf{S})} \leqslant c\|u - u^h\|_{L^2(\mathbf{S})}. \tag{6.23}$$

Let $W = \mathcal{L}^{-1}(w)$, then we get

$$\begin{aligned} \|u - u^h\|_{L^2(\mathbf{S})}^2 \leqslant{}& \big|\mathcal{A}\big(u - u^h, w - \pi_u(w)\big)\big| + \big|\mathcal{A}\big(u - u^h, \pi_u(w)\big) - \mathcal{A}^h\big(U - U^h, \Pi_u(W)\big)\big| \\ &+ \big|\mathcal{A}^h\big(U - U^h, \Pi_u(W)\big) - \mathcal{A}_G^h\big(U - U^h, \Pi_v(W)\big)\big| + \big|\mathcal{A}_G\big(u, \pi_v(w)\big) - \mathcal{A}_G^h(U^h, \Pi_v(W)\big| \\ &+ \big|\mathcal{A}_G^h\big(U, \Pi_v(W)\big) - \mathcal{A}_G\big(u, \pi_v(w)\big)\big|. \end{aligned} \tag{6.24}$$

First by Theorem 3, we have

$$\begin{aligned} \big|\mathcal{A}\big(u - u^h, w - \pi_u(w)\big)\big| &\leqslant c\|u - u^h\|_{H^1(\mathbf{S})}\|w - \pi_u(w)\|_{H^1(\mathbf{S})} \\ &\leqslant ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{H^2(\mathbf{S})}. \end{aligned} \tag{6.25}$$

By Lemma 8 and Theorem 3, we get

$$\begin{aligned} \big|\mathcal{A}\big(u - u^h, \pi_u(w)\big) - \mathcal{A}^h\big(U - U^h, \Pi_u(W)\big)\big| &\leqslant ch^2\|u - u^h\|_{H^1(\mathbf{S})}\|\Pi_u(W)\|_{H^1(\mathbf{S}^h)} \\ &\leqslant ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}. \end{aligned} \tag{6.26}$$

Lemma 12 directly tells us that

$$\big|\mathcal{A}^h\big(U - U^h, \Pi_u(W)\big) - \mathcal{A}_G^h\big(U - U^h, \Pi_v(W)\big)\big| \leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}. \tag{6.27}$$

From (5.21), we get

$$\big|\mathcal{A}_G\big(u, \pi_v(w)\big) - \mathcal{A}_G^h(U^h, \Pi_v(W)\big| \leqslant ch^2\|u\|_{H^2(\mathbf{S})}\|\Pi_v(W)\|_{L^2(\mathbf{S}^h)} \leqslant ch^2\|u\|_{H^2(\mathbf{S})}\|w\|_{W^{1,p}(\mathbf{S})}. \tag{6.28}$$

Lemma 11 gives us that

$$\big|\mathcal{A}_G^h(U, \Pi_v(W) - \mathcal{A}_G\big(u, \pi_v(w)\big)\big| \leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{H^1(\mathbf{S})}. \tag{6.29}$$

Combining (6.25)–(6.29) with (6.23), we finally get

$$\|u - u^h\|_{L^2(\mathbf{S})}^2 \leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|w\|_{H^2(\mathbf{S})} \leqslant ch^2\|u\|_{H^3(\mathbf{S})}\|u - u^h\|_{L^2(\mathbf{S})}$$

which deduces (6.22) directly. □

Remark 4. All results proved in Theorems 2–4 can be easily generalized to the case of $\partial\mathbf{S} = \emptyset$ with $b(\mathbf{x}) > \alpha_2 > 0$.

7. Some remarks

If a convection term is added into equation (2.1) such as

$$-\nabla_s \cdot \big(a(\mathbf{x})\nabla_s u(\mathbf{x})\big) + \nabla_s \cdot \big(\vec{\mathbf{v}}(\mathbf{x})u(\mathbf{x})\big) + b(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \text{for } \mathbf{x} \in \mathbf{S}, \tag{7.1}$$

where $\vec{\mathbf{v}} \in (W^{1,\infty}(\mathbf{S}))^3$ and $\nabla_s \cdot \vec{\mathbf{v}}(\mathbf{x}) + b(\mathbf{x}) \geqslant \alpha_3 > 0$. Let $\vec{\mathbf{V}} = \mathcal{L}^{-1}(\vec{\mathbf{v}})$. Then a generalized central finite volume scheme for the above diffusion–convection–reaction equation (7.1) is given by: find $U^h \in \mathcal{U}$ such that

$$\mathcal{A}_G^h\big(U^h, V^h\big) = \big(F, V^h\big)_{S_h}, \quad \forall V^h \in \mathcal{V}, \tag{7.2}$$

where

$$\mathcal{A}_G^h(U^h, V^h) = \sum_{i\in\sigma} V_i^h \mathcal{A}_G^h(U, \Psi_i^h)$$

with

$$\mathcal{A}_G^h(U^h, \Psi_i^h) = \int\limits_{\partial K_i^h} \left[-A(\mathbf{x})\nabla_{s_h} U^h(\mathbf{x}) + U^h(\mathbf{x})\vec{\mathbf{V}}(\mathbf{x})\right]\cdot \vec{\mathbf{n}}_{K_i^h}\, d\gamma_h + \int\limits_{K_i^h} B(\mathbf{x})U^h(\mathbf{x})\, ds_h.$$

For two adjacent vertices $\mathbf{x}_i$ and $\mathbf{x}_{i_j}$, let $\Gamma_{i,i_j} = K_i^h \cap \mathcal{K}_{i_j}^h$ and $\beta_{i,i_j} = \int_{\Gamma_{i,i_j}} \vec{\mathbf{V}}\cdot\vec{\mathbf{n}}_{K_i^h}\, d\gamma_h$, we may then approximate the convection term by

$$\int\limits_{\partial K_i^h} U^h(\mathbf{x})\vec{\mathbf{V}}(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_i^h}\, d\gamma_h \approx \sum_{i_j\in\chi_i} \frac{\beta_{i,i_j}}{2}\left(U_i^h + U_{i_j}^h\right).$$

Adding the above into the linear system (3.8), then we get the following system for the finite volume solution of (7.1):

$$\sum_{i_j\in\chi_i}\left[-p_{i,i_j}\left(U_{i_j}^h - U_i^h\right) + \frac{\beta_{i,i_j}}{2}\left(U_i^h + U_{i_j}^h\right)\right] + m\left(K_i^h\right)B_i U_i^h = m\left(K_i^h\right)F_i, \quad \text{for } i\in\sigma. \tag{7.3}$$

The scheme is expected to enjoy second order convergence in L^2 norm and first order in H^1 norm as in the case of planar problems when $|\vec{\mathbf{v}}/a|$ is not too large. For convection-dominated cases ($|\vec{\mathbf{v}}/a| \gg 1$), in order to eliminate the non-physical oscillations of the approximate solution, an upwind finite volume discretization for the convection term can be given by

$$\int\limits_{\partial K_i^h} U^h(\mathbf{x})\vec{\mathbf{V}}(\mathbf{x})\cdot\vec{\mathbf{n}}_{K_i^h}\, d\gamma_h \approx \sum_{i_j\in\chi_i}\left(\beta_{i,i_j}^+ U_i^h + \beta_{i,i_j}^- U_{i_j}^h\right),$$

where $\beta_{i,i_j}^+ = \max(0, \beta_{i,i_j})$ and $\beta_{i,i_j}^- = \max(0, -\beta_{i,i_j})$. The upwind scheme also leads to a linear system whose coefficient matrix is again an M-matrix. It is expected that at most linear convergence may be observed for the upwind scheme when the error is measured using either L^2 or H^1 norm.

A rigorous analysis of the convergence of the finite volume discretization (7.2) for Eq. (7.1) is not carried out here as it requires no essential difference in the analytic techniques than those considered in this paper. Numerical experiments, nevertheless, are given in the next section to demonstrate the effectiveness of the finite volume schemes.

8. Numerical experiments

To illustrate the method and to access the sharpness of the convergence rates proved in the proceeding sections, numerical experiments are performed for some model geometries with a given exact solution $u = u(\mathbf{x})$ of Eq. (2.1).

The design of a sequence of triangulations with increasing levels of resolutions is a challenging research subject in its own right. Here, to ensure the accurate evaluation of the convergence rate, all meshes of the surface $\mathbf{S}$ used in our experiments for discretization are generated by the so-called constrained centroidal Voronoi Delaunay triangulation (CCVDT) algorithm [14] with a uniform density function, see [14,20,21] and further discussions in the following section. If n_i denotes the number of nodes of the mesh at the ith level and $u^{h,i}$ the corresponding discrete solution, we calculate the convergence rate CR with respect to the norm $\|\cdot\|$ by

$$\mathrm{CR} = \frac{2\log(\|u - u^{h,i}\|/\|u - u^{h,i-1}\|)}{\log(n_{i-1}/n_i)}.$$

Applications to both the diffusion–reaction equation (2.1), and the diffusion–convection equation (7.1) are considered.

Example 1. The surface $\mathbf{S}$ is taken to be (see [23])

$$\mathbf{S} = \left\{\mathbf{x}\in\mathbb{R}^3 \mid \left(x_3 - x_2^2\right)^2 + x_1^2 + x_2^2 = 1,\, x_3 \geqslant x_2^2\right\} \tag{8.1}$$

with boundary

$$\partial\mathbf{S} = \left\{\left(x_1, x_2, x_2^2 + \sqrt{1 - x_1^2 + x_2^2}\right) \mid x_1^2 + x_2^2 = 1\right\}.$$

The outer normal at $\mathbf{x}\in\mathbf{S}$ is given by $\vec{\mathbf{n}}(\mathbf{x}) = \vec{t}/\|\vec{t}\|$ with $\vec{t} = (x_1, x_2(1 - 2(x_3 - x_2^2)), x_3 - x_2^2)$. Let the exact solution u be $u(\mathbf{x}) = x_1x_2$ with a corresponding Dirichlet boundary condition. Once the coefficients are specified, $f = f(\mathbf{x})$ is then obtained

Table 1
Computational results for the diffusion–reaction problem for Example 1.

Nodes	$h_{\max}$	$\|u-u^h\|_{L^\infty(\mathbf{S})}$	CR	$\|u-u^h\|_{L^2(\mathbf{S})}$	CR	$\|u-u^h\|_{H^1(\mathbf{S})}$	CR
64	0.4957	2.8357e−02	–	2.2658e−02	–	4.6767e−01	–
229	0.2719	8.7923e−03	1.84	5.8876e−03	2.11	2.2313e−01	1.16
865	0.1482	2.7364e−03	1.76	1.8197e−03	1.77	1.0807e−01	1.09
3361	0.0834	8.5325e−04	1.72	5.0669e−04	1.88	5.3484e−02	1.04
13249	0.0469	2.3333e−04	1.89	1.3132e−04	1.97	2.6639e−02	1.02

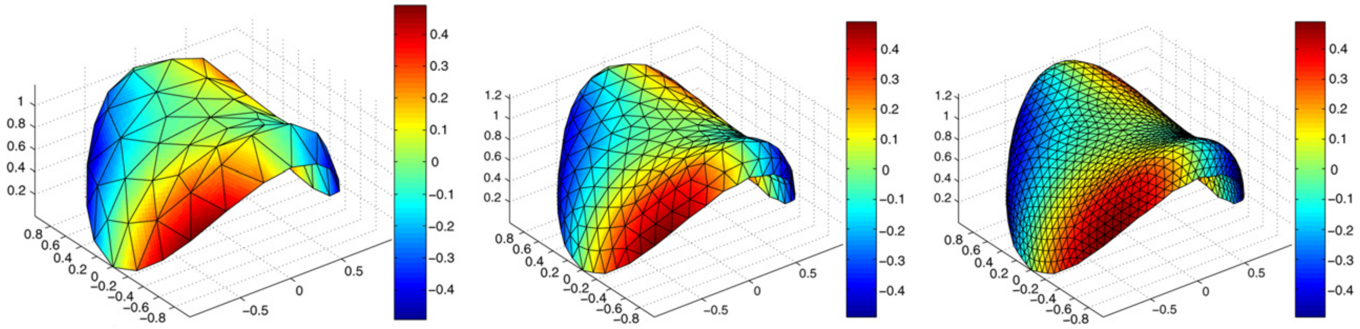

Fig. 2. Discrete solution u^h for the diffusion–reaction problem for Example 1 with 64, 229, 865 nodes respectively.

Table 2
Computational results for the problems having convection for Example 1.

Nodes	$\|u-u^h\|_{L^\infty(\mathbf{S})}$	CR	$\|u-u^h\|_{L^2(\mathbf{S})}$	CR	$\|u-u^h\|_{H^1(\mathbf{S})}$	CR
Case I						
64	2.4166e−02	–	1.5564e−02	–	4.2513e−01	–
229	8.2155e−03	1.69	4.3510e−03	2.00	2.1532e−01	1.07
865	1.7100e−03	2.36	8.8811e−04	2.39	1.0640e−01	1.06
3361	5.3521e−04	1.71	2.4013e−04	1.93	5.3208e−02	1.02
13249	1.5716e−04	1.79	5.5378e−05	2.14	2.6602e−02	1.01
Case II						
64	1.0921e−01	–	7.6897e−02	–	5.9924e−01	–
229	1.0120e−01	0.12	7.0858e−02	0.13	4.4851e−01	0.45
865	9.7551e−02	0.06	4.7239e−02	0.61	3.1186e−01	0.57
3361	5.8944e−02	0.74	2.6370e−02	0.86	1.7915e−01	0.82
13249	3.2107e−02	0.89	1.3978e−02	0.93	9.6005e−02	0.91

by substituting the exact solution into the equation. The mass-lumped scheme (3.8) is used in the implementation. We first let $a(\mathbf{x})=1$, $b(\mathbf{x})=1$ and $\vec{\mathbf{v}}(\mathbf{x})=\vec{0}$ (diffusion–reaction), followed by two cases with convection terms:

$$\text{Case I: } a(\mathbf{x})=1,\ \vec{\mathbf{v}}(\mathbf{x})=(2,1,1),\ b(\mathbf{x})=1.$$
$$\text{Case II: } a(\mathbf{x})=10^{-3},\ \vec{\mathbf{v}}(\mathbf{x})=(2,1,1),\ b(\mathbf{x})=1.$$

The central scheme is used for Case I, while for Case II, the upwind scheme is used.

The finite volume solutions are obtained on some CCVDT meshes with five different levels of resolution, namely, n_i is taken to be 64, 229, 865, 3361, 13249 respectively. The computational results are reported in Table 1 for the diffusion–reaction problem where h_{max} denotes the largest diameter of the surface mesh. The results match with our theoretical analysis nicely. Some meshes and corresponding discrete solutions are also plotted in Fig. 2, with the change in color representing the different values of the numerical solution.

The computational results for the problems having convection terms are reported in Table 2. It clearly shows that the approximate solutions of Case I (diffusion-dominated) solved by the central scheme have similar convergence rates as that of the diffusion–reaction problem. For the Case II (convection-dominated), due to the use of the upwind scheme, the convergence rates approach to 1 gradually in both L^2 and H^1 norms as the meshes are refined.

Example 2. Next, the surface $\mathbf{S}$ is chosen to be a torus such as

$$\mathbf{S}=\left\{\mathbf{x}\in\mathbb{R}^3 \;\middle|\; \left(x_1-\frac{(r_1+r_2)x_1}{2\rho}\right)^2+\left(x_2-\frac{(r_1+r_2)x_2}{2\rho}\right)^2+x_3^2=\frac{(r_2-r_1)^2}{4}\right\}, \tag{8.2}$$

Table 3
Computational results for the diffusion–reaction problem for Example 2.

Nodes	$h_{\max}$	$\|u-u^h\|_{L^\infty(\mathbf{S})}$	CR	$\|u-u^h\|_{L^2(\mathbf{S})}$	CR	$\|u-u^h\|_{H^1(\mathbf{S})}$	CR
74	0.4416	5.7660e−02	–	6.1200e−02	–	1.7190e+00	–
296	0.2398	2.2331e−02	1.37	2.0508e−02	1.58	8.8689e−01	0.95
1184	0.1359	5.2963e−03	2.08	4.5704e−03	2.17	4.4699e−01	0.99
4736	0.0749	1.5224e−03	1.81	1.1673e−03	1.97	2.2489e−01	0.99
18944	0.0408	3.9533e−04	1.95	2.9362e−04	1.99	1.1262e−01	1.00

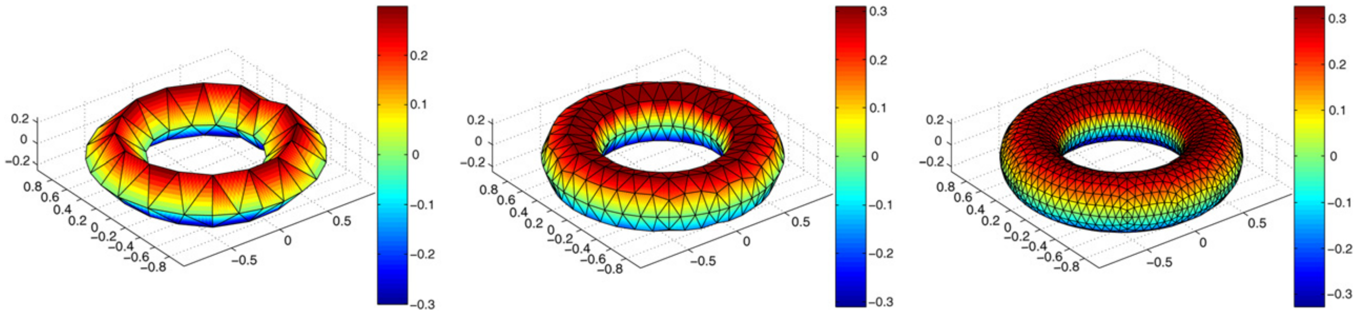

Fig. 3. Discrete solution u^h for the diffusion–reaction problem for Example 2 with 74, 296, 1184 nodes respectively.

Table 4
Computational results for the problems having convection for Example 2.

Nodes	$\|u-u^h\|_{L^\infty(\mathbf{S})}$	CR	$\|u-u^h\|_{L^2(\mathbf{S})}$	CR	$\|u-u^h\|_{H^1(\mathbf{S})}$	CR
Case I						
74	5.8692e−02	–	6.1233e−02	–	1.7191e+00	–
296	2.2404e−02	1.39	2.0798e−02	1.56	8.8667e−01	0.96
1184	5.3562e−03	2.06	4.6381e−03	2.16	4.4698e−01	0.99
4736	1.5672e−03	1.77	1.1968e−03	1.95	2.2489e−01	0.99
18944	3.9640e−04	1.98	2.9821e−04	2.00	1.1262e−01	1.00
Case II						
74	2.6547e−01	–	3.2088e−01	–	1.9942e+00	–
296	1.7306e−01	0.62	2.5127e−01	0.35	1.4635e+00	0.44
1184	1.2832e−01	0.43	1.6247e−01	0.63	1.1044e+00	0.41
4736	9.8184e−02	0.39	9.8201e−02	0.73	8.5308e−01	0.37
18944	7.1186e−02	0.46	5.6909e−02	0.79	6.7323e−01	0.35

where $\rho=\sqrt{x_1^2+x_2^2}$, $r_1=0.5$, $r_2=1$, and the outer normal at $\mathbf{x}\in\mathbf{S}$ is given by $\vec{\mathbf{n}}=\vec{t}/|\vec{t}|$ with

$$\vec{t}=\begin{pmatrix} (x_1-\tilde{x}_1)\big(1.0-\frac{r_1+r_2}{2\rho}+\frac{(r_1+r_2)x_1^2}{2\rho^3}\big)+(x_2-\tilde{x}_2)\big(\frac{(r_1+r_2)x_2x_1}{2\rho^3}\big) \\ (x_2-\tilde{x}_2)\big(1.0-\frac{r_1+r_2}{2\rho}+\frac{(r_1+r_2)x_2^2}{2\rho^3}\big)+(x_1-\tilde{x}_1)\big(\frac{(r_1+r_2)x_1x_2}{2\rho^3}\big) \\ x_3 \end{pmatrix},$$

where $\tilde{x}_1=(r_1+r_2)x_1/2\rho$, $\tilde{x}_2=(r_1+r_2)x_2/2\rho$. Clearly, this $\mathbf{S}$ has no boundary. We set the exact solution u to be $u(\mathbf{x})=x_2/\sqrt{x_1^2+x_2^2+x_3^2}$. We first solve the case with $a(\mathbf{x})=2+x_1x_2$, $b(\mathbf{x})=1+x_1^2+x_2^2+x_3^2$, and $\vec{v}=\vec{0}$, then we also test two cases of problems having convection terms:

Case I: $a(\mathbf{x})=2+x_1x_2$, $\vec{\mathbf{v}}(\mathbf{x})=(x_1,1+x_2,0)$, $b(\mathbf{x})=1+x_1^2+x_2^2+x_3^2$.
Case II: $a(\mathbf{x})=10^{-5}$, $\vec{\mathbf{v}}(\mathbf{x})=(x_1/4,1+x_2/4,0)$, $b(\mathbf{x})=1+x_1^2+x_2^2+x_3^2$.

The finite volume solution is solved on five levels of CCVDT meshes with $n_i=74$, 296, 1184, 4736 and 18944 respectively. The results are reported in Table 3 for the diffusion–reaction problem. Again they match with our theoretical analysis very well. The meshes and corresponding discrete solutions are plotted in Fig. 3.

The computational results for the cases having convection terms are reported in Table 4. It shows that the central scheme still works well for Case I. However the convergence rates of the upwind scheme are reduced to about 0.79 in L^2 norm and to about 0.34 at the last step for Case II, due to the very strong convection.

The above examples serve as numerical verifications of our theory. We note that the numerical schemes can naturally be implemented for more complex model equations on more general surfaces.

9. Conclusions and further discussions

In this paper, a finite volume method for solving second order elliptic PDEs on surfaces of arbitrary geometry has been studied using a piecewise linear complex representation of the surface. Optimal order error estimates have been proved under some mesh regularity assumptions and also been demonstrated through some numerical examples. For surface with complex geometry, a natural issue is how to generate a mesh with such regularity. To address this issues, let us briefly discuss the concept of constrained centroidal Voronoi tessellation (CCVT) [14] which are special Voronoi tessellations of the surface with the generators coincide with the constrained centroids of the corresponding Voronoi regions. Its duality is then the so-called constrained centroidal Voronoi Delaunay triangulation (CCVDT). The concept has been extended to the case constrained to a surface with the standard Euclidean metric [14] and also to the case of a one-sided distance function associated to a Riemannian metric [21]. Moreover, these extensions allow us to efficiently generate high quality surface unstructured meshes and triangulations. Applications to full 3d volume mesh generations and optimizations have also been explored, see [20].

The surface meshes produced using the CCVT technology tend to enjoy certain optimality properties. We refer to [22] for a review on the recent progress in this direction. For these surface meshes, the mesh regularity assumption is almost assured to be valid. Thus, they provide excellent surface meshes on which the finite volume methods can be further constructed. An example on the application of such meshes in connection to finite volume methods has been given in [16] where CCVT meshes on spherical surfaces have been used. Due to the excellent meshing quality, the finite volume solutions display superconvergent properties. We refer to recent works for extensive numerical experiments and applications [15–18].

There are some additional interesting questions related to the development of finite volume schemes of even higher order accuracy for smooth surfaces and solutions. Some research for the planar cases have been conducted in the literature, for example [38]. With singular surfaces and solutions, a posteriori error estimates and local mesh refinement can also be considered by generalizing the discussions in earlier works (see for instance [39]). Though it is shown here that the approximate surface representation does not degrade the optimal order error estimates of the finite volume methods for model second order elliptic equations, it is expected that similar conclusions hold for higher order equations [19] and other complex PDE systems. Connections with standard and mixed finite element methods [2,3,11], non-conforming and discontinuous finite element methods [1] can also be considered for problems on surfaces. Careful assessment on the performance of the different finite volume and finite element methods for some well designed bench-mark problems on surfaces will also be desirable for the future investigations.

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