NUMERICAL SOLUTION OF THE THREE-DIMENSIONAL GINZBURG–LANDAU MODELS USING ARTIFICIAL BOUNDARY

QIANG DU† AND XIAONAN WU‡

Abstract. For three-dimensional simulations of the vortex phenomena in superconductors based on the Ginzburg–Landau (GL) theory, the physical variables must be solved in the whole space in general. Exact boundary conditions on an artificial boundary are discussed to reformulate the problem in a finite domain. Approximations of these boundary conditions with high accuracy are given, and their convergence properties are examined. Error estimates are derived for the finite element approximations with the approximate boundary conditions.

Key words. superconductivity, Ginzburg–Landau equations, artificial boundary, unbounded domain, finite element method, convergence, $H^1$ and $L^2$ error estimates

AMS subject classifications. 65N99, 82D55

PII. S0036142997330317

1. Introduction. The Ginzburg–Landau (GL) models have been used extensively in studies of vortex phenomena in superconductivity [9, 26]. Let $\Omega \subset \mathbb{R}^3$ be a domain occupied by an isotropic, homogeneous superconducting sample, with $\kappa$ being its GL parameter, and the sample is subject to a constant applied magnetic field $H_0$. After nondimensionalization, the steady state GL model can be stated as a minimization problem of the free energy functional,

$$G(\psi, A) = \int_{\Omega} \frac{1}{2} (1 - |\psi|^2)^2 d\Omega + \int_{\Omega} \left| \frac{i}{\kappa} \nabla + A \right| \psi^2 d\Omega + \int_{\mathbb{R}^3} |\text{curl} A - H_0|^2 dx,$$

where the variable $\psi$ is the (complex-valued) order parameter and the variable $A$ is the magnetic vector potential.

Most of the numerical simulations using the GL models have focused on the solution of the two-dimensional versions of the GL equations in a finite domain. Such a reduction is valid when $\Omega$ is an infinite cylinder in the $z$ direction with a finite two-dimensional cross section and the applied field is constant and parallel to the $z$ axis. In the high-$\kappa$, high-field settings, as well as in the thin film setting, reductions of the three-dimensional problems in the whole space to the domain $\Omega$ have also been made [4, 8]. Nevertheless, for the general three-dimensional problems the interactions between the fields inside the superconducting sample and the external field are important. Various measurements of critical fields are also affected by the geometrical shape of the sample. To apply the GL theory in such situations, a coupled system of equations must be solved in both the sample and its exterior. When $\Omega$ is a bounded domain, this system of coupled equations consists of the nonlinear GL equations in...
the (bounded) interior of $\Omega$ and the linear Maxwell equations in the (unbounded) exterior $\Omega^e = \mathbb{R}^3 \setminus \Omega$ with far field conditions at infinity.

It has been shown that an effective approach for the solution of the exterior boundary value problems for general elliptic equations is to introduce an artificial boundary, impose some boundary conditions on the artificial boundary, and solve the problem in a bounded domain. The boundary condition on the artificial boundary can be imposed approximately or exactly. For related studies see, for example, [1, 2, 15, 16, 18, 19, 20, 21, 22, 24, 25] and the references cited therein.

In this paper, our main interests are the numerical approximations of the three-dimensional GL equations in the whole space. Let the boundary of $\Omega$ be denoted by $\Sigma$. Let $B_R = \{ x \mid |x| < R \}$ with $R > R_1$, the diameter of $\Omega$. We introduce an artificial boundary $\Sigma_R = \{ x \mid |x| = R \}$ in $\Omega^e$. Then the domain $\Omega^e$ is divided into two parts: the bounded part $\Omega_R = B_R \cap \Omega^e$ and the unbounded part $B_R^e$ (see Figure 1.1). Suitable artificial boundary conditions are constructed on the artificial boundary $\Sigma_R$, so that the original problem can be reduced to a problem in the bounded domain $\Omega_R$.

One may naturally attempt to apply a general theory for the interior-exterior coupled system of nonlinear PDEs to this problem. Nevertheless, the special features of the present problem will hardly be utilized. What makes the steady state GL models stand out from a typical variational problem is the so-called gauge invariance property (see the brief discussion in section 2 and in [9, 26]). The key idea in the present paper is to take advantage of the gauge invariance of the free energy functional to reduce the system of Maxwell equations in the exterior domain to a simple system of Laplace equations, thus allowing the application of known constructions of artificial boundary conditions for the Laplace equations. Meanwhile, this approach maintains the nice variational formulation of the original problem. The convergence of the finite element approximations with approximate artificial boundary conditions can be rigorously established along with both the $H^1$ and the $L^2$ error estimates for the approximations of the associated linearized equations as well as the original nonlinear GL models. The numerical methods discussed here can also be easily implemented.

The rest of the paper is organized as follows. In section 2, we briefly describe the properties of the GL models and their various equivalent forms. In section 3, we discuss the artificial boundary conditions on $\Sigma_R$ and verify the equivalence of the different formulations. In section 4, we study approximations of the artificial boundary conditions on $\Sigma_R$. The convergence property of such approximations is examined in section 5. A brief discussion of the finite element method for the solution of the reduced problem with the approximate artificial boundary conditions is presented in section 6. Error estimates for some associated linearized models are derived in section 7. Then in section 8, error estimates for the original GL models are given.
Numerical examples and solution algorithms are presented in section 9, and some concluding remarks are given in section 10.

2. Coupled interior-exterior problems. Although the basic mathematical properties of the GL models have been studied extensively, their formulation as a coupled interior-exterior problem has rarely been mentioned in the literature. Thus, we present some basic facts about the GL energy functional with emphasis on the coupling of interior and exterior domains. To begin our discussion, let us introduce an auxiliary variable $A_0$ such that \( \text{curl } A_0 = H_0 \) (the external applied magnetic field). Then, we redefine the functional \( G \) as

\[
G(\psi, A) = \int_\Omega \frac{1}{2} (1 - |\psi|^2) d\Omega + \int_\Omega \left| \left( \frac{i}{\kappa} \nabla + A + A_0 \right) \psi \right|^2 d\Omega + \int_{\mathbb{R}^3} |\text{curl } A|^2 dx,
\]

where \( \kappa \) is the GL parameter.

The GL functional has the so-called gauge invariance; that is, for any smooth function \( \phi \),

\[
G(\psi, A) = G(\psi e^{i\kappa \phi}, A + \nabla \phi).
\]

Numerical minimization of the free energy functional (even without the consideration of the external energy) is made difficult due to the gauge invariance. For this reason, it is convenient to look for minimizers in the so-called Coulomb gauge, so that \( \text{div } A = 0 \). Details may be found in [9, 4]. We use standard notation for Sobolev spaces and use notation such as \( \mathcal{H}^1(\Omega) \) and \( \mathcal{H}^1(\Omega) \) to denote the corresponding complex or vector-valued function spaces. Let \( \mathcal{H}(\mathbb{R}^3) \) be the completion of the space \( C_0^\infty(\mathbb{R}^3) \) under the norm

\[
||A||_H = \left( \int_{\mathbb{R}^3} |\nabla A|^2 dx \right)^{1/2}.
\]

The minimization of \( G \) is formalized as a search for minimizers in the space \( \mathcal{H}^1(\Omega) \times \mathcal{H}_d(\mathbb{R}^3) \), where \( \mathcal{H}_d \) is a subspace of \( \mathcal{H} \) defined by

\[
\mathcal{H}_d = \{ A \in \mathcal{H} \mid \text{div } A = 0 \}.
\]

Again, working with divergence-free spaces limits the numerical methods that can be used in the approximations. However, a nice remedy has been developed to avoid such pitfalls [9].

Define

\[
\tilde{F}(\psi, A) = G(\psi, A) + \int_{\mathbb{R}^3} |\text{div } A|^2 dx.
\]

Note that by integration by parts, we see that \( ||\text{div } A||_{L^2(\mathbb{R}^3)}^2 + ||\text{curl } A||_{L^2(\mathbb{R}^3)}^2 = ||A||_H^2 \). So, we may also write

\[
\tilde{F}(\psi, A) = \int_\Omega \frac{1}{2} (1 - |\psi|^2)^2 d\Omega + \int_\Omega \left| \left( \frac{i}{\kappa} \nabla + A + A_0 \right) \psi \right|^2 d\Omega + \int_{\mathbb{R}^3} |\nabla A|^2 dx.
\]

Similar to [4, 9], it can be shown that the following lemma holds.

**Lemma 2.1.** Any minimizer \( (\psi, A) \) of \( \tilde{F}(\psi, A) \) is also a minimizer of \( G(\psi, A) \) in \( \mathcal{H}^1(\Omega) \times \mathcal{H}_d \).
Proof. For any test function $\phi \in C_0^\infty (\mathbb{R}^3)$, let us consider $\tilde{F}(\psi e^{i \kappa \phi}, A + \nabla \phi)$. By gauge invariance, we have

$$\tilde{F}(\psi e^{i \kappa \phi}, A + \nabla \phi) = G(\psi, A) + \int_{\mathbb{R}^3} |\text{div} A + \Delta \phi|^2 dx.$$ 

Minimizing with respect to $\phi$, we deduce that $\text{div} A$ must be weakly harmonic in $\mathbb{R}^3$, and since it is also in $L^2(\mathbb{R}^3)$, we get that $\text{div} A = 0$; thus, $A \in H_d$. This easily leads to the conclusion given in the lemma. \qed

Thus, to solve for the minimizers of the original GL functional, we can solve for the critical points of the functional $\tilde{F}$. The Euler–Lagrange equations give the following coupled system of equations:

\begin{align*}
(\text{2.4}) & \quad \left( \frac{i}{\kappa} \nabla + A + A_0 \right)^2 \psi - \psi + |\psi|^2 \psi = 0 \quad \text{in} \ \Omega, \\
(\text{2.5}) & \quad -\Delta A = -\frac{i}{2\kappa} (\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 (A + A_0) \quad \text{in} \ \Omega, \\
(\text{2.6}) & \quad -\Delta A = 0 \quad \text{in} \ \Omega^c, 
\end{align*}

where $(\cdot)^*$ denotes the complex conjugate, along with natural boundary conditions on the boundary $\Gamma$

\begin{align*}
(\text{2.7}) & \quad \left( \frac{i}{\kappa} \nabla \psi + A \psi + A_0 \psi \right) \cdot n = 0 \quad \text{on} \ \Gamma = \partial \Omega, \\
(\text{2.8}) & \quad [\nabla A] = 0, \ [A] = 0 \quad \text{on} \ \Gamma, 
\end{align*}

and far field conditions

\begin{align*}
(\text{2.9}) & \quad \nabla A \to 0, \ A \to 0 \quad \text{at} \ \infty. 
\end{align*}

Here, $n$ is the unit outer normal vector to $\Gamma$ and $[\cdot]$ denotes the jump of function across the boundary. More general boundary conditions may also be considered; see [9].

3. Artificial boundary conditions on $\Sigma_R$. For a bounded domain $D$ with exterior $D^c$, let us define $H_\Delta(D)$ to be the completion of

$$\{ A \in C_0^\infty (\mathbb{R}^3) \mid \Delta A = 0 \ \text{in} \ D \}$$

under the norm $\| \cdot \|_H$. Clearly, functions in $H_\Delta(D)$ are weakly harmonic in $D$. Using the variational argument, it is easy to see the following.

**Lemma 3.1.** Let $(\psi, A)$ be a minimizer of

$$\min_{(\psi, A) \in H^1(\Omega) \times H(\mathbb{R}^3)} \tilde{F}(\psi, A).$$

Then, $A \in H_\Delta(\Omega^c)$. 

To proceed, we need some notation and properties of Legendre functions. For any point \( x = (x_1, x_2, x_3) \) in the Cartesian coordinates, \((r, \theta, \varphi)\) denotes the spherical coordinate, which is given by

\[
\begin{align*}
x_1 &= r \sin \theta \cos \varphi, \quad 0 \leq \theta \leq \pi, \\
x_2 &= r \sin \theta \sin \varphi, \quad 0 \leq \varphi \leq 2\pi, \\
x_3 &= r \cos \theta.
\end{align*}
\]

Let \( P_0^n(t) = P_n(t) \) be the Legendre polynomial of degree \( n \),

\[
P_0^n(t) = \frac{1}{2^n n!} \int_0^t d^n (t^2 - 1)^n.
\]

\( P_n^m(t) \) is the Legendre function,

\[
P_n^m(t) = (1 - t^2)^{\frac{m}{2}} \frac{d^m}{dt^m} P_n^m(t).
\]

By the addition theorem of Legendre functions, we have the following.

**Lemma 3.2.** Let \( 0 \leq \gamma \leq \pi \) satisfy \( \cos \gamma = \cos \xi \cos \theta + \sin \xi \sin \theta \cos(\psi - \varphi) \).

Then

\[
(3.2) \quad P_n^0(\cos \gamma) = P_0^n(\cos \xi) P_0^n(\cos \theta) + 2 \sum_{m=1}^n \frac{(n-m)!}{(n+m)!} P_m^n(\cos \xi) P_m^n(\cos \theta) \cos m(\psi - \varphi).
\]

Geometrically, \( \gamma \) denotes the angle between vectors \( x = (r, \theta, \varphi) \) and \( y = (r, \zeta, \psi) \) in the spherical coordinate. Let \( u(x) \) (\( u(r, \theta, \varphi) \) in the spherical coordinate) denote the solution of

\[
(3.3) \quad -\Delta u = 0 \quad \text{in} \quad B_R^n,
\]

\[
(3.4) \quad u|_{\Sigma_R} = u(R, \theta, \varphi),
\]

\[
(3.5) \quad u \to 0 \quad \text{when} \quad r \to \infty,
\]

where \( B_R^n = \mathbb{R}^3 \setminus B_R \) and \( u(R, \theta, \varphi) \) is the value of \( u \) on the artificial boundary. By [23] the solution \( u \) of (3.3)–(3.5) on the domain \( B_R^n \) is given by

\[
u = \frac{c_{00}}{2} \frac{R}{r} + \sum_{n=1}^{\infty} \left( \frac{R}{r} \right)^{n+1} \left[ \frac{c_{0n}}{2} P_0^n(\cos \theta) + \sum_{m=1}^n P_m^n(\cos \theta) (c_{nm} \cos m\varphi + d_{nm} \sin m\varphi) \right]
\]

with

\[
(3.7) \quad c_{nm} = \frac{(2n+1)(n-m)!}{2\pi (n+m)!} \int_0^{2\pi} \int_0^{\pi} u(R, \xi, \psi) P_{n}^m(\cos \xi) \cos m\psi \sin \xi \, d\xi \, d\psi
\]

\[
= \frac{(2n+1)(n-m)!}{2\pi R^2 (n+m)!} \int_{\Sigma_R} u(y) P_{n}^m(\cos \xi) \cos m\psi \, ds_y, \quad n, m = 0, 1, 2, \ldots,
\]

\[
(3.8) \quad d_{nm} = \frac{(2n+1)(n-m)!}{2\pi (n+m)!} \int_0^{2\pi} \int_0^{\pi} u(R, \xi, \psi) P_{n}^m(\cos \xi) \sin m\psi \sin \xi \, d\xi \, d\psi
\]

\[
= \frac{(2n+1)(n-m)!}{2\pi R^2 (n+m)!} \int_{\Sigma_R} u(y) P_{n}^m(\cos \xi) \sin m\psi \, ds_y, \quad n, m = 0, 1, 2, \ldots,
\]
where \( ds_y = R^2 \sin \xi \, d\psi \, d\xi \). Furthermore, let \( n \) be the unit exterior normal to the boundary \( \Sigma_R \) of \( B_R \). We have

\[
\frac{\partial u}{\partial n} \bigg|_{\Sigma_R} = \frac{c_{00}}{2R} + \sum_{n=1}^{\infty} \left( \frac{n+1}{R} \right) \left[ c_{n0} \frac{P_0^n}{2} (\cos \theta) + \sum_{m=1}^{n} P_n^m (\cos \theta) (c_{nm} \cos m\varphi + d_{nm} \sin m\varphi) \right].
\]

(3.9)

Substituting (3.7) and (3.8) into (3.9), we obtain

\[
\frac{\partial u}{\partial n} \bigg|_{\Sigma_R} = \frac{1}{4\pi R} \int_0^{2\pi} \int_0^\pi u(R,\xi,\psi) \sin \xi \, d\xi \, d\psi + \sum_{n=1}^{\infty} \left( \frac{(n+1)(2n+1)}{4\pi R} \right) \int_0^{2\pi} \int_0^\pi u(R,\xi,\psi) \cdot \left[ P_n^0 (\cos \xi) P_n^0 (\cos \theta) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m (\cos \xi) P_n^m (\cos \theta) \cos m(\psi - \varphi) \right] \sin \xi \, d\xi \, d\psi.
\]

By (3.3), we have

\[
\frac{\partial u}{\partial n} \bigg|_{\Sigma_R} = \frac{1}{4\pi R} \int_0^{2\pi} \int_0^\pi u(R,\xi,\psi) \sin \xi \, d\xi \, d\psi + \sum_{n=1}^{\infty} \left( \frac{(n+1)(2n+1)}{4\pi R} \right) \int_0^{2\pi} \int_0^\pi u(R,\xi,\psi) P_n (\cos \gamma) \sin \xi \, d\xi \, d\psi
\]

(3.10)

\[
= \int_{\Sigma_R} G(\gamma) u(y) \, ds_y,
\]

where \( \gamma \) is the angle between \( x = (R,\theta,\varphi) \) and \( y = (R,\zeta,\psi) \) given in the spherical coordinate and

\[
G(\gamma) = \frac{1}{4\pi R^3} + \sum_{n=1}^{\infty} \left( \frac{(n+1)(2n+1)}{4\pi R^3} \right) P_n (\cos \gamma).
\]

(3.11)

The function \( G(\gamma) \) is sometimes referred as the Hadamard kernel, which can also be calculated using the Green’s function defined on the domain \( B_R(0) \) [6]. This leads to a simpler expression

\[
G(\gamma) = \frac{1 - 4 \sin^2 (\gamma/2)}{32 \pi R^3 |\sin^3 (\gamma/2)|}.
\]

(3.12)

In terms of the position vectors \( x, y \), it can also be written as

\[
G(\gamma) = K(x, y) := \frac{3x \cdot y - 2}{4\pi |x - y|^3} \quad \forall \, x, y \in \Sigma_R.
\]

(3.13)

Thus, using integration by parts, we have the following.

**Lemma 3.3.** For \( A \in \mathcal{H}_\Delta(B_R) \),

\[
\int_{B_R} |\nabla A|^2 \, dx = \int_{\Sigma_R} \int_{\Sigma_R} G(\gamma) A(x) A(y) \, ds_x \, ds_y,
\]

(3.13)

where \( \Sigma_R = \partial B_R \), \( ds_x = R^2 \sin \theta \, d\varphi \, d\theta \), and \( ds_y = R^2 \sin \xi \, d\psi \, d\xi \).
Consequently, we may modify the energy functional as

\[
\mathcal{F}(\psi, A) = \int_{\Omega} \frac{1}{2} |\psi|^2 d\Omega + \int_{\Omega} \left( \frac{i}{\kappa} \nabla + A + A_0 \right)^2 |\psi|^2 d\Omega + \int_{B_R} |\nabla A|^2 dx + \int_{\Sigma_R} G(\gamma) A(x) A(y) ds_x ds_y
\]

and consider the problem

\[
\min_{(\psi, A) \in H^1(\Omega) \times H^1(B_R)} \mathcal{F}(\psi, A).
\] (3.14)

Minimizers of the above functional satisfy (2.4)–(2.8) in the region \(B_R\) with an artificial boundary condition which is the exact boundary condition satisfied by the solution \(u\) of the original equations on the given artificial boundary \(\Sigma_R\):

\[
\frac{\partial A}{\partial n} \bigg|_{\Sigma_R} = \mathcal{J}(A),
\] (3.15)

where for any \(x\) on \(\Sigma_R\),

\[
\mathcal{J}(A)(x) = \frac{1}{4\pi R^3} \int_{\Sigma_R} A(y) dy + \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{4\pi R^3} \int_{\Sigma_R} A(y) P_n(\cos \gamma) dy,
\] (3.16)

In the literature, this type of boundary condition is also referred as the \(DtN\) type boundary condition which has been studied by others, e.g., [18].

For any minimizer of (3.14), we may define a continuous extension of \(A\) such that

\[
\Delta A = 0 \quad \text{in} \quad B_R^c,
\]

\[
A \to 0 \quad \text{as} \quad x \to \infty.
\]

Then, by boundary condition (3.15), we get \(A \in H\). Furthermore, since for any \(A \in H_{\Delta}(\Omega)\), we have

\[
\mathcal{F}(\psi, A) = \tilde{\mathcal{F}}(\psi, A),
\]

it is therefore easy to verify the following.

**Theorem 3.4.** The above-defined extension of the minimizer \((\psi, A)\) is a minimizer of (3.1). The restriction of any minimizer \((\psi, A)\) of (3.1) on the domain \(B_R\) also gives a minimizer of (3.14).

In the following discussion, we look for numerical algorithms for the solution of (3.14).

4. **Approximate artificial boundary conditions on \(\Sigma_R\).** We now define the bilinear forms \(b(u, w)\) and \(b_n(u, w)\). For given functions \(u, w \in H^{1/2}(\Sigma_R)\), let

\[
b(u, w) = \int_{\Sigma_R} \int_{\Sigma_R} G(\gamma) u(x) w(y) ds_x ds_y.
\] (4.1)
Recalling the definition of $G$, we define

\[ b_N(u, w) = \int_{\Sigma_R} \int_{\Sigma_R} G_N(\gamma) u(x) w(y) \, ds_x \, ds_y, \]

where

\[ G_N(\gamma) = \frac{1}{4\pi R^3} + \sum_{n=1}^{N} \frac{(n+1)(2n+1)}{4\pi R^3} P_n(\cos \gamma). \]

For vector-valued functions $u$, $w$, we extend the bilinear forms $b(u, w)$ and $b_N(u, w)$ by simply using the scalar product in the definitions (4.1)–(4.2).

We may now approximate the energy functional

\[ F(\psi, A) = \int_{\Omega} \frac{1}{2} (1 - |\psi|^2)^2 \, d\Omega + \int_{\Omega} \left| \left( \frac{i}{\kappa} \nabla + A + A_0 \right) \psi \right|^2 \, d\Omega \]

by

\[ F_N(\psi, A) = \int_{\Omega} \frac{1}{2} (1 - |\psi|^2)^2 \, d\Omega + \int_{\Omega} \left| \left( \frac{i}{\kappa} \nabla + A + A_0 \right) \psi \right|^2 \, d\Omega \]

\[ + \int_{B_R} |\nabla A|^2 \, dx + b_N(A, A). \]

An approximate solution is given by

\[ \min_{(\psi, A) \in H^1(\Omega) \times H^1(B_R)} F_N(\psi, A). \]

Candidate minimizers satisfy the GL equations (2.4)–(2.8) with an approximate boundary condition on $\Sigma_R$:

\[ \frac{\partial A}{\partial n} \bigg|_{\Sigma_R} = J_N(A) := \int_{\Sigma_R} G_N(\gamma) A(y) \, ds_y. \]

Notice that for $N = 0$, the boundary condition coincides with the lowest order local approximation of the artificial boundary conditions given in [2].

5. Convergence of the approximate solutions. We first investigate the properties of the bilinear forms $b$ and $b_N$.

For any function given by

\[ w = \frac{e_00}{2} + \sum_{n=1}^{\infty} \left[ \frac{e_n0}{2} P_n^0(\cos \theta) + \sum_{m=1}^{n} P_m^0(\cos \theta) (e_{nm} \cos m\phi + f_{nm} \sin m\phi) \right], \]

we know that

\[ w \in H^{1/2}(\Sigma_R) \Leftrightarrow \]

\[ \|w\|^2 = \pi R e_0^2 + \sum_{n=1}^{\infty} \left\{ \pi R(n+1)e_n^2 + \sum_{m=1}^{n} \frac{2\pi R(n+1)(n+m)!}{(2n+1)(n-m)!} (e_{nm}^2 + f_{nm}^2) \right\} < +\infty, \]
where $\|w\|$ is an equivalent norm of $w$ in $H^{1/2}(\Sigma_R)$. For given $u, w \in H^1(\Omega_R)$ we know that the traces $u|_{\Sigma_R}, w|_{\Sigma_R} \in H^{1/2}(\Sigma_R)$, and we suppose that $w|_{\Sigma_R}$ is given by (5.1) and $u$ is given by (3.6). After some computation, we obtain

$$b(u, w) = \pi R c_{00} e_0 + \sum_{n=1}^\infty \left\{ \pi R(n+1) c_{n0} e_n + \frac{2\pi R(n+1)(n+m)!}{(2n+1)(n-m)!} (c_{nm} e_{nm} + d_{nm} f_{nm}) \right\},$$

$$b_N(u, w) = \pi R c_{00} e_0 + \sum_{n=1}^N \left\{ \pi R(n+1) c_{n0} e_n + \frac{2\pi R(n+1)(n+m)!}{(2n+1)(n-m)!} (c_{nm} e_{nm} + d_{nm} f_{nm}) \right\}.$$

Therefore, we have

$$|b(u, w)| \leq \|u|_{\Sigma_R}\|w|_{\Sigma_R}\|w\|_{1, \Omega_R},$$

$$|b_N(u, w)| \leq \|u|_{\Sigma_R}\|w|_{\Sigma_R}\|w\|_{1, \Omega_R},$$

where $M$ is a positive constant independent of $N$. For the bilinear forms $b(u, w)$ and $b_N(u, w)$ we also have the following.

**Lemma 5.1.** $b(u, w)$ and $b_N(u, w)$ are two symmetric, bounded bilinear forms on $H^1(\Omega_R) \times H^1(\Omega_R)$ and satisfy

$$b(w, w) = \|w|_{\Sigma_R}\|^2 \geq b_N(w, w) \geq b_0(w, w) \quad \forall w \in H^1(\Omega_R),$$

where $b_0(w, w) = \frac{1}{4\pi R^2} (\int_{\Sigma_R} w(x) \, ds_x)^2$.

The following inequality given in [23] will be used later.

**Lemma 5.2.** There exists a constant $C$, independent of $N$, such that for any $w \in H^1(B_R)$ and $u \in H^1(B_R) \cap H_\Delta(\Omega_{R1})$ for some $R_1 < R$ with $\Omega_{R1} = B_R \setminus B_{R_1}$ and $\Sigma_1 = \partial B_{R_1}$, we have

$$|b(u, w) - b_N(u, w)| \leq C \left( \frac{R_1}{R} \right)^{N+2} \|u\|_{1/2, \Sigma_{R_1}} \|w\|_{1/2, \Sigma_R}.$$

With an additional assumption, the above inequality may be strengthened.

**Lemma 5.3.** Let $R_1 < R$ with $\Omega_{R1} = B_R \setminus B_{R_1}$ and $\Sigma_1 = \partial B_{R_1}$. There exists a constant $C$, independent of $N$, such that for any $u, w \in H^1(B_R) \cap H_\Delta(\Omega_{R_1})$, we have

$$|b(u, w) - b_N(u, w)| \leq C \left( \frac{R_1}{R} \right)^{2N+3} \|u\|_{1/2, \Sigma_{R_1}} \|w\|_{1/2, \Sigma_R}.$$

**Proof of Lemma 5.3.** For $u, w \in H^1(B_R) \cap H_\Delta(\Omega_{R_1})$, let

$$u|_{\Sigma_R} = \frac{c_{00}}{2} + \sum_{n=1}^\infty \frac{c_{n0}}{2} P_n^0(\cos \theta) + \sum_{m=1}^n P_n^m(\cos \theta) (c_{nm} \cos m\varphi + d_{nm} \sin m\varphi),$$

$$w|_{\Sigma_R} = \frac{a_{00}}{2} + \sum_{n=1}^\infty \frac{a_{n0}}{2} P_n^0(\cos \theta) + \sum_{m=1}^n P_n^m(\cos \theta) (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi).$$
Then, with the notation that for any \( n \) and \( m \),
\[
\hat{c}_{nm} = c_{nm} \left( \frac{R}{R_1} \right)^{n+1}, \quad \hat{d}_{nm} = d_{nm} \left( \frac{R}{R_1} \right)^{n+1},
\]
\[
\hat{a}_{nm} = a_{nm} \left( \frac{R}{R_1} \right)^{n+1}, \quad \hat{b}_{nm} = b_{nm} \left( \frac{R}{R_1} \right)^{n+1},
\]
we get from (3.6) that
\[
\left| u \right|_{\Sigma_{R_1}} = \left\{ \sum_{n=1}^{\infty} \left[ \frac{\hat{c}_{00}}{2} P_n^{0}(\cos \theta) + \sum_{m=1}^{n} P_n^{m}(\cos \theta) (\hat{c}_{nm} \cos m \varphi + \hat{d}_{nm} \sin m \varphi) \right] \right\} \left( \frac{R}{R_1} \right)^{2N+3} \left\{ \sum_{n=1}^{\infty} \left( \pi R_1 (n+1) a_{n0}^2 + \sum_{m=1}^{n} \frac{2 \pi R_1 (n+1)(n+m)!}{(2n+1)(n-m)!} (\hat{c}_{nm}^2 + \hat{d}_{nm}^2) \right) \right\}^{\frac{1}{2}}
\]
\[
\left| w \right|_{\Sigma_{R_1}} = \left\{ \sum_{n=1}^{\infty} \left[ \frac{\hat{a}_{00}}{2} P_n^{0}(\cos \theta) + \sum_{m=1}^{n} P_n^{m}(\cos \theta) (\hat{a}_{nm} \cos m \varphi + \hat{b}_{nm} \sin m \varphi) \right] \right\} \left( \frac{R}{R_1} \right)^{2N+3} \left\{ \sum_{n=1}^{\infty} \left( \pi R_1 (n+1) b_{n0}^2 + \sum_{m=1}^{n} \frac{2 \pi R_1 (n+1)(n+m)!}{(2n+1)(n-m)!} (\hat{a}_{nm}^2 + \hat{b}_{nm}^2) \right) \right\}^{\frac{1}{2}}
\]
\[
\leq C \left( \frac{R}{R_1} \right)^{2N+3} \left\| u \right\|_{\Sigma_{R_1}} \left\| w \right\|_{\Sigma_{R_1}}
\]

where \( C \) is a constant independent of \( N \). We thus have proved the lemma. □

**Theorem 5.4.** Let \( (\psi_N, A_N) \) be a sequence of minimizers of \( F_N(\psi, A) \). Then

\[
\lim_{N \to \infty} F_N(\psi_N, A_N) = \min F(\psi, A).
\]

**Proof.** For large \( R \) and \( N \), there exists a constant \( C \), independent of \( N \), such that
\[
\left\| \nabla A_N \right\|_{0,B_R}^2 + b_N(A_N, A_N) \leq C.
\]

Using the Poincare–Friedrichs inequality, we get that \( \| A_N \|_{1,B_R} \) is uniformly bounded. Combining the uniform bound on the free energy and the Sobolev embedding theorem, this implies that \( \| \psi_N \|_{1,\Omega} \) is uniformly bounded. Using the weak compactness of \( H^1(\Omega) \) and \( H^1(B_n) \), we may, after extracting a subsequence, assume that
\[
\psi_N \rightharpoonup \psi
\]
and

$$A_N \rightharpoonup A$$

for some $\psi \in H^1(\Omega)$ and $A \in H^1(B_R)$. Since the embeddings $H^1(\Omega) \hookrightarrow L^p(\Omega)$ ($2 \leq p \leq 6$) and $H^1(B_R) \hookrightarrow H^{1/2}(\Sigma_R)$ are compact, after a further extracting subsequence, we have

$$\psi_N \rightharpoonup \psi \quad \text{in} \quad L^4(\Omega),$$

$$\psi_N A_N \rightharpoonup \psi A \quad \text{in} \quad L^2(\Omega),$$

and

$$A_N \rightharpoonup A \quad \text{in} \quad H^{1/2}(\Sigma_R).$$

Following from the norm equivalence of $\|\cdot\|$ with $\|\cdot\|_{1/2, \Sigma_R}$, the above strong convergence implies

$$|b_N(A_N, A_N) - b_N(A, A)| \to 0,$$

as $N \to \infty$. Meanwhile, we also have

$$|b_N(A, A) - b(A, A)| \to 0,$$

as $N \to \infty$. Thus

$$\lim_{N \to \infty} b_N(A_N, A_N) = b(A, A).$$

Consequently, one can easily show

$$\lim_{N \to \infty} F_N(\psi_N, A_N) \geq F(\psi, A).$$

Meanwhile,

$$\lim_{N \to \infty} F_N(\psi_N, A_N) \leq \lim_{N \to \infty} F_N(\psi, A) = F(\psi, A).$$

So, we have (5.4).

We now verify that $(\psi, A)$ is an energy minimizer of $F$. If not, there exists a minimizer $(\psi^*, A^*)$ such that

$$F(\psi^*, A^*) < F(\psi, A).$$

Since $A^* \in H_\Delta(\Omega)$, we get

$$|b_N(A^*, A^*) - b(A^*, A^*)| \to 0,$$

as $N \to \infty$. It follows that for large $N$,

$$F_N(\psi^*, A^*) < F_N(\psi_N, A_N).$$

This is a contradiction. The theorem is thus proved. \qed
6. Convergence of the finite element approximation. In this section we consider the finite element approximation of the variational problems (3.1) and (4.5). For problems where the variation of the exterior magnetic field is ignored, numerical approximations have been studied in [9, 10, 11, 12, 14] and the references therein. For simplicity we assume that \( \Omega \) is a convex polyhedral domain. Let \( \mathcal{V}_h \times \mathcal{V}_h \subset H^1(\Omega) \times H^1(B_R) \) be a finite element space. We consider the discrete problem

\[
\min_{(\psi^h, \mathbf{A}^h) \in \mathcal{V}_h \times \mathcal{V}_h} F_N(\psi^h, \mathbf{A}^h).
\]

Similar to the previous discussion, with the assumption that

\[
\bigcup_{h \to 0} \mathcal{V}_h \times \mathcal{V}_h \text{ is dense in } H^1(\Omega) \times H^1(B_R),
\]

we have the following.

**Theorem 6.1.** Let \((\psi^h_N, \mathbf{A}^h_N)\) be a sequence of minimizers of \(F_N(\psi, \mathbf{A})\) in \(\mathcal{V}_h \times \mathcal{V}_h\). Then

\[
\lim_{N \to \infty, h \to 0} F_N(\psi^h_N, \mathbf{A}^h_N) = \min_{H^1(\Omega) \times H^1(B_R)} F(\psi, \mathbf{A}).
\]

The proof follows along the same line as the proof of Theorem 5.4.

7. Error estimate for an associated linear problem. To prove an error estimate for the approximation given in (6.1), we use ideas similar to those presented in [9] for the finite element approximation of the GL equations in a bounded domain. To avoid excess repetition, we sketch only the main steps and emphasize the special features of the problems stated here, i.e., problem involving both the interior problem and the exterior problem. The framework given in [9] is based on results of [3] and [7] (see also [17]) concerning the approximation of a class of nonlinear problems.

As the first step, we study a related linear problem, for which the original nonlinear problem is treated as its compact perturbation in the appropriate sense. We first need the approximation properties of the finite element spaces given by

\[
\inf_{\psi^h \in \mathcal{V}_h} \| \psi - \psi^h \|_{1, \Omega} \leq C h^m \| \psi \|_{m+1, \Omega} \quad \forall \psi \in H^{m+1}(\Omega)
\]

for some appropriate integer \(m\) and

\[
\inf_{\mathbf{A}^h \in \mathcal{V}_h} \| \mathbf{A} - \mathbf{A}^h \|_{1, B_R} \leq C h^m \| \mathbf{A} \|_{m+1, B_R} \quad \forall \mathbf{A} \in H^{m+1}(B_R).
\]

One may consult [5] for conditions on the finite element partitions such that (7.1)–(7.2) are satisfied. Let us also use the following notation:

\[
X = H^1(\Omega) \times H^1(B_R), \quad X^h = \mathcal{V}_h \times \mathcal{V}_h,
\]

\[
Y = (H^1(\Omega))^\prime \times (H^1(\Omega))^\prime, \quad Z = L^{3/2}(\Omega) \times L^{3/2}(B_R),
\]

where \((\cdot)^\prime\) denotes the dual space. Note that \(Z \subset Y\) with a compact embedding.

Let the operator \(T \in \mathcal{B}(Y; X)\) (the space of bounded linear operators from \(Y\) into \(X\)) be defined in the following manner: \(T(\xi, \mathbf{P}) = (\theta, \mathbf{Q})\) for \((\xi, \mathbf{P}) \in Y\) and \((\theta, \mathbf{Q}) \in X\) if and only if

\[
\int \Omega (\nabla \theta \cdot \nabla \tilde{\psi}^* + \nabla \tilde{\psi} \cdot \nabla \theta^* + \theta \tilde{\psi}^* + \tilde{\psi} \theta^*) d\Omega = \int \Omega (\xi \tilde{\psi}^* + \xi^* \tilde{\psi}) d\Omega \quad \forall \tilde{\psi} \in H^1(\Omega).
\]
and

\begin{equation}
(7.4) \quad \int_{B_R} \nabla Q \nabla \tilde{A} \, dx + b(Q, \tilde{A}) = \int_{\Omega} P \cdot \tilde{A} \, d\Omega \quad \forall \; \tilde{A} \in H^1(B_R).
\end{equation}

It is easily seen that (7.3)–(7.4) are weak formulations of two uncoupled Poisson-type equations for \(\theta\) (with Neumann boundary condition) and \(Q\) (with the artificial boundary condition given before) and that \(T\) is the solution operator of these equations. Indeed, the corresponding PDEs can be written as

\[
-\Delta \theta + \theta = \xi \quad \text{in } \Omega,
\]
\[
\nabla \theta \cdot n = 0 \quad \text{on } \Gamma,
\]
\[
-\Delta Q = P \quad \text{in } \Omega,
\]
\[
-\Delta Q = 0 \quad \text{in } \Omega_R,
\]
\[
\frac{\partial Q}{\partial n} = J(Q) \quad \text{on } \Sigma_R.
\]

Similar to the approximation scheme given in the previous section, we consider the discrete operator \(T^h \in B(Y; X^h)\) as defined in the following manner:

\[
T^h(\xi, P) = (\theta^h, Q^h) \quad \text{for } (\xi, P) \in Y \text{ and } (\theta^h, Q^h) \in X^h
\]
and

\begin{equation}
(7.5) \quad \int_{\Omega} \lbrack \nabla \theta^h \cdot \nabla (\tilde{\psi}^h)^* + \nabla (\tilde{\psi}^h)^* \cdot \nabla (\theta^h)^* + \theta^h (\tilde{\psi}^h)^* + \tilde{\psi}^h (\theta^h)^* \rbrack \, d\Omega
\end{equation}

\[
= \int_{\Omega} \lbrack \xi (\tilde{\psi}^h)^* + \xi^* \tilde{\psi}^h \rbrack \, d\Omega \quad \forall \; \tilde{\psi}^h \in V^h
\]

and

\begin{equation}
(7.6) \quad \int_{B_R} \nabla Q^h \nabla \tilde{A}^h \, dx + b_{N(h)}(Q^h, \tilde{A}^h) = \int_{\Omega} P \cdot \tilde{A}^h \, d\Omega \quad \forall \; \tilde{A}^h \in V^h
\end{equation}

for some integer valued function \(N = N(h)\) which satisfies \(N(h) > 0\) and \(N(h) \to +\infty\) as \(h \to 0\).

The above equations consist of discretizations of the Poisson-type problems (7.3)–(7.4), respectively; also, \(T^h\) is the solution operator for these discrete problems. To analyze the approximation (7.6), we may use the techniques given in [23]. For the sake of completeness, we provide the following lemma.

**Lemma 7.1.** There exists some positive constant \(c > 0\) such that for some radius \(R_1 \in (0, R)\),

\begin{equation}
(7.7) \quad \|Q - Q^h\|_{1, B_R} \leq c \left\{ \inf_{\tilde{A}^h \in V^h} \|Q - \tilde{A}^h\|_{1, B_R} + \left( \frac{R_1}{R} \right)^N \|Q\|_{1, B_R} \right\}.
\end{equation}

**Proof.** For any \(\tilde{A}^h \in V^h\), using the fact that \(b_N(u, u) \geq b_0(u, u)\) for any \(u \in H^1(B_R)\) and the Poincare–Friedrichs inequality, we have, for some positive constant \(\alpha > 0\),

\[
\alpha \|\tilde{A}^h - Q^h\|^2_{1, B_R} \leq \int_{B_R} |\nabla (\tilde{A}^h - Q^h_N)|^2 \, dx + b_N(\tilde{A}^h - Q^h_N, \tilde{A}^h - Q^h_N)
\]
\[
\begin{align*}
&= \int_{B_R} \nabla (\tilde{A}^h - Q) \cdot \nabla (\tilde{A}^h - Q_N^h) \, dx + b_N(\tilde{A}^h - Q, \tilde{A}^h - Q_N^h) \\
&\quad + b_N(Q, \tilde{A}^h - Q_N^h) - b(Q, \tilde{A}^h - Q_N^h) \\
&\leq (1 + M)\|\tilde{A}^h - Q\|_{1,B_R} \|\tilde{A}^h - Q_N^h\|_{1,B_R} \\
&\quad + \left( \sup_{\tilde{A}^h \in V_h} \frac{|b_N(Q, \tilde{A}^h) - b(Q, \tilde{A}^h)|}{\|\tilde{A}^h\|_{1,B_R}} \right) \|\tilde{A}^h - Q_N^h\|_{1,B_R}.
\end{align*}
\]

Let \( R > R_1 > \text{diam}(\Omega) \). Using the inequality in Lemma 5.2, we have

\[
|b(Q, \tilde{A}^h) - b_N(Q, \tilde{A}^h)| \leq C(R_1, R) \left( \frac{R_1}{R} \right)^N \|Q\|_{1/2, \Sigma} \|\tilde{A}^h\|_{1,B_R},
\]

for some constant \( C(R_1, R) \), so the estimate in the lemma follows immediately from the trace theorem. \( \Box \)

We now consider also the \( L^2 \) error estimate for the problem (7.6). For standard finite element approximations such as the one given in (7.5), the \( L^2 \) error may be derived by the well-known duality argument. Because of the approximation of the bilinear form \( b(\cdot, \cdot) \), we need to modify the duality technique somewhat to derive the estimate for (7.6). As usual, we assume that the following \( H^2 \)-type regularity estimates hold: there exists a constant \( c \) such that, for any \( P \in L^2(B_R) \) with compact support, the solution of the problem

\[
-\Delta Q = P \quad \text{in } B_R, \quad \frac{\partial Q}{\partial n} = f(Q) \quad \text{on } \Sigma_R
\]

satisfies

\[
\|Q\|_{2,B_R} \leq c\|P\|_{0,B_R}.
\]

This assumption may follow from the regularity of the solution to the corresponding problem defined in the whole space.

The key to the \( L^2 \) estimate is to note that, to apply Lemma 5.2 or Lemma 5.3, one (or both) of the arguments in the bilinear form should be harmonic in the region next to the boundary. Thus, we let \( \phi \in C_0^\infty(B_R) \) be a smooth cut-off function, such that \( 0 \leq \phi(x) \leq 1 \) for any \( x \in B_R \) and \( R_0 \) is the diameter of the support of \( \phi \). Then we have the following.

**Lemma 7.2.** Given the function \( \phi \), for any radius \( R_1 \in (R_0, R) \), there exists some positive constant \( c > 0 \) such that

\[
\|\phi(Q - Q^h)\|_{0,B_R} \leq c \left\{ h^{m+1} + h^m \left( \frac{R_1}{R} \right)^N \right\} \|Q\|_{m+1,B_R} \\
+ c \left\{ h \left( \frac{R_1}{R} \right)^N + \left( \frac{R_1}{R} \right)^{2N} \right\} \|Q\|_{1,B_R}.
\]

**Proof.** Let \( E = Q - Q^h \). We consider the problem

\[
-\Delta A = \phi^2 E \quad \text{in } B_R, \quad \frac{\partial A}{\partial n} = f(A) \quad \text{on } \Sigma_R
\]
and its discretization
\[
\int_{B_R} \nabla A^h \nabla \tilde{A}^h \, dx + b_N(A^h, \tilde{A}^h) = \int_{B_R} \phi^2 E \cdot \tilde{A}^h \, d\Omega \quad \forall \tilde{A}^h \in \mathcal{V}^h.
\]

Then
\[
\|\phi E\|_{0,B_R}^2 = \int_{B_R} \nabla A \nabla E \, dx + b(A, E)
\]
\[
= \int_{B_R} \nabla (A - A^h) \nabla E \, dx + b(A, E) + \int_{B_R} \nabla A^h \nabla Q \, dx - \int_{B_R} \nabla A^h \nabla Q^h \, dx
\]
\[
= \int_{B_R} \nabla (A - A^h) \nabla E \, dx + b(A, E) + b_N(Q^h, A^h) - b(Q, A^h)
\]
\[
= \int_{B_R} \nabla (A - A^h) \nabla E \, dx + b_N(E, A - A^h) + b(A, E) - b_N(A, E)
\]
\[
+ b_N(Q, A) - b(Q, A) + b(Q, A - A^h) - b_N(Q, A - A^h).
\]

By the properties of the bilinear forms \(b\) and \(b_N\), we have
\[
\|\phi E\|_{0,B_R}^2 \leq c\|A - A^h\|_{1,B_R} \|E\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|A\|_{1,B_R} \|E\|_{1,B_R}
\]
\[
+ C(R_1, R) \left( \frac{R_1}{R} \right)^{2N} \|A\|_{1,B_R} \|Q\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|A - A^h\|_{1,B_R} \|Q\|_{1,B_R}
\]
\[
\leq ch \|A\|_{2,B_R} \|E\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|A\|_{1,B_R} \|E\|_{1,B_R}
\]
\[
+ C(R_1, R) \left( \frac{R_1}{R} \right)^{2N} \|A\|_{1,B_R} \|Q\|_{1,B_R}
\]
\[
+ C(R_1, R)h \left( \frac{R_1}{R} \right)^N \|E\|_{2,B_R} \|Q\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|E\|_{1,B_R} \|Q\|_{1,B_R}
\]
\[
\leq ch \|\phi E\|_{0,B_R} \|E\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|\phi E\|_{0,B_R} \|E\|_{1,B_R}
\]
\[
+ C(R_1, R)h \left( \frac{R_1}{R} \right)^N \|\phi E\|_{0,B_R} \|Q\|_{1,B_R} + C(R_1, R) \left( \frac{R_1}{R} \right)^N \|\phi E\|_{0,B_R} \|Q\|_{1,B_R},
\]

where we have used the regularity estimate and the fact that \(0 \leq \phi \leq 1\). Thus,
\[
\|\phi E\|_{0,B_R} \leq \left\{ ch + C(R_1, R) \left( \frac{R_1}{R} \right)^N \right\} \|E\|_{1,B_R}
\]
\[
+ C(R_1, R) \left[ h \left( \frac{R_1}{R} \right)^N + \left( \frac{R_1}{R} \right)^{2N} \right] \|Q\|_{1,B_R}
\]
\[
\leq ch^{m+1} \|Q\|_{m+1,B_R} + C_1(R_1, R) \left( \frac{R_1}{R} \right)^N h^m \|Q\|_{m+1,B_R}
\]
\[
+ C_2(R_1, R)h \left( \frac{R_1}{R} \right)^N \|Q\|_{1,B_R} + C_3(R_1, R) \left( \frac{R_1}{R} \right)^N \|Q\|_{1,B_R},
\]
where \(C_i(R_1, R) (i = 1, 2, 3)\) are all constants independent of \(h\) and \(N\). \(\square\)
Notice that the above proof does not provide a uniform bound for all cut-off functions; thus, the resulting $L^2$ estimate can be viewed only as an interior estimate.

To conclude, concerning the operators $T$ and $T^h$, we have the following result.

**Lemma 7.3.** The operator $T$ is well defined by (7.3)–(7.4). Let the finite element subspaces satisfy the inclusions $V^h \subset H^1(\Omega)$ and $V^h \subset H^1(B_R)$. Then, the operator $T^h$ is also well defined by (7.5)–(7.6). Let the finite element spaces satisfy the density property (6.2). Then, as $h \to 0$,

$$(7.13) \quad \|(T - T^h)(\xi, P)\|_X \to 0.$$  

Also, if the finite element spaces satisfy the approximation properties (7.1)–(7.2) and if $(\vartheta, Q) = T(\xi, P)$ satisfies $(\vartheta, Q) \in \mathcal{H}^{m+1}(\Omega) \times \mathcal{H}^{m+1}(B_R)$, then for some positive radius $R_1 < R$, we have

$$(7.14) \quad \|(T - T^h)(\xi, P)\|_X \leq c h^m(\|\vartheta\|_{m+1, \Omega} + \|Q\|_{m+1, B_R}) + C(R_1, R) \left(\frac{R_1}{R}\right)^{N(h)} \|Q\|_{1, B_R},$$

where $c$ and $C(R_1, R)$ are both positive constants independent of $h$.

**Proof.** By well defined we mean, for example, that $T$ does indeed belong to $\mathcal{B}(Y; X)$. The left-hand side of (7.5) defines a Hermitian, positive definite sesquilinear form on $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$, and the left-hand side of (7.6) defines a symmetric, positive definite bilinear form on $\mathcal{H}^1(B_R) \times \mathcal{H}^1(B_R)$. Moreover, whenever $(\xi, P) \in Y$, the right-hand sides of (7.5) and (7.6) define bounded linear functionals on $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^1(B_R)$, respectively. Thus, by the Lax–Milgram theorem, both (7.5) and (7.6) have unique solutions and the solution operator is bounded, i.e., the operator $T$ is well defined. Similarly, one can show that the operator $T^h$ is also well defined.

Standard finite element arguments applied to the pairs (7.3) and (7.5) imply that

$\|\vartheta - \vartheta^h\|_{1, \Omega} \leq \inf_{\tilde{\vartheta}^h \in V^h} \|\vartheta - \tilde{\vartheta}^h\|_{1, \Omega},$

and by Lemma 7.1,

$\|Q - Q^h\|_{1, B_R} \leq c \inf_{\tilde{\vartheta}^h \in V^h} \|Q - \tilde{\vartheta}^h\|_{1, B_R} + C(R_1, R) \left(\frac{R_1}{R}\right)^{N(h)} \|Q\|_{1, B_R}$

for some constants $c$ and $C(R_1, R)$ independent of $h$ as $h$ is small. Then, for $\vartheta \in \mathcal{H}^{m+1}(\Omega)$ and $Q \in \mathcal{H}^{m+1}(B_R)$, the theorem follows from the approximation properties (7.1)–(7.2). □

**8. Error estimates for the nonlinear problem.** Let $\Lambda$ be a compact subset of $\mathbb{R}_+$. We now briefly state the relevant results on the approximation of nonlinear problems in [3, 7, 17], specialized to our needs. The nonlinear problems under consideration are of the type

$$(8.1) \quad F(\kappa, u) \equiv u + TG(\kappa, u) = 0,$$

where $T \in \mathcal{B}(Y; X)$; $G$ is a $C^2$ mapping from $\Lambda \times X$ into $Y$.

Let $\Lambda$ be a compact interval of $\mathbb{R}$. We say that $\{(\kappa, u(\kappa)) : \kappa \in \Lambda\}$ is a branch of solutions of (8.1) if $\kappa \to u(\kappa)$ is a continuous function from $\Lambda$ into $X$ such that $F(\kappa, u(\kappa)) = 0$. The branch is called a regular branch if we also have that $D_\kappa F(\kappa, u(\kappa))$
is an isomorphism from \( X \) into \( X \) for all \( \kappa \in \Lambda \). Here, \( D_u \) denotes the Frechét derivative with respect to \( u \).

Approximations are defined by introducing a subspace \( X^h \subset X \) and an approximating operator \( T^h \in \mathcal{B}(Y; X^h) \). Then, we seek \( u^h \in X^h \) such that

\[
F^h(\kappa, u^h) \equiv u^h + T^h G(\kappa, u^h) = 0.
\]

If there exists another Banach space \( Z \), contained in \( Y \), with continuous embedding, such that

\[
D_u G(\kappa, u) \in \mathcal{B}(X; Z) \quad \forall \kappa \in \Lambda \text{ and } u \in X
\]

and for \( T^h \)

\[
\lim_{h \to 0} \| (T^h - T)v \|_X = 0 \quad \forall v \in Y;
\]

and, consequently, by the compact embedding \( Z \hookrightarrow Y \),

\[
\lim_{h \to 0} \| (T^h - T) \|_{\mathcal{B}(Z; X)} = 0,
\]

then by [3, 7, 17] we have the following.

**Theorem 8.1.** Let \( X \) and \( Y \) be Banach spaces and \( \Lambda \) a compact subset of \( \mathbb{R} \). Assume that \( G \) is a \( C^2 \) mapping from \( \Lambda \times X \) into \( Y \) and that all second Frechét derivatives of \( G \) are bounded on all bounded sets of \( \Lambda \times X \). Assume that (8.3)–(8.4) hold and that \( \{(\kappa, u(\kappa)); \kappa \in \Lambda\} \) is a branch of nonsingular solutions of (8.1). Then there exists a neighborhood \( \mathcal{O} \) of the origin in \( X \) and, for \( h \) sufficiently small, a unique \( C^2 \) function \( \kappa \mapsto u^h(\kappa) \in X^h \), such that \( \{(\kappa, u^h(\kappa)); \kappa \in \Lambda\} \) is a branch of nonsingular solutions of (8.2) and \( u^h(\kappa) - u(\kappa) \in \mathcal{O} \) for all \( \kappa \in \Lambda \). Moreover, there exists a constant \( C > 0 \), independent of \( h \) and \( \kappa \), such that

\[
\| u(\kappa) - u^h(\kappa) \|_X \leq C \|(T^h - T)G(\kappa, u(\kappa))\|_X \quad \forall \kappa \in \Lambda.
\]

Next, we define the *nonlinear* mapping \( G: \Lambda \times X \to Y \) as follows: \( G(\kappa, (\theta, Q)) = (\xi, P) \) for \( \kappa \in \Lambda \), \((\theta, Q) \in X\), and \((\xi, P) \in Y\) if and only if

\[
\int_{\Omega} \xi \tilde{\psi}^* \, d\Omega = \int_{\Omega} \left[ \kappa^2 |\theta|^2 - 1 + |Q + A_0|^2 - 1 \right] \theta \tilde{\psi}^* \, d\Omega
\]

\[
- i \int_{\Omega} (Q + A_0) \cdot \left[ \theta \nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \theta \right] \, d\Omega \quad \forall \tilde{\psi} \in H^1(\Omega)
\]

and

\[
\int_{\Omega} P \cdot \tilde{A} \, d\Omega = \int_{\Omega} \left( |\theta|^2 (Q + A_0) \cdot \tilde{A} + \frac{i}{2} (\theta^* \nabla \theta - \theta \nabla \theta^*) \cdot \tilde{A} \right) \, d\Omega \quad \forall \tilde{A} \in H^1(\mathcal{B}_h).
\]

It is easily seen, with the association \( u = (\psi, \kappa A) \), that the nonlinear system of GL equations with artificial boundary conditions is equivalent to

\[
(\psi, \kappa A) + T G(\kappa, (\psi, \kappa A)) = 0
\]

and that the equations satisfied by the solution of the discrete approximate problem are equivalent to

\[
(\psi^h, \kappa A^h) + T^h G(\kappa, (\psi^h, \kappa A^h)) = 0.
\]
Thus, we have cast our problem into the above framework. Throughout we make the assumption that the system (8.6) has a branch of regular solutions for \( \kappa \) belonging to a compact interval of \( \mathbb{R}_+ \).

Let \( D_2G(\kappa, (\psi, \kappa A)) \) denote the Frechét derivative of \( G \) with respect to the second argument \( (\psi, \kappa A) \). For given \((\theta, Q) \in X\), a direct computation yields that \((\xi, P) \in Y\) satisfies

\[
(\xi, P) = D_2G(\kappa, (\theta, Q))(\hat{\theta}, \hat{Q})
\]

for \((\hat{\theta}, \hat{Q}) \in X\) if and only if

\[
\int_{\Omega} \xi \tilde{\psi}^* \, d\Omega = \int_{\Omega} \left( [\kappa^2(\theta^2\hat{\theta}^* + 2\theta\hat{\theta}^* \hat{\theta} - \hat{\theta}) + (2\theta(Q + A_0) \cdot \hat{Q} + |Q + A_0|^2 \hat{\theta} - \hat{\theta})] \tilde{\psi}^* \right) \, d\Omega
\]

\[
- \int_{\Omega} \left( i(Q + A_0) \cdot [\hat{\theta}\nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \hat{\theta}] + i\hat{Q} \cdot [\theta \nabla \tilde{\psi}^* - \tilde{\psi}^* \nabla \theta] \right) \, d\Omega \quad \forall \tilde{\psi} \in \mathcal{H}^1(\Omega)
\]

and

\[
\int_{\Omega} P \cdot \hat{A} \, d\Omega = \int_{\Omega} \left( [\theta^2 \hat{Q} + (\theta \theta^* + \theta \hat{\theta}^*) (Q + A_0)] \cdot \hat{A} \right)
\]

\[
+ \left( \frac{i}{2} (\hat{\theta}^* \nabla \theta - \theta \nabla \hat{\theta}^* + \theta^* \nabla \hat{\theta} - \hat{\theta} \nabla \theta^*) \cdot \hat{A} \right) \, d\Omega \quad \forall \hat{A} \in \mathcal{H}^1(B_R).
\]

Similar to the derivation given in [9], we may verify that, for any \((\theta, Q) \in X\),

\[
D_2G(\kappa, (\theta, Q)) \in \mathcal{B}(Z; X),
\]

and, moreover, all second Frechét derivatives of \( G \) are bounded on bounded sets of \( \Lambda \times X \).

Hence, using Theorem 8.1, we are led to the following result.

**Theorem 8.2.** Assume that \( \Lambda \) is a compact interval of \( \mathbb{R}_+ \) and that there exists a branch \( \{\kappa, (\psi, A) : \kappa \in \Lambda\} \) of regular solutions of the system (8.6). Assume that the finite element spaces \( V^h \) and \( V^h \) satisfy the conditions (6.2). Then, there exists a neighborhood \( \mathcal{O} \) of the origin in \( X = \mathcal{H}^1(\Omega) \times \mathcal{H}^1(B_R) \) and, for \( h \) sufficiently small, a unique branch \( \{\kappa, (\psi^h, A^h) : \kappa \in \Lambda\} \) of solutions of the discrete system (8.7) such that \((\psi, A) - (\psi^h, A^h) \in \mathcal{O} \) for all \( \kappa \in \Lambda \). Moreover,

\[
\|\psi(\kappa) - \psi^h(\kappa)\|_{1, \Omega} + \|A(\kappa) - A^h(\kappa)\|_{1, B_R} \to 0
\]

as \( h \to 0 \), uniformly in \( \kappa \).

If, in addition, the solution of the system (8.6) satisfies \((\psi, A) \in \mathcal{H}^{m+1}(\Omega) \times \mathcal{H}^{m+1}(B_R) \) and the spaces \( V^h \) and \( V^h \) satisfy the conditions (7.1)–(7.2), then as \( h \to 0 \),

\[
\|\psi(\kappa) - \psi^h(\kappa)\|_{1, \Omega} + \|A(\kappa) - A^h(\kappa)\|_{1, B_R}
\]

\[
\leq C_1 h^m(\|\psi(\kappa)\|_{m+1, \Omega} + \|A(\kappa)\|_{m+1, B_R}) + C_2 \left( \frac{R_1}{R} \right)^{N(h)} \|A(\kappa)\|_{1, B_R}
\]

uniformly in \( \kappa \) for some constants \( C_1 \) and \( C_2 = C_2(R_1, R) \), independent of \( h \).
Remark. From the above error estimates, we see that as long as we use the expansion up to term
\[ (8.8) \quad N(h) \approx m \frac{\log(1/h)}{\log(R/R_1)} \quad \text{as } h \to 0, \]
in the approximation of the artificial boundary condition, the order of error remains comparable with the standard finite element approximations of problems in a bounded domain.

Remark. Using some additional results (that may be found in [3, 7, 17]) concerning the approximation of problems of the type (8.1) and the estimates for the associated linear problem given in the previous section, we may derive error estimates for \((\psi, \mathbf{A})\) in the \([\mathcal{L}^2(\Omega) \times \mathcal{L}^2(B_R)]\)-norm:
\[
\|\psi(\kappa) - \psi^h(\kappa)\|_{0,\Omega} + \|\phi(\mathbf{A}(\kappa) - \mathbf{A}^h(\kappa))\|_{0,B_R}
\leq C h^{m+1} \|\psi(\kappa)\|_{m+1,\Omega} + C h^{m+1} \|\mathbf{A}(\kappa)\|_{m+1,B_R}
+ C(R_1, R) h^m \left( \frac{R_1}{R} \right)^{N(h)} \|\mathbf{A}(\kappa)\|_{m+1,B_R}
+ C(R_1, R) \left\{ h \left( \frac{R_1}{R} \right)^{N(h)} + \left( \frac{R_1}{R} \right)^{2N(h)} \right\} \|\mathbf{A}(\kappa)\|_{1,B_R}
\]
for some constants \(C\) and \(C(R_1, R)\) and the cut-off function \(\phi\) defined in the last section with \(R_0\) being the diameter of its support and \(R_1 \in (R_0, R)\). For \(N(h)\) given by (8.8), we have the \(L^2\) error on the order of \(h^{m+1}\). In practice, we are most interested in the behavior of the solution inside the superconducting sample. Therefore, even though the estimate depends on the support of \(\phi\), by letting the support of \(\phi\) include the domain \(\Omega\), the estimate actually gives full order of accuracy of the \(L^2\) error in \(\Omega\).

9. Solution algorithms. Naturally, many other standard iterative methods for the solution of nonlinear systems of equations may also be employed to solve the nonlinear discrete system. For example, Newton’s method is defined as follows. Given a value for \(\kappa\) and an initial guess \((\psi^0, \mathbf{A}^0)\) for \((\psi^h, \mathbf{A}^h)\), the sequence of Newton iterates \(\{(\psi(s), \mathbf{A}(s))\}_{s \geq 1}\) is defined by
\[
T^{-1}(\psi(s+1), \mathbf{A}(s+1)) = D_2G(\kappa, (\psi(s), \mathbf{A}(s)))(\psi(s+1), \mathbf{A}(s+1))
= G(\kappa, (\psi(s), \mathbf{A}(s)) - D_2G(\kappa, (\psi(s), \mathbf{A}(s)))(\psi(s), \mathbf{A}(s)) \quad \text{for } s = 0, 1, \ldots .
\]
It can be shown (using techniques similar to those employed for the Navier–Stokes equations [17]) that if the initial guess \((\psi^0, \mathbf{A}^0)\) is “sufficiently” close to a nonsingular solution \((\psi^h, \mathbf{A}^h)\) of the discrete system, then the Newton iterates converge to \((\psi^h, \mathbf{A}^h)\) with a quadratic rate of convergence.

The implementation of Newton’s method requires the simultaneous solution of both the order parameter and the magnetic potential. The coefficient matrices also need to be updated at each iteration. In our numerical testing, we adopted the following scheme:
\[
\left( \frac{\psi^{n+1} - \psi^n}{\delta t}, \phi^h \right) + \left( \left( \frac{\nabla}{\kappa} - i(\mathbf{A}^n + \mathbf{A}_0) \right) \psi^{n+1}, \left( \frac{\nabla}{\kappa} - i(\mathbf{A}^n + \mathbf{A}_0) \right) \phi^h \right)
+ ((|\psi^n|^2 - 1)\psi^{n+1}, \phi^h) = 0 \quad \forall \phi^h \in V^h
\]
and
\[ \left\langle \frac{A^{n+1} - A^n}{\delta t}, Q^h \right\rangle + \langle \nabla A^{n+1}, \nabla Q^h \rangle + b_N(A^{n+1}, Q^h) = - \left( \frac{i}{2k}(\psi^{n+1} \nabla \psi^{n+1} - \psi^{n+1} \nabla \psi^{n+1}) + |\psi^{n+1}|^2(A^n + A_0), Q^h \right) \quad \forall Q^h \in V^h. \]

for \( n = 0, 1, \ldots \). Here, \( \delta t \) may be viewed as a time step or relaxation parameter which can be increased when the solution is approaching a steady state. \( \langle \cdot, \cdot \rangle \) is the inner product on \( \Omega \) while \( \langle \cdot, \cdot \rangle \) is the inner product on \( B_R \).

So, at each step \( n + 1 \), we may solve for the real and imaginary components of \( \psi \) simultaneously. Then, we may solve the components of \( A \) one by one. With respect to the latter, they share the same coefficient matrix and the matrix remains the same at all iterations. This gives significant savings in both computing time and memory for the assembly of the matrices.

It is not difficult to show that for given \( (\psi^n, A^n) \in V^h \times V^h \), if \( \delta t \) is small enough, then the solution map of the above iteration can be viewed as a contraction map; thus it gives a unique solution \( (\psi^{n+1}, A^{n+1}) \in V^h \times V^h \). We have implemented the above algorithm and, computationally, the iterations remain bounded even if we take a small, but constant, \( \delta t \) at all iteration steps. Consequently, the iteration converges to the finite element approximation of (8.7) as \( n \to \infty \).

We now present some preliminary results on the experimentation using the finite element approximation with approximate artificial boundary conditions. We use \( \ell \) to denote the coherent length [8, 12, 13, 26]. For convergence studies, the domain \( \Omega \) is taken to be of cubic shape: \( \Omega = [-0.9 \times \ell, 0.9 \times \ell] \). This implies that the dimension of \( \Omega \) is about twice the coherent length (hence a tiny sample). The GL parameter \( \kappa \) is set to be 3. The applied field is taken to be \( h\vec{t} \), where \( \vec{t} = (t_1, t_2, t_3) \) is a unit vector determining the orientation of the applied field and \( h \) is a positive scalar determining the strength of the field. For practical applications, it is often interesting to consider large values of \( \kappa \) over much larger physical samples. Our experimentation here is merely directed to demonstrate the convergence of the finite element approximation via the artificial boundary condition.

In the numerical testing, the triangulation of \( \Omega \) (with \( m \) grid points in each direction), along with a slightly larger cube \( C \) (with \( n \) grid points in each direction), is done first. The grid points fall in \( C \setminus \Omega \) and are then mapped into \( B_R \setminus \Omega \). Together, they provide a triangulation of \( B_R \). The continuous piecewise linear finite element spaces are used for both \( \psi \) and \( A \). Notice that the unknown \( \psi \) is only solved in \( \Omega \). The resulting linear system of equations was solved by direct solvers available in LAPACK for small grids and by a BICG routine for large grids.

To verify the convergence of our numerical methods, we have computed the solution using various sizes of the radius, \( R \), different numbers of grid points, \( m \) and \( n \), and with different numbers, \( N \), of boundary terms. We use a fixed applied magnetic field \( (0, 0, -8) \).

We first compute the numerical solution on a coarse grid \( (m = 7, n = 9) \). For the given grid and given \( R = 6\sqrt{3}l/5 \), the three-dimensional contour plots of the magnitude of the order parameter are given in Figure 9.1 for \( N = 0, 2, 4, 6 \). A single vortex tube [9, 26] is clearly evident in all the plots.

The same experiments were repeated for a finer grid \( (m = 13, n = 17) \) with the same radius \( R = 6\sqrt{3}l/5 \). The three-dimensional contour plots of the magnitude of the order parameter are given in Figure 9.2 for \( N = 0, 2, 4, 6 \).
Next, we fixed the number of boundary terms used as $N = 4$ and let $R$ vary. In Figure 9.3, on a coarse grid ($m = 7, n = 9$), the three-dimensional contour plots of the magnitude of the order parameter are given for $R = 1.8\sqrt{3}l, 1.5\sqrt{3}l, 1.2\sqrt{3}l$, and they are compared with the solutions obtained in the high-$\kappa$ model (see [8, 12]) on the same grid.

Similar results with a finer grid ($m = 13, n = 17$) are given in Figure 9.4, where the three-dimensional contour plots of the magnitude of the order parameter are given for $R = 1.2\sqrt{3}l, 1.05\sqrt{3}l$, and the high-$\kappa$ solution.

It has been demonstrated in [8, 12] that the high-$\kappa$ solution obtained from the high-$\kappa$, high-field model can give good approximations to the solution of the original GL models when the applied magnetic field is strong and $\kappa$ is large. Using the present model with artificial boundary conditions, numerical studies of the vortex phenomena can be made for strong as well as weak applied fields.
The visual differences can be seen in some of the figures while in others they cannot be easily noticed. To give qualitative information, in Tables 9.1 and 9.2 we compare the $L^2$ difference of the solutions for $N = 0, 2, 4, 6$ as well as the high-$\kappa$ solution with the solution for $N = 8$. The three errors $e_\psi, e_{\text{field}},$ and $e_{\text{sc}}$ are given for the differences of the magnitude of the order parameter, the magnetic field curl ($\nabla \times (\mathbf{A} + \mathbf{A}_0)$), and the supercurrent $\frac{1}{2\kappa}(\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2(\mathbf{A} + \mathbf{A}_0)$.

### Table 9.1
$L^2$ differences of the solutions ($m = 7, n = 9$).

<table>
<thead>
<tr>
<th>Error</th>
<th>$N = 0$</th>
<th>$N = 2$</th>
<th>$N = 4$</th>
<th>$N = 6$</th>
<th>High-$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_\psi$</td>
<td>2.12E-02</td>
<td>2.99E-04</td>
<td>2.11E-05</td>
<td>&lt; 1.0E-06</td>
<td>1.26E-01</td>
</tr>
<tr>
<td>$e_{\text{field}}$</td>
<td>2.50E-01</td>
<td>1.21E-02</td>
<td>2.51E-03</td>
<td>&lt; 1.0E-06</td>
<td>3.72E-01</td>
</tr>
<tr>
<td>$e_{\text{sc}}$</td>
<td>6.93E-02</td>
<td>1.01E-03</td>
<td>8.92E-05</td>
<td>&lt; 1.0E-06</td>
<td>1.14E-00</td>
</tr>
</tbody>
</table>

### Table 9.2
$L^2$ differences of the solutions ($m = 13, n = 17$).

<table>
<thead>
<tr>
<th>Error</th>
<th>$N = 0$</th>
<th>$N = 2$</th>
<th>$N = 4$</th>
<th>$N = 6$</th>
<th>High-$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_\psi$</td>
<td>1.84E-02</td>
<td>1.32E-04</td>
<td>1.37E-05</td>
<td>&lt; 1.00E-06</td>
<td>1.27E - 01</td>
</tr>
<tr>
<td>$e_{\text{field}}$</td>
<td>2.71E-01</td>
<td>1.15E-02</td>
<td>3.42E-03</td>
<td>&lt; 1.00E-06</td>
<td>4.33E-01</td>
</tr>
<tr>
<td>$e_{\text{sc}}$</td>
<td>5.19E-02</td>
<td>1.19E-03</td>
<td>1.00E-04</td>
<td>&lt; 1.00E-06</td>
<td>1.19E-00</td>
</tr>
</tbody>
</table>

### Table 9.3
$L^2$ differences of the solutions using different $R$s.

<table>
<thead>
<tr>
<th>Error</th>
<th>$R = 1.2 l$</th>
<th>$R = 1.5 l$</th>
<th>$R = 1.8 l$</th>
<th>$R = 2.1 l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_\psi$</td>
<td>4.03E-03</td>
<td>1.88E-03</td>
<td>8.46E-04</td>
<td>3.08E-04</td>
</tr>
<tr>
<td>$e_{\text{field}}$</td>
<td>4.72E-02</td>
<td>1.75E-02</td>
<td>7.00E-03</td>
<td>3.64E-03</td>
</tr>
<tr>
<td>$e_{\text{sc}}$</td>
<td>1.38E-02</td>
<td>6.40E-03</td>
<td>2.88E-03</td>
<td>1.04E-03</td>
</tr>
</tbody>
</table>

Comparison of solutions using different radii $R$ for the artificial boundary is also made. In Table 9.3, we present the differences in the $L^2$ norms between the solution with $R = 2.4 l$ and solutions with $R = 2.1 l, 1.8 l, 1.5 l, 1.2 l$. The same grid $(m = 7)$ is used in $\Omega$ while the number of grids in each direction $(n)$ grows proportionally with $R$ in $\Omega_R$.

To compare solutions using the same radius $R$ for the artificial boundary and the same number of boundary terms but on different grids, we computed the solutions for the above problems using $R = 1.2 l$ and $N = 4$ on grids $(m, n) = (9, 7), (17, 13)$, and (25, 19). The contour plots of the order parameter are given in Figure 9.5. Since the exact solution is unknown, we replace it by the extrapolation of the solutions on the finer grids and present the errors of the order parameter in Table 9.4.

Apart from the convergence study and the comparison of various solutions, we have also plotted other physically interesting variables such as the magnetic field, the magnetization curl $\mathbf{A}$, and the supercurrent.

In Figure 9.6, a cross section of the three-dimensional vector field plot of the magnetic field curl $(\mathbf{A} + \mathbf{A}_0)$ inside $B_R$ is given along with a plot of the magnetization curl $\mathbf{A}$ (magnified 20 times). In Figure 9.7, the three-dimensional vector field plots of the superconducting current near the surface of $\Omega$ are given. Here, the two plots...
Fig. 9.5. Contour plot of $|\psi|$ with $n = 4, R = 1.2l$ but with different grids.

Table 9.4
$L^2$ differences of the solutions using different grids.

<table>
<thead>
<tr>
<th>Error</th>
<th>$(m, n) = (9, 7)$</th>
<th>$(m, n) = (17, 13)$</th>
<th>$(m, n) = (25, 19)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_\psi$</td>
<td>$5.51E-02$</td>
<td>$1.76E-02$</td>
<td>$8.24E-03$</td>
</tr>
<tr>
<td>$e_\psi/h^2$</td>
<td>$2.45$</td>
<td>$3.13$</td>
<td>$3.29$</td>
</tr>
</tbody>
</table>

Fig. 9.6. The magnetic field and the magnetic induction (magnified 20 times).

Fig. 9.7. The supercurrent near the surface of $\Omega$.

represent solutions of three different faces of $\Omega$, respectively. Both Figures 9.6 and 9.7 are computed on the finer grid with $R = 6\sqrt{3}/5l$.

Finally, one may conduct many other interesting experiments, such as the study of vortex tubes with tilted applied field. For example, the three-dimensional contour plots of the magnitude of the order parameter are given in Figure 9.8 for solutions with applied field parallel to $\vec{t}_1 = (0.0, 0.0, 1.0)$ and $\vec{t}_2 = (0.6, 0.0, 0.8)$. Here, $\kappa = 5$, $R = 1.5l$, and $\Omega = (-0.75l, 0.75l)^3$. The vortex tubes turn to align themselves with the applied magnetic field. Other experiments are underway and they will be reported elsewhere.
10. Conclusions. The numerical methods we developed in this paper are useful in the simulation of superconductors in three-dimensional space. They have even greater significance in the case where the geometrical effect of the superconducting sample becomes accountable. Using gauge invariance and results on the exterior problem for the Laplace equations, we obtain the boundary conditions on the artificial boundary for the coupled interior-exterior boundary value problem in three dimensions. This approach has greatly reduced the complexity of the original problems. Convergence and error estimates are also provided along some numerical experiments. More numerical studies of the vortex phenomena in type-II superconductors with much larger sample sizes and involving many more vortex tubes are underway. Approximations of the artificial boundary conditions by local boundary conditions and generalizations to other models in superconductivity, such as the time-dependent models and the $s + d$ wave models, will be pursued in the future.

REFERENCES


[14] Q. Du, R. A. Nicolaides, and X. Wu, *Analysis and convergence of a covolume approxima-


