The Spectral Analysis of Frobenius-Perron Operators

J. Ding

Department of Mathematics, University of Southern Mississippi, Hattiesburg, Mississippi 39406-5045

AND

Q. Du and T. Y. Li

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

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It is known that the Frobenius-Perron operator \( P_S : L^1(0, 1) \to L^1(0, 1) \) associated with a transformation \( S \) from \([0, 1]\) to itself with \( \inf |S'| > 1 \) is quasi-compact as an operator on the Banach space \( BV[0, 1] \) of functions of bounded variation in \( L^1(0, 1) \), and thus \( P_S : BV[0, 1] \to BV[0, 1] \) possesses only the finite peripheral spectrum and in particular \( 1 \) is an isolated eigenvalue of \( P_S \). In this paper, we show that under mild conditions on \( S \), the spectrum of \( P_S : L^1(X) \to L^1(X) \) is either the closed unit disk \( \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \) or a cyclic subset of \( \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). © 1994 Academic Press, Inc.

1. Introduction

Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and \( L^1(X) \) be the Banach space of all \( \mu \)-integrable complex functions defined on \( X \). Let \( S : X \to X \) be a measurable non-singular transformation, i.e., \( \mu(S^{-1}(A)) = 0 \) for all \( A \in \mathcal{A} \) such that \( \mu(A) = 0 \). The operator \( P_S : L^1(X) \to L^1(X) \) defined by

\[
\int_A P_S f(x) \, d\mu(x) = \int_{S^{-1}(A)} f(x) \, d\mu(x)
\]

is called the Frobenius-Perron operator associated with \( S \). In ergodic theory, \( P_S \) is usually defined only for real functions by means of (1). The extension of the operator to the complex case given here is for the purpose

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of the spectral analysis of $P_S$. It is obvious from (1) that for $f \in L^1(X)$, the
complex measure $\mu_f$ defined by
$$
\mu_f(A) = \int_A f(x) \, d\mu(x) \quad \text{for all } A \in \mathcal{A},
$$
which is absolutely continuous with respect to $\mu$, is invariant under $S$ if and
only if $f$ is a fixed point of $P_S$. Here, the invariance of the measure $\mu_f$
(under $S$) means that $\mu_f(S^{-1}(A)) = \mu_f(A)$ for every measurable set $A$.
Hence the existence of absolutely continuous invariant measure for a non-
singular transformation is equivalent to the fixed point problem of the
Corresponding Frobenius-Perron operator. This is one of the main
problems in modern ergodic theory and dynamical systems [15].

Much progress has been achieved in studying the existence of absolutely
continuous invariant measures. Based on a fundamental inequality, Lasota
and Yorke have established the existence of invariant densities of
Frobenius-Perron operators associated with piecewise $C^2$ and stretching
mappings of the interval $[0, 1]$ [16]. Later on, Wong [23] extended this
result to piecewise $C^1$ and stretching mappings $S$ such that $|S|\sigma$ is of
bounded variation. In [18], Li and Yorke gave a sufficient condition for
the uniqueness of the invariant density and thus for the ergodicity of the
mapping. For the family of logistic models $S_\lambda(x) = 2\lambda(1 - x)$ with
$\lambda \in [1, 4]$ which do not satisfy the conditions of the Lasota–Yorke
theorem, Misiurewicz [19] proved that, for uncountably many $\lambda$’s of
positive Lebesgue measure, $S_\lambda$ preserves an absolutely continuous measure.
More general results have been obtained in, for example, the work of
Jakobson [10]. For some other existence results, see the monograph of
Lasota and Mackey [15].

For more general constractive Markov operators $P$, Lasota, Li, and
Yorke [14] and Komornicki [13] established the spectral decomposition
theorem which gives the asymptotic periodicity of iterates of $P$. The same
results for Frobenius-Perron operators associated with stretching mappings
were also obtained by Keller [11] who used an ergodic theorem of Ionescu
Tuless and Marinescu [9]. With the observation that the subspace
$BV[0, 1]$ of functions of bounded variation in $L^1(0, 1)$ is invariant under
$P_S$ from the Lasota–Yorke inequality [16] for stretching mappings, and
with the help of the same theorem in [9], Hofbauer and Keller [7]
showed that $P_S: BV[0, 1] \to BV[0, 1]$ is quasi-compact, that is,
$|P_S - K|_p < 1$ for some compact operator $K$ and some positive integer $r$.
Here the Banach space $BV[0, 1]$ is equipped with the norm $\|f\|_p = \|f\|_1 +$ $\sqrt{\int f}$ so that the closed unit ball of $BV$ is compact in $L^1(0, 1)$. Thus, as a
consequence of the uniform ergodic theory [6], there are only a finite
number of points with unit modulus in the spectrum of $P_S: BV[0, 1] \to$ $BV[0, 1]$ each of which is an isolated eigenvalue of order $l$ with the
corresponding finite dimensional eigenspace. A similar result was obtained by Rychlik [20].

For numerical computation of the invariant measure, Ulam [22] first proposed a piecewise constant approximation method and conjectured that the approximate densities converge to the fixed density. This conjecture was proved by Li [17]. Later on, in order to improve the convergence rate of Ulam's method, Kohda and Murao [12] proposed a general piecewise polynomial Galerkin scheme to approximate the fixed point of $P_S$. In [4, 5], Ding and Li presented two classes of numerical methods. One is again based on the Galerkin method, and the other uses the Markov finite approximations of the Frobenius-Perron operator. The convergence of these methods was proved for piecewise $C^2$ stretching mappings. A general high order approximation method was presented in [3]. Error estimates of the finite Markov approximations were obtained by Chiu, Du, and Li in [2], using the technique of eigenspace projections of linear operators. A more general framework was presented in [3].

Recently, Hunt and Miller [8] investigated the convergence rate of Ulam’s method for quasi-compact Frobenius-Perron operators. Their analysis is based on the assumption that $l$ is an isolated eigenvalue of $P_S: L^1(X) \to L^1(X)$. Thus they can use the concept of strongly stable approximations of linear operators developed in [1] to analyze the approximation of $P_S$ in $L^1(X)$. Here, the spectral analysis of the Frobenius-Perron operator $P_S$ plays an essential role in the numerical analysis of $P_S$ as well as the error analysis of approximation methods. However, to the best of our knowledge, the spectral analysis of $P_S$ in $L^1(X)$ has not been investigated completely in the literature. The purpose of this paper is to show that the spectrum of $P_S: L^1(X) \to L^1(X)$ is either the unit disk \( \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \) or a cyclic subset of \( \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \) under some mild conditions on the mappings and the measures defined on $X$. Considering the fact that the peripheral spectrum of $P_S: BV[0, 1] \to BV[0, 1]$ for stretching mappings is only a finite point spectrum and that $BV[0, 1]$ is dense as a subspace of $L^1(0, 1)$, our result is somewhat astonishing. First, our result indicates that, even for a stretching mapping $S$, the eigenvalue $l$ of $P_S: L^1(0, 1) \to L^1(0, 1)$ may not be isolated even if $P_S: BV[0, 1] \to BV[0, 1]$ is quasi-compact. Second, to apply the technique in [1] to the analysis of the convergence rate of the discrete methods, one must restrict discussion in the Banach space $BV(X)$ of functions of bounded variation for quasi-compact Frobenius-Perron operators $P_S: BV(X) \to BV(X)$.

The paper is organized as follows. Section 2 gives some necessary material for our analysis. In Section 3, we prove two main theorems concerning the spectrum of $P_S$ which cover the cases where $\mu$ is regular or the $\sigma$-subalgebra $S^{-1}.\mathcal{A}$ equals $\mathcal{A}$. Some consequences and applications of our results are presented in Section 4. We conclude in Section 5.
2. Preliminaries

Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and let \(S: X \to X\) be a measurable non-singular transformation. Throughout this paper, we assume that the measure space is complete, i.e., \(\mu(A) = 0\) implies \(B \in \mathcal{A}\) for any \(B \subset A\). Let \(L^1(X)\) be the Banach space of all \(\mu\)-integrable complex functions defined on \(X\) with the norm \(\|f\|_1 = \int_X |f(x)| \, d\mu(x)\), and \(L^\infty(X)\) be the Banach space of all bounded almost everywhere complex measurable functions on \(X\) with the norm \(\|g\|_\infty = \text{ess sup} \, |g(x)|\). Here functions which are equal almost everywhere with respect to \(\mu\) are considered identical. \(L^\infty(X)\) is the adjoint of \(L^1(X)\), i.e., \((L^1(X))^* = L^\infty(X)\), the space of all bounded complex linear functionals on \(L^1(X)\). For \(f \in L^1(X)\) and \(g \in L^\infty(X)\), their adjoint relation is given by

\[
\langle f, g \rangle = \int_X f(x) \, g(x) \, d\mu(x).
\]

Let \(L^1_r(X)\) and \(L^\infty_r(X)\) denote all real functions in \(L^1(X)\) and \(L^\infty(X)\), respectively. The linear operator \(P_S: L^1_r(X) \to L^1_r(X)\) defined by

\[
\int_A P_S f(x) \, d\mu(x) = \int_{S^{-1}(A)} f(x) \, d\mu(x), \quad A \in \mathcal{A}
\]

is called the Frobenius-Perron operator associated with \(S\). By the Radon-Nikodym theorem [6], \(P_S\) is well-defined. Let \(f = f_1 + i f_2 \in L^1(X)\) with \(f_1, f_2 \in L^1_r(X)\), define

\[
P_S f = P_S f_1 + i P_S f_2.
\]

Then \(P_S\) is extended to \(L^1(X)\). The linear operator \(U_S: L^\infty_r(X) \to L^\infty_r(X)\) defined by

\[
U_S g(x) = g(S(x))
\]

is called the Koopman operator with respect to \(S\). Note that since \(S\) is non-singular, \(U_S g\) is well-defined. Let \(g = g_1 + ig_2 \in L^\infty(X)\) with \(g_1, g_2 \in L^\infty_r(X)\), define

\[
U_S g = U_S g_1 + i U_S g_2.
\]

Then \(U_S\) is well-defined on \(L^\infty(X)\). Some basic properties of \(P_S\) and \(U_S\) are listed in the following lemma.

**Lemma 2.1.** (i) \(P_S f \geq 0\) if \(f \geq 0\).

(ii) \(\langle P_S f, 1 \rangle = \langle f, 1 \rangle\).
(iii) \( \langle P_S f, g \rangle = \langle f, U_S g \rangle \) for \( f \in L^1(X) \) and \( g \in L^\infty(X) \).

(iv) If \( S : X \to X \) and \( T : X \to X \) are both non-singular, then so is \( T \circ S \) and \( P_{T \circ S} = P_T P_S \).

Proof. Parts (i) and (ii) follow directly from the definition. For \( f \in L^1(X) \) and \( g \in L^\infty(X) \), (iii) is just a standard result in ergodic theory [15]. Now let \( f = f_1 + if_2 \in L^1(X) \) and \( g = g_1 + ig_2 \in L^\infty(X) \). Then

\[
\langle P_S f, g \rangle = \langle P_S f_1 + iP_S f_2, g_1 + ig_2 \rangle \\
= \langle P_S f_1, g_1 \rangle + i\langle P_S f_2, g_1 \rangle + i\langle P_S f_1, g_2 \rangle - \langle f_2, U_S g_2 \rangle \\
= \langle f_1, U_S g_1 \rangle + i\langle f_2, U_S g_1 \rangle + i\langle f_1, U_S g_2 \rangle - \langle f_2, U_S g_2 \rangle \\
= \langle f_1 + if_2, U_S g_1 + U_S g_2 \rangle = \langle f, U_S g \rangle.
\]

This proves (iii). Part (iv) follows from

\[
\int_A P_{T \circ S} f(x) \, d\mu(x) = \int_{(T \circ S)^{-1}(A)} f(x) \, d\mu(x) \\
= \int_{T^{-1}(A)} P_S f(x) \, d\mu(x) \\
= \int_A P_T P_S f(x) \, d\mu(x).
\]

Q.E.D.

Remark 2.1. Parts (i) and (ii) imply that \( P_S \) is a Markov operator. That is, \( P_S \) maps positive functions to positive functions and preserves their norms. Part (iii) means that \( U_S \) is adjoint to \( P_S \), i.e., \( U_S = (P_S)^\star \). Thus \( U_S \) and \( P_S \) have the same spectrum. Part (iv) implies that if \( S \) is injective, so is \( P_S \) and if \( S \) is surjective, so is \( P_S \), hence, if \( S \) is bijective, so is \( P_S \).

By Lemma 2.1, it follows immediately that

**Lemma 2.2.** \( \|P_S\|_1 = \|U_S\|_\infty = 1 \).

Now we introduce some terminologies to be used later. A measurable transformation \( S : X \to X \) is said to be *onto* if there does not exist any \( A \in \mathcal{A} \) with \( \mu(A) > 0 \) such that \( S(X) \cap X - A \), the supplementary set of \( A \) in \( X \). \( S \) is said to be \( \mathcal{A} \)-invariant if \( S\mathcal{A} \subseteq \mathcal{A} \), where \( S\mathcal{A} = \{ S(A) : A \in \mathcal{A} \} \). If \( \mu(S^{-1}(A)) > 0 \) whenever \( \mu(A) > 0 \), then \( \mu \) is said to be *regular*. \( \mu \) is said to be *normal* if \( \mu(A) = 0 \) implies \( S(A) \in \mathcal{A} \) and \( \mu(S(A)) = 0 \). Note that \( S \) is non-singular and \( \mu \) is regular if and only if the measure \( \mu \circ S^{-1} \) and the measure \( \mu \) are equivalent, that is, \( \mu \circ S^{-1}(A) = 0 \) if and only if \( \mu(A) = 0 \). It is easy to show that if \( \mu \) is regular and \( S^{-1}\mathcal{A} = \mathcal{A} \), then \( \mu \) is normal. Here \( S^{-1}\mathcal{A} = \{ S^{-1}(A) : A \in \mathcal{A} \} \).
For any $f \in L^1(X)$, let $\text{supp } f = \{ x \in X : f(x) \neq 0 \}$ which is called the support of $f$. The following lemma gives the relation between the supports of $f$ and $P_S f$ and will be frequently used in the sequel.

**Lemma 2.3.** Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $S : X \to X$ be a non-singular transformation. Let $P_S$ be the Frobenius-Perron operator associated with $S$. If $f \geq 0$, then $f(x) = 0$ for all $x \in S^{-1}(A)$ if and only if $P_S f(x) = 0$ for all $x \in A$, and in particular,

$$S^{-1}(\text{supp } P_S f) \supseteq \text{supp } f.$$

**Proof.** By the definition of the Frobenius-Perron operator, we have

$$\int_X \chi_A(x) P_S f(x) \, d\mu(x) = \int_X \chi_{S^{-1}(A)}(x) f(x) \, d\mu(x),$$

where $\chi_A$ denotes the characteristic function of $A$. Suppose $f \in L^1(X)$ is nonnegative. Then $P_S f(x) = 0$ on $A$ implies that $f(x) = 0$ on $S^{-1}(A)$ and vice versa. Since

$$\int_{\text{supp } f} f(x) \, d\mu(x) = \int_{\text{supp } P_S f} P_S f(x) \, d\mu(x) = \int_{S^{-1}(\text{supp } P_S f)} f(x) \, d\mu(x),$$

we have $S^{-1}(\text{supp } P_S f) \supseteq \text{supp } f$. Q.E.D.

The following lemma gives the relations between the several terminologies introduced above.

**Lemma 2.4.** Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $S : X \to X$ be a non-singular transformation. If $\mu$ is regular, then $S$ is onto. Conversely, if $S$ is onto and $\mu$ is normal, then $\mu$ is regular.

**Proof.** Suppose there exists $A \in \mathcal{A}$ such that $\mu(A) > 0$ and $S(X) \subseteq X - A$. Then obviously $\mu(S^{-1}(A)) = 0$ since $S^{-1}(A)$ is actually a null set. Therefore, $\mu$ is not regular.

Conversely, let $\mu$ be normal. Suppose for some $A \in \mathcal{A}$, we have $\mu(A) > 0$ and $\mu(S^{-1}(A)) = 0$. Let $f \geq 0$ be in $L^1(X)$. By the definition of the Frobenius-Perron operator,

$$\int_A P_S f(x) \, d\mu(x) = \int_{S^{-1}(A)} f(x) \, d\mu(x) = 0.$$

Thus $\text{supp } P_S f \subseteq X - A$. Since $S(\text{supp } f) \subseteq \text{supp } P_S f$ by Lemma 2.3, we obtain

$$S(\text{supp } f) \subseteq \text{supp } P_S f \subseteq X - A.$$
Now, since $X$ is $\sigma$-finite, we can construct a nonnegative function $f \in L^1(X)$ such that $\text{supp } f = X$. Hence,

$$S(X) \subset X - A.$$ 

This means that $S$ is not onto. Q.E.D.

We briefly present some standard results from functional analysis for the use in the following sections. Let $T : E \to E$ be a bounded linear operator on a complex Banach space $E$. Its spectrum $\sigma(T)$ is defined to be the set of all complex numbers $\lambda$ such that $T - \lambda I$ does not have the bounded inverse defined on $E$, where $I$ is the identity operator from $E$ to $E$. The complement of $\sigma(T)$ in $\mathbb{C}$ is called the resolvent set of $T$ and is denoted by $\rho(T)$. $\sigma(T)$ is a compact subset of the closed disk $\{ \lambda \in \mathbb{C} : |\lambda| \leq r(T) \}$, where $r(T) = \lim_{n \to \infty} (\| T^n \|)^{1/n}$ is the spectral radius of $T$, and $\sigma(T)$ is the disjoint union of the point spectrum $\sigma_p(T)$, the continuous spectrum $\sigma_c(T)$, and the residual spectrum $\sigma_r(T)$. The boundary of $\sigma(T)$ is denoted by $\partial \sigma(T)$. $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum $\sigma_a(T)$ if there exists a sequence $\{ x_n \} \in E$ such that $\| x_n \| = 1$ for all $n$ and $\| (T - \lambda I) x_n \| \to 0$. Obviously $\sigma_a(T) \subset \sigma(T)$. The following lemma is a standard result concerning $\sigma_a(T)$ (see [1]).

**Lemma 2.5.** $\sigma_a(T)$ contains the union of $\sigma_p(T)$, $\sigma_c(T)$, and $\partial \sigma(T)$.

From now on, we let $\mathcal{D} = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$ and $\partial \mathcal{D} = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$.

**Lemma 2.6.** $\sigma(P_S)$ is a cyclic subset of the unit disk $\mathcal{D}$, that is, $\lambda \in \sigma(P_S)$ with $\lambda = |\lambda| \gamma$ implies $|\lambda| \gamma^k \in \sigma(P_S)$ for all integers $k$.

**Proof.** By Remark 2.1 and Lemma 2.2, $\sigma(P_S) = \sigma(U_S) \subset \mathcal{D}$. From the theory of Banach lattice [21], $L^\infty(X)$ is a Banach lattice with the natural ordering of functions. Since

$$|U_S g(x)| = U_S |g|(x)$$

for all $x \in X$ by definition, $U_S$ is a lattice homomorphism (Proposition II.2.5 in [21]). Thus from Theorem V.4.4 in [21], $\sigma(U_S)$ is cyclic. Q.E.D.

3. The Spectrum of $P_S$

We first study the spectrum of $P_S$ in the case when $\mu$ is regular. By Lemma 2.4, in this case, $S$ is onto in the measure-theoretic sense, i.e., $S(X)$ is not contained in any $X - A$ with $\mu(A) > 0$ and with the additional assumption that $\mu$ is normal, $S$ being onto implies that $\mu$ is regular. The main results are Theorem 3.1 and Theorem 3.2. There are two important
special cases. The first case is where $S$ is measure preserving, i.e., $\mu(S^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$. This implies that $\mu$ is regular and the conclusion is given in Corollary 3.1. The second case, discussed in Corollary 3.2, is where $P_S$ has a fixed density which is positive $\mu$-almost everywhere. In this case, one can use the invariant measure to deduce that $\mu$ is regular. Notice also that the Lebesgue measure is often regular with respect to piecewise continuous mappings from the unit interval onto itself.

The next lemma plays a key role in this part of the spectrum analysis.

**Lemma 3.1.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $S: X \to X$ be a non-singular transformation. Then $\mu$ is regular if and only if the Koopman operator $U_S: L^\infty(X) \to L^\infty(X)$ is an isometry, i.e.,

$$\|U_S g\|_\infty = \|g\|_\infty.$$

In other words, $\mu \circ S^{-1}$ and $\mu$ are equivalent if and only if $U_S$ is isometric.

**Proof.** Suppose $\mu$ is regular. Let $g \in L^\infty(X)$ be a nonzero function. Given any $\varepsilon > 0$, let

$$A = \{x \in X : |g(x)| > \|g\|_\infty - \varepsilon\}.$$

Then $\mu(A) > 0$. So, $\mu(S^{-1}(A)) > 0$, i.e.,

$$\{x \in X : |U_S g(x)| > \|g\|_\infty - \varepsilon\}$$

is not a null set, namely,

$$\|U_S g\|_\infty > \|g\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|U_S g\|_\infty \geq \|g\|_\infty$. Then Lemma 2.2 implies that $U_S$ is an isometry.

Conversely, suppose $U_S$ is isometric. Let $\mu(A) > 0$. Then the characteristic function $\chi_A \in L^\infty(X)$ of $A$ is nonzero. Now since $U_S$ is injective, $\chi_{S^{-1}(A)} = U_{S\chi_A}$ is nonzero, which implies $\mu(S^{-1}(A)) > 0$. Thus, $\mu$ is regular.

Q.E.D.

**Remark 3.1.** We actually proved that $U_S$ is injective if and only if $\mu$ is regular.

We are ready to prove the main result of this section.

**Theorem 3.1.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $S: X \to X$ be a non-singular transformation such that $\mu$ is regular. Let $P_S$ be the Frobenius-Perron operator associated with $S$. If $0 \in \sigma(P_S)$, then $\sigma(P_S) = \mathcal{D}$ and if $0 \in \rho(P_S)$, then $\sigma(P_S) \subset \partial \mathcal{D}$. 

Proof. Let $U_S: L^\infty(X) \to L^\infty(X)$ be the Koopman operator with respect to $S$. From Lemma 3.1, $U_S$ is an isometry.

Let $|\lambda| < 1$ be given. If $U_S g - \lambda g = 0$ for some $g \in L^\infty(X)$, then

$$\|g\|_\infty = \|U_S g\|_\infty = |\lambda| \|g\|_\infty.$$ 

Thus, $\|g\|_\infty = 0$, i.e., $U_S - \lambda I$ is injective. From

$$\|U_S g - \lambda g\|_\infty \geq \|U_S g\|_\infty - |\lambda| \|g\|_\infty = (1 - |\lambda|) \|g\|_\infty,$$

we see that $U_S - \lambda I$ is bounded below, and thus $\lambda$ is not in the approximate point spectrum of $U_S$. Hence for $|\lambda| < 1$, $\lambda \notin \partial \sigma(U_S)$ by Lemma 2.5. In particular, $0 \notin \partial \sigma(U_S)$.

Now consider first the case $0 \in \sigma(U_S)$. If there exists $|\lambda| < 1$ such that $\lambda \notin \sigma(U_S)$, then it is easy to see that there exists $\lambda_1 \in \partial \sigma(U_S)$ such that $|\lambda_1| < 1$, which is a contradiction to the fact that $\lambda \notin \partial \sigma(U_S)$ for $|\lambda| < 1$. Therefore,

$$|\lambda| < 1 \Rightarrow \lambda \in \sigma(U_S).$$

Since $\sigma(U_S)$ is a closed subset of $\mathcal{D}$, we have

$$\sigma(P_S) = \sigma(U_S) = \mathcal{D}.$$

Consider now the case $0 \in \rho(P_S)$. If there exists $|\lambda| < 1$ such that $\lambda \in \sigma(U_S)$, then there exists a $\lambda_1 \in \partial \sigma(U_S)$ with $|\lambda_1| < 1$, which also contradicts the fact that $\lambda \notin \partial \sigma(U_S)$ for $|\lambda| < 1$. Therefore,

$$|\lambda| < 1 \Rightarrow \lambda \notin \sigma(U_S).$$

In this case, $\sigma(P_S) \subset \partial \mathcal{D}$. Q.E.D.

Remark 3.2. Actually, we have proved the following general result: If $T: E \to E$ is an isometry from a complex Banach space $E$ into itself, then $\sigma(T) = \mathcal{D}$ if $0 \in \sigma(T)$ and $\sigma(T) \subset \partial \mathcal{D}$ if $0 \in \rho(T)$. This remark will be useful when we prove Theorem 3.2.

Remark 3.3. From the proof we see that $\lambda \notin \sigma_p(U_S)$ for $|\lambda| < 1$ if $\sigma \in \sigma(P_S)$. If $0 \in \rho(P_S)$, then by Lemma 2.6, $\sigma(P_S) \subset \partial \mathcal{D}$ is cyclic.

Remark 3.4. If $0 \in \rho(P_S)$, then one can complete the proof of the above theorem by showing that the mapping $F: L^\infty(X) \to L^\infty(X)$.

$$F(f) = \lambda U_S^{-1} f + U_S^{-1} g$$

is a contraction for any $|\lambda| < 1$. Thus, $U_S - \lambda I$ is bijective. Then the Banach Inverse Mapping Theorem implies that $U_S - \lambda I$ has the bounded inverse defined on $L^\infty(X)$. Thus, $\lambda \in \rho(U_S)$. Therefore,

$$\sigma(P_S) = \sigma(U_S) \subset \partial \mathcal{D}.$$
Corollary 3.1. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and \(S: X \to X\) be measure preserving. If \(0 \in \sigma(P_S)\), then \(\sigma(P_S) = \mathcal{D}\) and if \(0 \in \rho(P_S)\), then \(\sigma(P_S) \subset \partial \mathcal{D}\).

Proof. Since \(\mu(S^{-1}(A)) = \mu(A)\) for all \(A \in \mathcal{A}\), it is obvious that \(\mu\) is regular. By Theorem 3.1, the conclusion is achieved. Q.E.D.

Corollary 3.2. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and \(S: X \to X\) be a non-singular transformation. Suppose \(P_S\) has a nonnegative fixed density \(\tilde{f}\) such that \(\text{supp}\ \tilde{f} = X\). If \(0 \in \sigma(P_S)\), then \(\sigma(P_S) = \mathcal{D}\); if \(0 \in \rho(P_S)\), then \(\sigma(P_S) \subset \partial \mathcal{D}\).

Proof. Define \(\tilde{\mu}(A) = \int_A \tilde{f}(x) \, d\mu(x)\). Then the measure \(\tilde{\mu}\) is invariant under \(S\). Let \(A \in \mathcal{A}\) be such that \(\mu(A) > 0\). Since \(\tilde{f}(x) > 0\), we have \(\tilde{\mu}(A) > 0\). So \(\tilde{\mu}(S^{-1}(A)) > 0\) which implies \(\mu(S^{-1}(A)) > 0\). Hence \(\mu\) is regular. Q.E.D.

Example 3.1. Let \(S: X \to X\) be the identity map, i.e., \(S(x) = x\). Then \(\sigma(P_S) = \{1\} \subset \partial \sigma(\mathcal{D})\).

Example 3.2. Let \(X = [0, 1]\) and \(m\) be the Lebesgue measure. Let \(S: [0, 1] \to [0, 1]\) be defined by \(S(x) = 2x \ (\text{mod} \ 1)\). Then \(m\) is invariant under \(S\). It is easy to see that

\[
P_S f(x) = \frac{1}{2} \left[ f\left(\frac{x}{2}\right) + f\left(\frac{1}{2} + \frac{x}{2}\right)\right].
\]

Let

\[
f(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1/2 \\
-1 & \text{if } 1/2 \leq x \leq 1.
\end{cases}
\]

Then \(P_S f = 0\), i.e., \(0 \in \sigma(P_S)\). By Theorem 3.1, \(\sigma(P_S) = \{ \lambda \in \mathcal{C} : |\lambda| \leq 1\}\).

By Lemma 2.4 and Theorem 3.1, we also have

Corollary 3.3. Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and \(S: X \to X\) be a non-singular transformation such that \(S\) is onto and \(\mu\) is normal. If \(0 \in \sigma(P_S)\), then \(\sigma(P_S) = \mathcal{D}\) and if \(0 \in \rho(P_S)\), then \(\sigma(P_S) \subset \partial \mathcal{D}\).

The above discussion does not include the most general case, in particular, the case when \(S\) is not onto. An example of maps which are not onto in the measure-theoretic sense is \(S: [0, 1] \to [0, 1]\) defined by \(S(x) = x/2\). To do the spectrum analysis of the corresponding Frobenius-Perron operators for this type of maps, we now consider the second case.
Notice that the result of Theorem 3.1 is achieved mainly by the fact that\( \mathcal{U}_S \) is isometric. It follows from this observation and Remark 3.2 that the same result holds when \( \mathcal{P}_S \) is isometric. It turns out that if \( \sigma \)-subalgebra \( S^{-1}\mathcal{A} \) equals \( \mathcal{A} \), then \( \mathcal{P}_S \) is an isometry. Theorem 3.2 in the following is parallel to Theorem 3.1. To prove it, we first need

**Lemma 3.2.** Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and \( S: X \to X \) be a non-singular transformation. Let \( \mathcal{P}_S: L^1(X) \to L^1(X) \) be the Frobenius-Perron operator associated with \( S \). Then \( S^{-1}\mathcal{A} = \mathcal{A} \) if and only if \( \mathcal{P}_S \) is injective.

**Proof.** Suppose \( S^{-1}\mathcal{A} = \mathcal{A} \). Let \( \mathcal{P}_S f = 0 \) for some \( f \in L^1(X) \). From the definition of \( \mathcal{P}_S \), for all \( A \in \mathcal{A} \),

\[
\int_{S^{-1}(A)} f(x) \, d\mu(x) = \int_A \mathcal{P}_S f(x) \, d\mu(x) = 0.
\]

By the given condition, each \( B \in \mathcal{A} \) is of the form \( S^{-1}(A) \) for some \( A \in \mathcal{A} \). Thus,

\[
\int_B f(x) \, d\mu(x) = 0 \quad \text{for all} \quad B \in \mathcal{A}.
\]

This is equivalent to \( f = 0 \).

Conversely, suppose \( S^{-1}\mathcal{A} \neq \mathcal{A} \). Since \( S^{-1}\mathcal{A} \) is a \( \sigma \)-subalgebra of \( \mathcal{A} \), we can find a \( B \in \mathcal{A} \) such that \( \mu(B) > 0 \) and \( B \) is disjoint with any \( S^{-1}(A) \).

Now, since \((X, \mathcal{A}, \mu)\) is \( \sigma \)-finite, we can construct a nonnegative \( f_B \in L^1(X) \) such that \( \text{supp} \ f = B \). From

\[
\int_A \mathcal{P}_S f_B(x) \, d\mu(x) = \int_{S^{-1}(A)} f_B(x) \, d\mu(x) = 0
\]

for all \( A \in \mathcal{A} \), we have \( \mathcal{P}_S f_B = 0 \). Thus \( \mathcal{P}_S \) is not injective. Q.E.D.

**Theorem 3.2.** Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and \( S: X \to X \) be a non-singular transformation. Let \( \mathcal{P}_S: L^1(X) \to L^1(X) \) be the Frobenius-Perron operator associated with \( S \). Suppose that \( S^{-1}\mathcal{A} = \mathcal{A} \). If \( 0 \in \sigma(\mathcal{P}_S) \), then \( \sigma(\mathcal{P}_S) = \mathcal{P} \). If \( 0 \in \rho(\mathcal{P}_S) \), then \( \sigma(\mathcal{P}_S) \subset \partial \mathcal{P} \).

**Proof.** From Lemma 3.2, \( \mathcal{P}_S \) is injective. We show that \( \mathcal{P}_S: L^1(X) \to L^1(X) \) is an isometry. Let \( g \in \text{Range}(\mathcal{P}_S) \subset L^1(X) \) be nonnegative. Then there is a unique \( f \in L^1(X) \) such that \( \mathcal{P}_S f = g \). Since

\[
0 \leq \int_A g(x) \, d\mu(x) = \int_A \mathcal{P}_S f(x) \, d\mu(x) = \int_{S^{-1}(A)} f(x) \, d\mu(x)
\]
for all \( A \in \mathcal{A} \) and by the assumption \( S^{-1} \mathcal{A} = \mathcal{A} \), so,

\[
\int_B f(x) \, d\mu(x) \geq 0 \quad \text{for all } B \in \mathcal{A}.
\]

Thus \( f \geq 0 \). This means that \( P_S^{-1} : \text{Range}(P_S) \to L^1(X) \) is a positive operator. Since \( P_S \) keeps the norm of positive functions invariant, so does \( P_S^{-1} \). Hence Jensen’s inequality [15] guarantees that \( \| P_S^{-1} g \|_1 \leq \| g \|_1 \) for all \( g \in \text{Range}(P_S) \). Therefore, for any \( f \in L^1(X) \),

\[
\| P_S f \|_1 = \| f \|_1.
\]

This shows that \( P_S \) is an isometry.

The remaining part of the proof is exactly the same as in Theorem 3.1 if \( L^\infty(X) \) is replaced by \( L^1(X) \) and \( U_S \) is replaced by \( P_S \). Q.E.D.

From Remark 2.1, we obtain the following

**Corollary 3.4.** Under the same assumption as above, if \( S : X \to X \) is bijective in the measure-theoretic sense, then \( \sigma(P_S) \subset \partial \Phi \).

A few sufficient conditions for \( S^{-1} \mathcal{A} = \mathcal{A} \) are given as follows.

**Proposition 3.1.** If the Koopman operator \( U_S : L^\infty(X) \to L^\infty(X) \) is surjective, then \( S^{-1} \mathcal{A} \) equals to \( \mathcal{A} \).

**Proof.** Given \( B \in \mathcal{A} \), since \( U_S \) is surjective, there is \( g \in L^\infty(X) \) such that \( U_S g = \chi_B \). Let \( A = g^{-1}(\{1\}) \). Then \( A \in \mathcal{A} \) since \( g \) is \( \mathcal{A} \)-measurable. It is easy to see that \( g(x) = \chi_A(x) \) \( \mu \)-a.e. Hence \( S^{-1}(A) = B \). Q.E.D.

It is also easy to verify the following.

**Proposition 3.2.** Let \( S : [0, 1] \to [0, 1] \) be non-singular and one-to-one. Assume that \( S^{-1} : S([0, 1]) :\to [0, 1] \) is measurable; then \( S^{-1} \mathcal{A} \) equals \( \mathcal{A} \).

Notice that, however, the condition \( S^{-1} \mathcal{A} = \mathcal{A} \) does not hold for non-singular stretching mappings on the unit interval which are not one-to-one in general. To end this section, we give the following result which is a consequence of Lemma 3.2.

**Proposition 3.3.** If \( S^{-1} \mathcal{A} \neq \mathcal{A} \), then \( 0 \in \sigma(P_S) \).

4. Further Results and Some Consequences

It is known that for the class of piecewise \( C^2 \) and stretching mappings \( S \) from \([0, 1]\) into itself, the corresponding Frobenius-Perron operator \( P_S \)
keeps the subspace $BV[0,1]$ of functions of bounded variation invariant, and under a new norm $\|\cdot\|_e$ which makes the unit ball of $BV[0,1]$ compact in $L^1(0,1)$, $P_S: BV[0,1] \to BV[0,1]$ is quasi-compact, i.e., there is a positive integer $r$ such that $\|P_S^r - K\|_e < 1$ for some compact operator $K$ (see [7] for more details). As a consequence, $P_S: BV[0,1] \to BV[0,1]$ possesses only finitely many peripheral spectral points each of which is an isolated eigenvalue of order 1 with the corresponding finite-dimensional eigenspace. The following result indicates that it is not so for $P_S: L^1(0,1) \to L^1(0,1)$ in general.

**Theorem 4.1.** Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and $S: X \to X$ be a non-singular transformation. If 1 is not isolated in the spectrum of $P_S: L^1(X) \to L^1(X)$, $P_S$ is not quasi-compact. Moreover, if $1 \notin \sigma_p(P_S)$, or if $1 \in \sigma_p(P_S)$ but $\dim N(P_S - I) = \infty$, $P_S$ is not quasi-compact.

**Proof.** Since $P_S$ is a positive operator, $1 = r(P_S) \in \sigma(P_S)$ [21]. If $P_S$ is quasi-compact, then Theorem VIII.8.3 of [6] implies that 1 is an isolated eigenvalue of $P_S$ with the finite dimensional eigenspace which contradicts the assumption. Q.E.D.

As a simple example, let $S(x) = x$ for all $x \in [0,1]$. Then $P_S = I$. Since $N(P_S - I) = L^1(0,1)$, $P_S$ is not quasi-compact. Another example is the following

**Example 4.1.** Let $S: [0,1] \to [0,1]$ be defined by $S(x) = 4x(1-x)$. Then a simple computation gives

$$P_S f(x) = \frac{1}{4 \sqrt{1-x}} \left[ f \left( \frac{1}{2} (1 - \sqrt{1-x}) \right) + f \left( \frac{1}{2} (1 + \sqrt{1-x}) \right) \right].$$

In 1947, Ulam and von Neumann already found a fixed point of this operator:

$$\tilde{f}(x) = \frac{1}{\pi \sqrt{x(1-x)}}.$$

Let $f$ be as in Example 3.2. Then $P_S f = 0$. Hence $\sigma(P_S) = \emptyset$ by Theorem 3.1. Thus, $P_S: L^1(0,1) \to L^1(0,1)$ is not quasi-compact.

The observation of this example leads to the following result. We use $m$ to denote the Lebesgue measure on $[0,1]$.

**Proposition 4.1.** Suppose $S: [0,1] \to [0,1]$ is non-singular and $m$ is regular. If $S(x) = S(1-x)$ for all $x \in [0,1]$, then $\sigma(P_S) = \emptyset$. 
Proof. Let \( S_1 = S \mid_{[0, 1/2]} \) and \( S_2 = S \mid_{(1/2, 1]} \). Since \( S(x) \) is symmetric about 1/2, it is easy to see that for any \( x \in [0, 1] \),

\[
m(S_1^{-1}[0, x]) = m(S_2^{-1}[0, x]).
\]

Now, from the definition of the Frobenius-Perron operator, for any \( f \in L^1(0, 1) \),

\[
P_S f(x) = \frac{d}{dx} \left( \int_{S^{-1}[0, x]} f(t) \, dm(t) \right)
\]

\[
= \frac{d}{dx} \left\{ \left[ \int_{S_1^{-1}[0, x]} f(t) \, dm(t) \right] + \left[ \int_{S_2^{-1}[0, x]} f(t) \, dm(t) \right] \right\}.
\]

Let

\[
f(x) = \begin{cases} 
1 & \text{if } 0 \leq x < 1/2 \\
-1 & \text{if } 1/2 \leq x \leq 1.
\end{cases}
\]

Since \( S_1^{-1}[0, x] \subset [0, 1/2] \) and \( S_2^{-1}[0, x] \subset [1/2, 1] \) for all \( x \in [0, 1] \), we obtain

\[
P_S f(x) = \frac{d}{dx} \left\{ \left[ \int_{S_1^{-1}[0, x]} \, dm(t) \right] + \left[ \int_{S_2^{-1}[0, x]} \, dm(t) \right] \right\}
\]

\[
= \frac{d}{dx} \left\{ m(S_1^{-1}[0, x]) - m(S_2^{-1}[0, x]) \right\} = 0.
\]

Hence, \( 0 \in \sigma(P_S) \). The conclusion follows from Theorem 3.1. Q.E.D.

Similarly, motivated by Example 3.2, we can prove

Proposition 4.2. Suppose \( S: [0, 1] \to [0, 1] \) is non-singular and \( m \) is regular. If \( S(x) = S(1/2 + x) \) for \( x \in [0, 1/2] \), then \( \sigma(P_S) = \mathcal{D} \).

More generally, we have

Theorem 4.2. Suppose \( S: [0, 1] \to [0, 1] \) is non-singular and \( m \) is regular. If there is a partition \( 0 = a_0 < a_1 < \cdots < a_k = 1 \) of \( [0, 1] \) such that

\[
\frac{d}{dx} \left\{ \sum_{i=1}^{k} (-1)^{x_i} m(S_i^{-1}[0, x]) \right\} = 0 \quad m\text{-a.e.},
\]

where \( S_i = S \mid_{[a_{i-1}, a_i]} \) and \( \tau \) is a mapping from \( \{1, 2, \ldots, k\} \) into itself, then \( \sigma(P_S) = \mathcal{D} \).
Proof. Since

\[ P_S f(x) = \frac{d}{dx} \int_{S_{i}^{-1}[0, x]} f(t) \, dm(t) = \frac{d}{dx} \left\{ \sum_{i=1}^{k} \int_{S_{i}^{-1}[0, x]} f(t) \, dm(t) \right\}, \]

if we let \( f = \sum_{i=1}^{k} (-1)^{i} \chi_{[a_{i-1}, a_{i})} \), then \( f \neq 0 \) and \( P_S f = 0 \). Hence \( \sigma(P_S) = \emptyset \). Q.E.D.

Remark 4.1. From the proof we see that a general sufficient condition for \( \sigma(P_S) = \emptyset \) is that the equation

\[ \frac{d}{dx} \left\{ \sum_{i=1}^{k} \int_{S_{i}^{-1}[0, x]} f(t) \, dm(t) \right\} = 0 \quad m - \text{a.e.} \]

have a nonzero solution \( f \in L^1(0, 1) \).

Remark 4.2. Based on the above theorem and the Lasota–Yorke theorem, it is easy to construct stretching mappings \( S \colon [0, 1] \to [0, 1] \) such that the operators \( P_S \colon BV[0, 1] \to BV[0, 1] \) corresponding to those mappings are quasi-compact, but \( P_S \colon L^1(0, 1) \to L^1(0, 1) \) are not. The mapping defined in Example 3.2 is one of such examples.

Finally, to conclude this section, let us note that the spectrum analysis in the previous section is not the most general one. In fact, it does not cover interesting mappings such as \( S \colon [0, 1] \to [0, 1] \) defined by \( S(x) = \alpha x(1-x) \), \( 0 < \alpha < 4 \), with respect to the Lebesgue measure. The following is another example.

Example 4.2. Let \( X = \{0, 1\} \) with \( \mu\{0\} = \mu\{1\} = 1/2 \) and \( S(0) = S(1) = 1 \). Since \( U_S f(x) = f(1), \forall x \in X \) and \( \forall f \in L^\infty(X) \), one can easily get \( \sigma(P_S) = \{0, 1\} \).

Based on this example, we conjecture that in the general case, \( \sigma(P_S) \), the spectrum of \( P_S \) in \( L^1(X) \), may either be the unit disc \( \emptyset \) or a cyclic subset of \( \partial \emptyset \cup \{0\} \).

5. Conclusions

Motivated by the numerical analysis of approximation methods for solving the fixed point problem of Frobenius-Perron operators, we systematically investigated the spectral structure of the Frobenius-Perron operator \( P_S \) associated with a non-singular transformation \( S \) on a \( \sigma \)-finite measure space. Our results indicate that under very mild conditions, the spectrum of \( P_S \) is either the closed unit disk of the complex plane or a cyclic subset of the unit circle.
Our analysis also shows that even though \( P_S \) is quasi-compact on the space of functions of bounded variation, it is not always so on \( L^1(X) \) in general. Thus, we cannot take advantage of the quasi-compactness of \( P_S \) is the convergence rate analysis of numerical methods in the \( L^1 \) space if we want to use such standard techniques as those in \([1]\). Rather we should restrict our error estimates analysis in the \( BV \) space if all the fixed points of \( P_S \) are functions of bounded variation, in order to apply the quasi-compactness.

Since Frobenius-Perron operators are Markov operators and they have many similar properties, and since Markov operators are widely used in stochastic analysis, applied probability, and other applied fields, it is natural to analyze the spectrum for different classes of general Markov operators. Such a spectral analysis will lay some theoretic basis for designing corresponding numerical algorithms and for error estimates of such methods.

Some problems are still open even for the spectral analysis of Frobenius-Perron operators. A main unsolved problem is determining the spectrum of \( P_S \) in general when neither \( \mu \) is regular nor \( S^{-1}(\mathcal{A}) = \mathcal{A} \). Another problem involves exploring the relation between the spectral structure and such properties as ergodicity, mixingness, and exactness of \( \mu \).

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