REPRESENTATIONS OF THE POINCARÉ GROUP FROM POSITIVE ENERGY REPRESENTATIONS OF SO(2,3) †

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We describe representations of the simply connected covering group of the Poincaré group, \( \mathfrak{P} \), which are associated with certain positive energy representations of \( SO_0(2, 3)^+ \), the universal cover of \( SO_0(2, 3) \). The translation generators of these representations of \( \mathfrak{P} \) can be viewed as solutions of certain algebraic equations with coefficients from a commutative algebraic extension of the skew field of \( \mathfrak{P} \). These representations of \( \mathfrak{P} \) depend upon a parameter \( \lambda \) that is essentially the reciprocal of the radius of anti-deSitter space, and they go over into the Segal-Is"oní-Wigner contractions of the corresponding representations of \( SO_0(2, 3)^+ \), as \( \lambda \to 0 \).

Explicit results are given for the Dirac and Rac representations and the representations of \( SO_0(2, 3)^+ \), which extend to massless, unitary irreducible representations of \( SU(2, 2) \), the four-fold cover of the conformal group of Minkowski space-time.


1. Introduction

There are three well-known space-times of general relativity which carry local causal structures identical to that of Minkowski space \( \mathbb{M}_0 \), and into which \( \mathbb{M}_0 \) may be conformally embedded, namely, the Einstein universe and the two spaces of constant curvature \( \mathbb{M}_\lambda = \mathbb{C} \lambda / \mathcal{L} \), where \( \mathcal{L} \) is \( SO_0(1, 3) \) and \( \mathfrak{C}_\lambda \) is isomorphic to \( SO_0(2, 3) \) for \( \lambda \in \mathbb{R} \) and \( \mathfrak{C}_\lambda \) is isomorphic to \( SO_0(1, 4) \) for \( \lambda \in i \mathbb{R} \). Here we concentrate on the second one. We consider positive energy representations of \( SO_0(2, 3)^+ \) and describe a relationship between these representations and certain representations of \( \mathfrak{P} \). The representations of \( \mathfrak{P} \) are constructed out of skew symmetric solutions \( \mathbf{P}_\mu \) to eqns. (6), solutions which are explicitly described in Theorem 3.2. We have also obtained results identical to those in Theorems 3.2 and 3.3 for \( q \) deformations \( U_q(\mathfrak{s}(p, q)) \) with small \( p \) and \( q \) values, and applying our results to the representations of \( U_q(\mathfrak{s}(p, q)) \) (for \( q \) both generic and a root of unity), we get representations of classical Lie algebras which depend upon the
deformation parameter \(q\). It seems that our results should be related to those of Currong and Zachos in the simplest case of type \(A_1\) i.e. that associated with \(U_q(so(2, 1))\). We have also obtained some analogous results in the classical case for higher dimensional \(SO_0(p, q)\) and associated Poincaré groups.\(^2\)

2. \(SO(2, 3)\), the Poincaré group and their Lie algebras

Let \(\beta_0 = \text{diag}(1, -1, -1, -1, 1)\), where the right hand side of this equation denotes a diagonal matrix with diagonal entries as shown inside the parentheses. \(\Phi = SO(2, 3)\) is the component connected to identity of the group

\[
SO(2, 3) = \{ g \in SL(5, \mathbb{R}) \mid g \beta_0 g^\dagger = \beta_0 \}.
\]

(\(g^\dagger\) denotes transpose of a matrix.) Denote by \(G\) the Lie algebra of \(\Phi\). A realization of \(G\) is provided by the set of all matrices \((a_{ij})\) (0 ≤ \(i, j \leq 4\)) such that \(a_{ii} = 0\) (0 ≤ \(i \leq 4\)), \(a_{ij} = -a_{ji}\) (1 ≤ \(i \leq j \leq 3\)), \(a_{0j} = a_{j0}\) (1 ≤ \(j \leq 3\)), \(a_{4j} = a_{j4}\) (1 ≤ \(j \leq 3\)) and \(a_{04} = -a_{40}\).

Let \(E_{ij}\) be the matrix such that the \((i, j)\) component is equal to 1 and the other components are all equal to 0. Let \(L_{ij} = E_{ij} - E_{ji}\) (0 ≤ \(i \leq j \leq 4\)). \(L_{0i} = E_{i0} + E_{0i}\) (1 ≤ \(i \leq 3\)), \(L_{04} = E_{i4} + E_{4i}\) (1 ≤ \(i \leq 3\)) and \(L_{04} = E_{04} - E_{40}\). The \(L_{ab}\) (\(a, b = 0, 1, 2, 3, 4\)), viewed abstractly, are a basis for \(G\).

A Cartan-Weyl basis for \(G_C\) is given by

\[
-iH_1 = 2L_{21}, \quad H_2 = (L_{40} - L_{21}), \quad -iX^\pm_1 = L_{31} \mp iL_{23},
\]

\[
-iX^\pm_2 = \pm 1 \frac{1}{2} (L_{41} + L_{20}) - i \frac{1}{2} (L_{10} - L_{42}).
\]

\(H_1\) and \(H_2\) are a basis for a Cartan subalgebra \(H_C\) of \(G_C\). The commutation relations for the basis vectors of the simple roots are

\[
[H_j, H_k] = 0, \quad [H_j, X^\pm_k] = \pm a_{jk} X^\pm_k, \quad [X^+_j, X^-_k] = \delta_{jk} H_k
\]

(No summation is intended in these equations.) \((a_{jk}) = 2\frac{\alpha_j}{\alpha_j + \alpha_k}\) is the Cartan matrix of \(so(5, C)\), and \(\alpha_1, \alpha_2\) are the simple roots with products: \((\alpha_1, \alpha_1) = 2\), \((\alpha_2, \alpha_2) = 4\), \((\alpha_1, \alpha_2) = -2\). The Cartan-Weyl basis vectors corresponding to the non-simple roots are given by:

\[
X^\pm_3 = \pm [X^\pm_1, X^\pm_2], \quad X^\pm_4 = \pm \frac{1}{2} [X^\pm_1, X^\pm_2].
\]

Let \(\Psi\) be the Poincaré group, and denote its Lie algebra by \(P\). A basis for the Poincaré Lie algebra is given by the \(L_{ij}\) and \(L_{ij}\) \((i, j = 1, 2, 3)\) of \(so(1, 3)\) together with a commuting Lorentz vector operator \(P_\mu\) which generates the translations.

Denote by \(E(G)\) and \(E(P)\) the universal enveloping algebras of \(G\) and \(P\), respectively. We introduce the following elements of \(E(G)\): \(Q_2 = L_{01}^2 + L_{02}^2 + L_{03}^2 - L_{12}^2 - L_{23}^2 - L_{31}^2\), \(Q_4 = (L_{12}L_{30} + L_{23}L_{40} + L_{31}L_{20})^2\), \(C_2 = -L_{04}^2 + L_{01}^2 + L_{02}^2 + L_{03}^2 - L_{12}^2 - L_{23}^2 - L_{31}^2 + L_{14}^2 + L_{24}^2 + L_{34}^2\).
and 
\[ C_4 = - \left( \frac{1}{2}(\lambda - \rho) \right)^2 - (L_{12} L_{30} + L_{23} L_{10} + L_{31} L_{20})^2 + \sum_{ijk\ell m=1}^3 (\epsilon_{ijk} (\frac{1}{2} L_{04} L_{jk} - L_{0j} L_{4k})) (\epsilon_{\ell m} (\frac{1}{2} L_{04} L_{\ell m} - L_{0\ell} L_{4m})) \]

with \( \frac{1}{2}(\lambda - \rho) = L_{12} L_{34} + L_{23} L_{14} + L_{31} L_{24} \). The center \( Z(\mathcal{F}) \) of \( \mathcal{F}(\mathcal{G}) \) is generated by \( C_2 \) and \( C_4 \). The center \( Z(\mathcal{P}) \) of \( \mathcal{F}(\mathcal{P}) \) is generated by the following set of elements:
\[
P^2 = \sum_{k=0}^3 P_k P^k = P_0^2 - P_1^2 - P_2^2 - P_3^2,
\]
and \( W = \sum_{\mu=0}^3 \sum_{\nu=0}^3 (P_{\mu} P_{\nu} L_{\mu \nu} L_{\nu \mu} - \frac{1}{2} P_{\mu} P_{\nu} L_{\mu \nu} L_{\nu \mu}) \).

3. Algebraic Results

Let \( \mathcal{K}(\mathcal{P}) \) be the skew field of \( \mathcal{P} \) and \( \mathcal{K}(\mathcal{G}) \) be the skew field of \( \mathcal{G} \). We define the following commutative algebraic extensions of \( \mathcal{K}(\mathcal{P}) \) and \( \mathcal{K}(\mathcal{G}) \):
\[
\tilde{\mathcal{K}}(\mathcal{P}) = \left\{ a + b Y \mid a, b \in \mathcal{K}(\mathcal{P}) \right\}, \quad \tilde{\mathcal{K}}(\mathcal{G}) = \left\{ a + b \tilde{Y} + c \tilde{Y}^2 + d \tilde{Y}^4 \mid a, b, c, d \in \mathcal{K}(\mathcal{G}) \right\}
\]
de where \( \tilde{Y} \) commutes with all elements of \( \tilde{\mathcal{K}}(\mathcal{G}) \) and satisfies the equation \( \tilde{Y}^4 + C_2 \tilde{Y}^2 + C_4 = 0 \) with \( C_2 = \left( C_2 + \frac{1}{4} I \right) \) and \( C_4 = \left( C_4 + \frac{1}{4} C_2 + \frac{9}{16} I \right) \). (I is the identity in \( \tilde{\mathcal{K}}(\mathcal{G}) \).) Now define a mapping \( \tau_\lambda \) from \( \mathcal{G} \) to \( \tilde{\mathcal{K}}(\mathcal{P}) \) by
\[
\tau_\lambda(L_{\mu \nu}) = L_{\mu \nu}, \quad \tau_\lambda(L_{4\mu}) = \frac{i}{2Y} \left[ Q_2, P_\mu \right] + P_\mu.
\]

The \( \lambda^{-1} \tau_\lambda(L_{4\mu}) \) and \( \tau_\lambda(L_{\mu \nu}) \) satisfy the commutation relations of the generators of \( \mathcal{G} \). The \( \tau_\lambda(L_{4\mu}) \) and \( \tau_\lambda(L_{\mu \nu}) \) are a basis for an isomorphic copy \( \mathcal{G}_\lambda \) of \( \mathcal{G} \), which differs from \( \mathcal{G} \) by a scaling factor \( \lambda \) in the \( L_{4\mu} \) directions, and hence generate \( \mathcal{G}_\lambda \).

We henceforth consider for simplicity the case \( \lambda = 1 \) and let \( \tau_{\lambda=1}(\tilde{Y}) = Y \), then \( \tau = \tau_{\lambda=1} \) can be extended to a homomorphism of \( \tilde{\mathcal{K}}(\mathcal{G}) \) into \( \tilde{\mathcal{K}}(\mathcal{P}) \) in an obvious way, which, because of Theorem 3.2, is actually surjective. Denote this extension also by \( \tau \). Elements of \( \tilde{\mathcal{K}}(\mathcal{G}) \) are denoted with a tilde to keep them distinct from elements of \( \tilde{\mathcal{K}}(\mathcal{P}) \).

**Theorem 3.1.** Let \( \mathcal{G} \) be the deformation of \( \mathcal{P} \) having basis elements \( L_{ij} \in \mathcal{G} \) and \( L_{4\mu} \in \tilde{\mathcal{K}}(\mathcal{P}) \) defined by eqns. (6). Then (for \( \lambda = 1 \)) the following holds:
\[
C_2 = - Y^2 - \left[ \frac{W}{Y^2} + \frac{9}{4} I \right], \quad C_4 = \left[ Y^2 + \frac{1}{4} \right] \frac{W}{Y^2}.
\]

Now we view the second set of eqns. in (6) as algebraic equations in \( \tilde{\mathcal{K}}(\mathcal{P}) \) and solve them for the \( P_\mu \).

**Theorem 3.2.** Solutions \( P_\mu \) to eqns. (6) (\( \lambda = 1 \)) are given by:
\[
P_\mu = D^{-1} A_\mu \nu L^{\mu \nu}
\]

with \( A_\mu \nu = \left[ Q_2 + \frac{1}{4} C_2 \right] \delta_\mu \nu - \frac{1}{2} L_\mu \nu L^{\mu \nu} - Q_4 \epsilon_\mu \nu \rho \rho L^{\mu \nu} \) Y
\[-\left[ (Q_2 + \frac{1}{4} C_2) \delta_\mu \nu - L_\mu \nu L^{\mu \nu} \right] Y^2 + i \left( \frac{1}{2} \delta_\mu \nu \right) Y^3, \quad D = Q_4 +
\[ \frac{1}{8}Q_2 - C_4^t + \frac{3}{16} I + i (Q_2 + \frac{1}{2}) Y - (Q_2 - C_2^t - \frac{1}{2}) Y^2 + 2iY^3. \] Furthermore \( Y^2 \) satisfies the equation

\[ Y^4 + C_4^t Y^2 + C_4^t = 0. \]

An easy way to prove Theorem 3.2 is to show that \( P_0 = D^{-1} A_0 L_{\nu 4} \) satisfies eqns. (6), and then use \( P_i = [L_{0i}, P_0] \).

**Lemma 3.1.** \( \ker(\tau) \bigg|_{\mathcal{E}(\mathcal{G})} = 0. \)

**Proof:** Let \( I = \{ \tilde{u} \in \mathcal{E}(\mathcal{G}) | \tau(\tilde{u}) = 0 \} \). \( I \) is a two sided ideal in \( \mathcal{E}(\mathcal{G}) \). Suppose \( I \neq 0 \). Then \( \mathcal{G} \) semisimple \( \Rightarrow I \cap \mathbb{Z}(\mathcal{G}) \neq \{ 0 \} \). Thus \( \exists \tilde{z} \in I \cap \mathbb{Z}(\mathcal{G}), \tilde{z} \neq 0, \tilde{z} = \sum \alpha_{ij} (C_i^2)^i (C_i^t)^j \tau(z) = \sum \alpha_{ij} (-Y^2 - \frac{x}{y} + \frac{1}{4})i \left( \frac{y}{x} + \frac{1}{4} \right)^j = 0. \)

Now evaluate this last equation in an arbitrary representation \( \pi \) of \( \mathcal{P} \). Since \( d\pi \) is arbitrary it must be that the \( \alpha_{ij} \) vanish identically, i.e. \( \alpha_{ij} = 0 \), and thus \( \ker(\tau) \bigg|_{\mathcal{E}(\mathcal{G})} = 0. \)

Now we introduce \( \ast \) structures on \( \mathcal{K}(\mathcal{P}) \) and \( \mathcal{K}(\mathcal{G}). \)

**Lemma 3.2.** If \( \tilde{L}_{\mu}^\dagger = -\tilde{L}_{\nu}^\dagger, \tilde{L}_{4\mu} = -\tilde{L}_{4\mu} \) and if \( \tilde{Y}^\dagger = \tilde{Y} \), then \( \tilde{P}_\mu = (\tilde{D}^{-1} \tilde{A}_{\mu} \tilde{L}_{4\mu}) \) and also \( \tilde{P}_\mu = \left( (\tilde{D}^{-1} \tilde{A}_{\mu} \tilde{L}_{4\mu}) \right) = \tilde{L}_{4\mu} \tilde{A}_{\mu} \tilde{L}_{4\mu} = -\tilde{P}_\mu. \)

Furthermore \( [\tilde{P}_\mu, \tilde{P}_\nu] = 0. \)

**Proof:** Only commutativity of the \( \tilde{P}_\mu \) is difficult: \( \tilde{P}_\mu \tilde{P}_\nu - \tilde{P}_\nu \tilde{P}_\mu = -\left( \tilde{P}_\mu \tilde{P}_\nu - \tilde{P}_\nu \tilde{P}_\mu \right) = (\tilde{D}^{-1} (\tilde{F}_{\nu}^\dagger - \tilde{F}_{\mu}^\dagger) \tilde{P}_{\mu} = 0 \Rightarrow \tau(\tilde{F}_{\nu}^\dagger - \tilde{F}_{\mu}^\dagger) = 0 \Rightarrow \tilde{F}_{\nu}^\dagger - \tilde{F}_{\mu}^\dagger = 0 \) by Lemma 3.1. Thus \( 0. \)

Although a representation of \( \mathcal{G} \) always gives a representation of the enveloping algebra \( \mathcal{E}(\mathcal{G}). \) it does not necessarily give a representation of the skew field. However, the following follows immediately from Lemma 3.2.

**Theorem 3.3.** Let \( (d\pi, \mathcal{H}) \) be an infinitesimally unitarizable representation of \( \mathcal{G} \) on an Hilbert space \( \mathcal{H}, \) and let \( \tilde{Y} \) be a self adjoint operator on \( \mathcal{H} \) which satisfies \( \tilde{Y}^4 + d\pi(\tilde{C}_i^2) \tilde{Y}^2 + d\pi(\tilde{C}_i^t) = 0. \) Then, if both \( d\pi(\tilde{D})^{-1} \) and \( d\pi(\tilde{D})^\dagger \) exist on a suitable, dense domain in \( \mathcal{H}, \) there exists a skew symmetric representation \( d\tilde{\pi} \) of \( \mathcal{P} \) on \( \mathcal{H} \) defined by: \( d\tilde{\pi}(L_{ij}) = d\pi(\tilde{L}_{ij}), d\tilde{\pi}(P_0) = d\pi(\tilde{D})^{-1}, d\tilde{\pi}(\sum_{i=0}^2 \tilde{A}_i \tilde{L}_{n,i}) \) and \( d\tilde{\pi}(P_i) = [d\pi(L_{0i}), d\tilde{\pi}(P_0)](i = 1, 2, 3). \)

Notice it is crucial that both operators \( d\pi(\tilde{D})^{-1} \) and \( d\pi(\tilde{D})^\dagger \) exist on the suitable dense domain in \( \mathcal{H} \) in order to be assured of the existence of mutually commuting translation operators \( d\tilde{\pi}(P_\mu). \)
4. Representations

**Definition 4.1.** (For brevity, we let here and below $\hat{X} = d\pi(\hat{X})$ for $\hat{X} \in \hat{\mathcal{G}}(\mathcal{G})$) or $\mathcal{E}(\mathcal{G}_C)$.) Let $V$ be an $\mathcal{E}(\mathcal{G}_C)$ module with action $d\pi$ of $\mathcal{E}(\mathcal{G}_C)$ on $V$. Let: i) $V = \sum_\lambda V_\lambda$ with $V_\lambda = \{ v \in V \mid H_i v = \lambda(H_i) v \}$; 2) $\exists \ v_0 \ \exists \ a \ \ x_i^+(x_i^-) v_0 = 0$, b) $H_i v_0 = \lambda(H_i) v_0$, c) $\mathcal{E}(\mathcal{G}_C)v_0 = V$. Then, if $d\pi$ comes from a unitary representation of $\mathcal{G}_C$, $V$ is an infinitesimally unitarizable lowest (highest) weight representation with lowest (highest) weight $\lambda$.

**Definition 4.2.** An infinitesimally unitarizable, irreducible representation $d\pi$ of $\mathcal{G}_C$ on an Hilbert space $\mathcal{H}$ is a representation with positive energy if $\{ X \in \mathcal{G}_C \mid i \ d\pi(X) \text{ is a non-negative self-adjoint operator} \}$ is a non-zero and proper cone in $\mathcal{G}_C$.

**Theorem 4.1.** (Peclet, Ohashi)\(^5\) Let $d\pi$ be an irreducible, infinitesimally unitarizable representation of $\mathcal{G}$ on an Hilbert space $\mathcal{H}$. Then $d\pi$ is positive energy $\Leftrightarrow d\pi$ is a lowest (highest) weight representation.

A classification of the infinitesimally unitarizable, irreducible lowest weight representations of $\mathcal{G}$ using a modern representation theoretic approach involving Verma modules and singular vectors is given in Ref. 6. Similar classifications for other noncompact Lie groups and supergroups are given in Ref. 7. The results in Ref. 6 are: let $D(E_0, s_0)$ be a given such lowest weight representation with lowest weight $\Lambda \ni \Lambda(H_1) = s_0$ and $\Lambda(H_2) = E_0$. Then we have: i) $D(E_0, s_0) = D(\frac{1}{2}, 0) \text{ (Rac)}$; ii) $D(E_0, s_0) = D(1, \frac{1}{2}) \text{ (Di)}$; iii) $D(E_0 > \frac{1}{2}, s_0 = 0)$; iv) $D(E_0 > 1, s_0 = \frac{1}{2})$; v) $D(E_0 \geq s_0 + 1, s_0 \geq 1)$.

The massless representations are $D(E_0 = s_0 + 1, s_0 \geq 1)$.

**Proposition 4.1.** The massless representations $D(E_0 = s_0 + 1, s_0 \geq 1)$ give representations of $\mathcal{P}$, none of which are skew symmetric.

Proof: We have $\tilde{C}_2 = 2(s_0^2 + 1) \cdot I$ and $\tilde{C}_4 = \left[ s_0^2 - 1 \right] . I$. (I is the identity in $\mathcal{H}$.) Thus $\tilde{Y}^2 = \tilde{C}_2 \cdot \tilde{Y}^2 + \tilde{C}_4 = 0 \Rightarrow \left[ \tilde{Y}^2 + (s^2 + \frac{1}{2}) \cdot I \right]^2 = s^2 \cdot I \Rightarrow \tilde{Y} = \left[ \tilde{Y} + (s^2 + \frac{1}{2}) \cdot I \right] = \pm s \cdot I \Rightarrow \tilde{Y}^2 = -(s \pm \frac{1}{2})^2 \cdot I \Rightarrow$ according to Theorem 3.3 that $d\pi(\mathcal{P}_\mu)$ is not skew symmetric. However, $d\pi(\mathcal{D})^{-1}$ and $(d\pi(\mathcal{D}))^{-1}$ exist as continuous operators on the representation spaces for all massless representations, so that we still get from Theorem 3.3 non-unitary representations of the Poincaré group.

Similar calculations for the other representations give essentially the same results, except in the case of the Di and Rac. What goes wrong in these cases is more serious. For both the Di and Rac is $\tilde{Y}^2 = -1, -\frac{1}{2}$ which implies not only that $d\pi(\mathcal{P}_\mu)$ are not skew-symmetric operators, but, e.g. for the Rac $d\pi(\mathcal{D}) = 0$ (with $Y = -i$). Thus the $d\pi(\mathcal{P}_\mu)$ do not even exist as skew symmetric operators on the Hilbert space of the representation. Hence, the Di and the Rac do not have contractions to representations of the Poincaré group. This analysis sharpens the
meaning of kinematical confinement by Flato, Fronsdal, Sternheimer and their coworkers, a notion which one must admit is one of the most fascinating and powerful ideas of theoretical physics in modern times.

References