A Theory of Stable Market Segmentations

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Market Segmentation

Where do segmentations come from?

Segment = a coalition of consumers
Market Segmentation

Where do segmentations come from?

Segment = a coalition of consumers

$3

$2
Where do segmentations come from?
Where do segmentations come from?

If consumers choose?

Segment = a coalition of consumers
Where do segmentations come from?

If consumers choose?

- Segment = a coalition of consumers
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives
Platforms, clubs, cooperatives

Platform 1

Platform 2

Platform 3
“Stable” segmentations
“Stable” segmentations have “good welfare properties”
Coalitions, segments, and segmentations
Coalitions, segments, and segmentations

Measure 0.4 0.6
Values 1 2
Consumers 0 0.4 1
Coalitions, segments, and segmentations

Values

Consumers

0 0.4 1

\[ S = \{ (C_1, 1), (C_2, 2) \} \]

\[ \forall c \in C_1, CS(c, S) = \max \{ v(c) - 1, 0 \} \]

\[ \forall c \in C_2, CS(c, S) = \max \{ v(c) - 2, 0 \} \]
Coalitions, segments, and segmentations

Segmentation $S = \{(C_1, 1), (C_2, 2)\}$ s.t. coalitions partition $[0, 1]$

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Consumers Values Measures

$0$ $1$ $2$

$0$ $0.4$ $1$

$(\{1, 2\}) C_1$

$0$ $0.3$ $0.7$ $1$

$(\{1, 2\}) C_2$

$0.3$ $0.7$
Coalitions, segments, and segmentations

Values

Consumers

\( \{1, 2\} \) \( C_1 \)

\( \{1, 2\} \) \( C_2 \)

\( v(1) = 1 \), \( v(2) = 2 \)

\( CS(c, S) = \max \{ v(c) - 1, 0 \} \)

\( CS(c, S) = \max \{ v(c) - 2, 0 \} \)
Coalitions, segments, and segmentations

$(C_1, 1)$: a segment  $(C_1, 2)$: a segment
$(C_2, 1)$: not a segment  $(C_2, 2)$: a segment
Coalitions, segments, and segmentations

\(\{C_1, 1\}\): a segment \hspace{1cm} \(\{C_1, 2\}\): a segment
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Outline

1. Core
2. Stability
The core

Definition (Objection)

A segment \((C, p)\) objects to segmentation \(S\) if

\[
\max \{v(c) - p, 0\} \geq CS(c, S) \text{ for all } c \in C
\]

\[
\text{max} \{v(c) - p, 0\} > CS(c, S) \text{ for some } (\text{measure } > 0) c \in C
\]

Note: Objecting segment \((C, p) / \in S\)

Definition (Core)

\(S\) is in the core if \(\not\exists\) segment \((C, p)\) that objects to \(S\)
The core

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A segment \((C, p)\) objects to segmentation \(S\) if

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The core

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Definition (Core)

\(S\) is in the core if \(\not\exists\) segment \((C, p)\) that objects to \(S\)
Core is empty in non-trivial cases
Core is empty in non-trivial cases

Let $v_1$ be the lowest possible value

**Proposition**

1. If price $v_1$ is *revenue-maximizing* to sell to $[0, 1]$,

2. If price $v_1$ is *not revenue-maximizing* to sell to $[0, 1]$,
Core is empty in non-trivial cases

Let $v_1$ be the lowest possible value

**Proposition**

1. *If price $v_1$ is revenue-maximizing to sell to $[0,1]$, then $(([0,1], v_1)) \in \text{core}$ and “essentially unique”*

2. *If price $v_1$ is not revenue-maximizing to sell to $[0,1]$, then Core is empty*
Core is empty in non-trivial cases

Let $v_1$ be the lowest possible value

**Proposition**

1. If price $v_1$ is revenue-maximizing to sell to $[0, 1]$, 
   
   $\{([0, 1], v_1)\} \in $ core and “essentially unique”

2. If price $v_1$ is not revenue-maximizing to sell to $[0, 1]$, 
   
   Core is empty

Essentially unique: If $S'$ in core, then $S' \approx \{([0, 1], v_1)\}$

$\triangleright$ $S' \approx S$: $CS(c, S') = CS(c, S)$ for (almost) all $c \in [0, 1]$
Two type illustration

If $\delta < 0.8$:

$S = \{ (C_1, 1), (C_2, 2) \}$ not in core

If $\delta = 0.8$:

$S = \{ (C_1, 1), (C_2, 2) \}$ not in core

Segment $(C'_1, 1)$ objects

But $(C_1, 1) \in S$ also objects to resulting $S' = \{ (C'_1, 1), (C'_2, 2) \}$

Values 1 2

Consumers 0 0.4 1

$\delta (\{1, 2\}) (C_1, 0) = 0.8 (\{1, 2\}) (C_2, 0.4) = 0.8 (\{1, 2\}) (C'_1, 0) = 0.4 (\{1, 2\}) (C'_2, 0.8)$
Two type illustration

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- Segment $(C_1', 1)$ objects

\[
\begin{array}{c|cc|c}
\text{Values} & 1 & \cdot & 2 \\
\hline
\text{Consumers} & & & \\
0 & 0.4 & 1 \\
(\{1, 2\}) \ C_1 & & \qquad \delta = 0.8 & \\
0 & 1 \\
(\{2\}) \ C_2 & & 0.8 & 1 \\
(\{1, 2\}) \ C_1' & & 0.4 & 0.8 & 1 \\
\end{array}
\]
Two type illustration

If $\delta < 0.8$: $S = \{(C_1, 1), (C_2, 2)\}$ not in core
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- Segment $(C_1', 1)$ objects
- But $(C_1, 1) \in S$ also objects to resulting $S' = \{(C_1', 1), (C_2', 2)\}$
Stability

Definition (Stability)

$S$ is stable if $\forall S' \not\approx S$, $\exists (C, p) \in S$ that objects to $S'$

Existing coalitions have sovereignty.
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Existing coalitions have sovereignty.
Two type illustration and stability

\( S = \{(C_1, 1), (C_2, 2)\} \) is stable

- \((C_1, 1)\) objects to any \(S' \not\approx S\)

\[
S = \{(C_1, 1), (C_2, 2)\}
\]

\((C_1, 1)\) objects to any \(S' \not\approx S\)
Two type illustration and stability

$S = \{(C_1, 1), (C_2, 2)\}$ is stable

- $(C_1, 1)$ objects to any $S' \neq S$

$S' = \{(C'_1, 1), (C'_2, 2)\}$ is not stable

- $S$ objects to $S'$ but $S'$ doesn’t object to $S$

Values

<table>
<thead>
<tr>
<th>Consumers</th>
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<tr>
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<th>2</th>
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Comparing solution concepts

Definition (Stability)

$S$ is stable if there is no deviation from it

Definition (Core)

$S$ is in the core if there is no deviation from it
Comparing solution concepts

**Definition (Stability)**

$S$ is stable if there is no deviation from it

$\rightarrow$ $S$ does not contain an objection to $S$

Objection in $S$ has the power to prevent a move

**Definition (Core)**

$S$ is in the core if there is no deviation from it

$\rightarrow S \rightarrow S'$ if $S'$ contains an objection to $S$

Objection in $S'$ has the power to force a move
Comparing solution concepts

**Definition (Stability)**

$S$ is **stable** if there is no deviation from it

$\Rightarrow S \rightarrow S'$ if $S$ does not contain an objection to $S'$

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Characterization of stable segmentations

Proposition

1. Segmentation is stable iff its induced canonical segmentation is stable
2. Canonical segmentation $S$ is stable iff it is efficient and saturated
Characterization of stable segmentations

Proposition

1. Segmentation is stable iff its induced canonical segmentation is stable
2. Canonical segmentation $S$ is stable iff it is efficient and saturated
Segmentations

efficient

Stable = efficient + saturated

Core

?
Stable segmentations exist? An example
Stable segmentations exist? An example

Values: 1, 2, 3

Consumers: 0, 1/3, 1/2, 1
Stable segmentations exist? An example

\( S = \{(C_1, 1), (C_2, 2), (C_3, 3)\} \)
Stable segmentations exist? An example

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Maximal equal-revenue (MER) segmentation
Maximal equal-revenue (MER) segmentation

Is defined recursively. Let $\bar{C} = [0, 1]$, $S = \emptyset$

1. $C := \text{largest coalition where all prices (among remaining values in } \bar{C} \text{) are revenue-maximizing}$
2. Add $(C, \nu(C))$ to $S$
3. Remove $C$ from $\bar{C}$
4. Repeat until $\bar{C} = \emptyset$

Proposition
The MER segmentation is stable

Bergemann, Brooks, Morris (2015):

$\triangleright$ The MER segmentation maximizes consumer surplus
$\triangleright$ But is not the only one
Maximal equal-revenue (MER) segmentation

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In each step $|\{\nu|\exists c \in \bar{C}, \nu(c) = \nu\}|$ reduces by at least 1
Maximal equal-revenue (MER) segmentation

Is defined recursively. Let $\bar{C} = [0, 1]$, $S = \emptyset$

1. $C := \text{largest coalition where all prices (among remaining values in } \bar{C}\text{) are revenue-maximizing}$
2. Add $(C, v(C))$ to $S$
3. Remove $C$ from $\bar{C}$
4. Repeat until $\bar{C} = \emptyset$

In each step $|\{v | \exists c \in \bar{C}, v(c) = v\}|$ reduces by at least 1

Proposition

*The MER segmentation is stable*
Maximal equal-revenue (MER) segmentation

Is defined recursively. Let $\bar{C} = [0, 1], S = \emptyset$

1. $C :=$ largest coalition where all prices (among remaining values in $\bar{C}$) are revenue-maximizing
2. Add $(C, v(C))$ to $S$
3. Remove $C$ from $\bar{C}$
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In each step $|\{v| \exists c \in \bar{C}, v(c) = v\}|$ reduces by at least 1

Proposition

The MER segmentation is stable

Bergemann, Brooks, Morris (2015):

- The MER segmentation maximizes consumer surplus
- But is not the only one
Stability $\not\Rightarrow$ maximizing consumer surplus
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Values

Consumers

0 $\frac{1}{3}$ $\frac{2}{3}$ 1
Stability $\neq$ maximizing consumer surplus

$S = \{(C_1, 1), (C_2, 3)\}$ is efficient and saturated $\Rightarrow$ stable
Stability $\not\iff$ maximizing consumer surplus
Stability $\not\equiv$ maximizing consumer surplus

$S = \{(C_1, 1), (C_2, 2)\}$ maximizes consumer surplus

![Graph showing consumer surplus](image-url)
Stability $\Leftrightarrow$ maximizing consumer surplus

$S = \{(C_1, 1), (C_2, 2)\}$ maximizes consumer surplus

- Efficient allocation
- $p = 3$ is revenue-maximizing for $C_1, C_2, [0, 1]$

$S$ is not saturated and so not stable:

<table>
<thead>
<tr>
<th>Values</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumers</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
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$\{1, 3\}$

$\{2, 3\}$

$C_1$

$C_2$
Stability \not\Rightarrow\text{ maximizing consumer surplus}

\[ S = \{(C_1, 1), (C_2, 2)\} \text{ maximizes consumer surplus} \]

- Efficient allocation
- price 3 is revenue-maximizing for \( C_1, C_2, [0, 1] \)

\( S \) is not saturated and so not stable:

\[
\begin{array}{cccc}
\text{Values} & 1 & 2 & 3 \\
\text{Consumers} & 0 & \frac{1}{3} & \frac{1}{2} & 1 \\
\end{array}
\]

\[
\{(1, 3)\} \quad C_1 \\
\begin{array}{cccc}
0 & \frac{1}{3} & \frac{5}{6} & 1 \\
\end{array}
\]

\[
\{(2, 3)\} \quad C_2 \\
\begin{array}{cccc}
\frac{1}{3} & \frac{5}{6} \\
\end{array}
\]
Segmentations

efficient

Stable

MCS

MER
Segmentations

efficient

Pareto undominated

Stable

MCS

MER
Pareto undominance

Definition (Pareto undominance)

A set $S$ is Pareto undominated if there does not exist a set $S'$ such that $CS(c, S') \geq CS(c, S)$ for all $c \in [0, 1]$ and $CS(c, S') > CS(c, S)$ for some (measure $> 0$) $c \in [0, 1]$

Proposition

Stable $\subset$ Pareto undominated $\subset$ efficient
Pareto undominance

Definition (Pareto undominance)

S is Pareto undominated if \( \forall S' \) s.t.

\[
CS(c, S') \geq CS(c, S) \quad \text{for all} \quad c \in [0, 1] \\
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Pareto undominance

Definition (Pareto undominance)

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\[ CS(c, S') > CS(c, S) \quad \text{for some (measure > 0)} \quad c \in [0, 1] \]

Proposition

Stable ⊂ Pareto undominated ⊂ efficient
Related work

Markets as coalitional games
► Shapley (1959); Shubik (1959); . . .; Peivandi and Vohra (2021)
► Core vs. CE: Edgeworth (1881); Debreu and Scarf (1963)

Third degree price discrimination
► Pigou (1920); Robinson (1969); Schmalensee (1981); Varian (1985);
  Aguirre, Cowan, Vickers (2010); Cowan (2016); . . .

Decentralized Exchanges
► Malamud and Rostek (2017); Chen and Duffie (2021)

Information design
► All segmentations: Bergemann, Brooks, Morris (2015)
► Maximize CS: Hidir and Vellodi (2018); Ichihashi (2020)

Other solutions concepts
► Stable sets (vNM, Harsanyi, Ray and Vohra)
► Bargaining set
Problem: market power leads to inefficiency
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Tools:

1. Antitrust
2. Regulated natural monopolist
Problem: market power leads to inefficiency

Tools:
1. Antitrust
2. Regulated natural monopolist
3. This paper: market segmentation

- Stable segmentations: efficient, Pareto un-dominated (for consumers)
  - One of them maximizes average consumer surplus
- "Perfect" segmentation: efficient, eliminates consumer surplus

How to implement stable segmentations?
- Ensure coalitional sovereignty
Problem: market power leads to inefficiency

Tools:
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   ▶ Stable segmentations: efficient, Pareto un-dominated (for consumers)
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How to implement stable segmentations?
▶ Ensure coalitional sovereignty
Consumer’s control over their data
The Commission recognizes the need for flexibility to permit [...] uses of data that benefit consumers.

("Consumer Privacy in an Era of Rapid Change", FTC, 2012)

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Consumer’s control over their data

- Data cooperatives
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Consumer’s control over their data

▶ Data cooperatives
Conclusions

Market segmentation as a tool for achieving efficiency

Market segmentation subject to “coalitional sovereignty”

- Stable segmentations are efficient and saturated
  - They are all Pareto un-dominated
  - One of them maximizes consumer surplus
Segmentations

- Efficient
- Pareto undominated
- Stable
- MCS
- MER
Thanks!
Recall: Stability

**Definition**

*S* is **stable** if it objects to any *S' ∼ S*
Stable set (von Neumann and Morgenstern 1944)

Definition

A set of segmentations $\mathcal{S}$ is a \textit{stable set} if

1. **Internal Stability:** \(\forall S \in \mathcal{S}, \nexists S' \in \mathcal{S} \text{ that objects to } S\)

2. **External Stability:** \(\forall S \notin \mathcal{S}, \exists S' \in \mathcal{S} \text{ that objects to } S\)

If $S$ is stable then \(\{S' \mid S' \approx S\}\) is a stable set:

- $S'$ does not object to $S$
- $S$ objects to any $S'' \not\approx S$

**Proposition**

$S$ is stable set iff $S = \{S' \mid S' \approx S\}$, s.t. $S$ weakly objects to any $S'' \not\approx S$. 

Stable set (von Neumann and Morgenstern 1944)

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A set of segmentations $S$ is a stable set if

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If $S$ is stable then $\{ S' : S' \approx S \}$ is a stable set:

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Other stable sets

Definition

- **S Harsanyi-objects to S’** if exists $S’ = S^0, S^1 \ni C^1, \ldots, S^k = S \ni C^k$ s.t. $CS(c, S^{i-1}) \leq CS(c, S)$ for all $c \in C^i$ ($<$ for some).

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Proposition

The following are equivalent for any set of segmentations $S$:

- $S$ is a Harsanyi stable set
- $S$ is a RV stable set
- $S = \{S’: S’ \approx S\}$ where $S$ is Pareto undominated.
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Other solution concepts: Bargaining set

For each objection, ∃ stronger objection to same segmentation

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**Stability:** for each objection, ∃ objection to resulting segmentation

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Formally: ∀ objection \((C, p), ∃\) counter-objection \((C', p')\):

- \(CS(c, (C', p')) \geq CS(c, S)\) for all \(c \in C' \setminus C\)
- \(CS(c, (C', p')) \geq CS(c, (C, p))\) for all \(c \in C' \cap C\)

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Other solution concepts

kernel, nucleolus

- Similar to bargaining set
- Not applicable to NTU games
  - need to measure “dissatisfaction” of coalitions