When Is Pure Bundling Optimal?

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Joint work with Jason Hartline (Northwestern)

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Multi-product Monopolist’s Optimal Selling Strategy?

- Sell Separately
- Pure Bundling
- Mixed Bundling

This paper: When is Pure Bundling Optimal?
Multi-product Monopolist’s Optimal Selling Strategy?

- **Sell Separately:** Offer each product for a price
Multi-product Monopolist’s Optimal Selling Strategy?

- **Sell Separately:** Offer each product for a price
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![Logos](image)
Multi-product Monopolist’s Optimal Selling Strategy?

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- **Mixed Bundling:** Offer a menu of bundles and prices
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The Model
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Single seller, products 1 to $k$, single buyer
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- Bundle $b \subseteq \{1, \ldots, k\}$
- $v_b$ value for bundle $b$
- Type $\nu = (v_b)_{b \subseteq \{1, \ldots, k\}}$
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Mechanism:

- menu of (price, bundle)

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</tr>
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<tbody>
<tr>
<td>$4$</td>
<td>$b$</td>
</tr>
<tr>
<td>$5$</td>
<td>$b'$</td>
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Pure Bundling Mechanism:

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Examples: Additive Values \( v_{1,2} = v_1 + v_2 \)
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Stigler ’63, Adams & Yellen ’76: Bundle if values negatively correlated
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Mixed Bundling:

Stigler ’63, Adams & Yellen ’76: Bundle if values negatively correlated
McAfee et al. ’89: Pure bundling generically not optimal
Special Case: Two Identical Products

Values ($v_1, v_{gb}$) $\sim$ $\mu$ $\left( v_{gb} \neq 2v_1 \right)$

Mechanism:

Pure Bundling (PB) Mechanism:

Two Units $\$5$

One Unit $\$3$

Two Units $\$4$

Main Result:

$\mu$ PB optimal if $v_1 / v_{gb}$ "stochastically nondecreasing" in $v_{gb}$

High $v_{gb}$ implies high "relative utility"

$\mu$ PB not optimal if $v_1 / v_{gb}$ "stochastically decreasing" in $v_{gb}$
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Intuition

PB optimal if high $v_{gb}$ implies high “relative utility” $v_1/v_{gb}$
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$$v_1$$

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\downarrow \\
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PB optimal if high $v_{gb}$ implies high “relative utility” $v_1/v_{gb}$
A Special Case: Types “on a Path”

\[ \text{Ratio (relative utility)} r(v_{gb}) := \frac{v_1}{v_{gb}}; \text{ e.g., } r(v_{gb}) = \hat{r} \]

\[ \text{Proposition} \]

Given "Path" \( V_1 \), \( PB \) is optimal \( \forall \mu \) iff \( r \) monotone nondecreasing.
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\[ v_{gb} = Pr[v']v'_{gb} \]
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Proposition

Given “Path” \( V_1 \), PB is optimal \( \forall \mu \) iff \( r \) monotone nondecreasing.

Stokey’79, Acquisti and Varian’05:

- PB optimal if \( r \) constant
Main Theorem (Two Identical Products)

Ratio (relative utility) $r := \frac{v_1}{v_{gb}}$

Theorem PB is optimal if $r$ stochastically nondecreasing in $v_{gb}$.

Not optimal if $r$ stochastically decreasing in $v_{gb}$.

$r$ stochastically nondecreasing in $v_{gb}$:

$\text{Pr}(r \geq \hat{r} | v_{gb})$ nondecreasing in $v_{gb}$ (stochastic dominance)
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[Diagram showing a graph with \( v_1 \) and \( v_{gb} \) axes, and a shaded region indicating the range of \( r \).]
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Curve:
Main Theorem: Any Number of Products

Products 1 to $k$, $(v_b)_{b \subseteq \{1, \ldots, k\}}$

- $\forall b$, define ratio $r_b = \frac{v_b}{v_{gb}} \in [0, 1]$. Let $r = (r_b)_{b \subseteq \{1, \ldots, k\}}$. 

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$r$ **stochastically nondecreasing in** $v_{gb}$:

- $Pr(r \in \hat{R} \mid v_{gb})$ nondecreasing in $v_{gb}$ for all “upper sets” $\hat{R}$.
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Example 1: Complementarities

Two products, values \( v\{1\}, v\{2\}, v\{1,2\} \)
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Two products, values $v_{\{1\}}, v_{\{2\}}, v_{\{1,2\}}$ described by $x, y_1, y_2$

$$v_b = x \cdot (y_1 1_{1\in b} + y_2 1_{2\in b} + (1 - y_1 - y_2) 1_{1,2\in b})$$
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\[ v_b = x \cdot (y_1 1_{1 \in b} + y_2 1_{2 \in b} + (1 - y_1 - y_2) 1_{1,2 \in b}) \]

- $x$: intensity
- $y_1$: values for product 1 only ($y_2$ for product 2)
- $y_1 + y_2 > 1 \Rightarrow$ substitutes: $v_{\{1\}} + v_{\{2\}} > v_{\{1,2\}}$. 

Corollary

$PB$ is

- optimal if $(y_1, y_2)$ stochastically nondecreasing in $x$.
- not optimal if $(y_1, y_2)$ stochastically decreasing in $x$.

$PB$ optimal if high value consumers consider products more substitutable.
Example 1: Complementarities

Two products, values \( v_{\{1\}} \), \( v_{\{2\}} \), \( v_{\{1,2\}} \) described by \( x, y_1, y_2 \)

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- \( x \): intensity
- \( y_1 \): values for product 1 only (\( y_2 \) for product 2)
- \( y_1 + y_2 > 1 \Rightarrow \) substitutes: \( v_{\{1\}} + v_{\{2\}} > v_{\{1,2\}} \).
  \( y_1 + y_2 > 1 \Rightarrow \) complements; \( y_1 + y_2 = 1 \Rightarrow \) additive

Corollary

PB is optimal if \((y_1, y_2)\) stochastically nondecreasing in \( x \).

PB is not optimal if \((y_1, y_2)\) stochastically decreasing in \( x \).

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- $y_1 + y_2 > 1 \Rightarrow$ substitutes: $v_{\{1\}} + v_{\{2\}} > v_{\{1,2\}}$. ($y_1 + y_2 > 1 \Rightarrow$ complements; $y_1 + y_2 = 1 \Rightarrow$ additive)

Corollary

$PB$ is

- optimal if $(y_1, y_2)$ stochastically nondecreasing in $x$.
- not optimal if $(y_1, y_2)$ stochastically decreasing in $x$. 
Example 1: Complementarities

Two products, values $v\{1\}$, $v\{2\}$, $v\{1,2\}$ described by $x, y_1, y_2$

$$v_b = x \cdot (y_1 1_{1 \in b} + y_2 1_{2 \in b} + (1 - y_1 - y_2) 1_{1,2 \in b})$$

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PB optimal if high value consumers consider products more substitutable
Recall Additive Example

Additivity & perfect negative correlation

\[ \Rightarrow v_{gb} \]

\[ \Rightarrow r_{trivially \ stochastically\ nondecreasing\ in\ v_{gb}} \]

\[ \Rightarrow PB\ optimal \]

\[ \begin{array}{cc}
0.8 & 0.2 \\
0.2 & 0.8 \\
\end{array} \]

Folklore: Bundle if \( v\{1\} \), \( v\{2\} \) negatively correlated

\[ \Rightarrow v_{1}, v_{2} \]: disutility from getting smaller bundle (compared to \( \{1, 2\} \))

Reinterpretation: Bundle if disutilities negatively correlated

Our result: Bundle if \( v_{1}/v_{gb} \) and \( v_{gb} \) positively correlated

\[ 1 - v_{1}/v_{gb} : relative \ disutility \ from \ getting \ smaller \ bundle \]

\[ \Rightarrow \] Bundle if relative disutility and \( v_{gb} \) negatively correlated
Recall Additive Example

Additivity & perfect negative correlation $\Rightarrow v_{gb}$ constant
$\Rightarrow r$ trivially stochastically nondecreasing in $v_{gb} \Rightarrow$ PB optimal
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Example 2: Cobb Douglas Utilities

- $k$ divisible products $1, \ldots, k$
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Envelope Analysis and Virtual Values

Single dimension: $\phi(v) = v - \text{revenue loss}$

"virtual value" $\phi(v) = v$ - revenue loss
Envelope Analysis and Virtual Values

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Lemma (Myerson’81)
Revenue of any IC mechanism is \( E_v[x(v) \cdot \phi(v)] \)

Theorem (Myerson’81; Riley and Zeckhauser’83)
Posting a price for the item is the optimal mechanism
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Single dimension:

\[ \phi(v) = v - \text{revenue loss} = v - \frac{1 - F(v)}{f(v)} \]

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```
\max_{\text{mechanism } (x,p)} \quad E_v[x(v) \cdot \phi(v)] \\
\text{s.t. } 0 \leq x(v) \leq 1, \forall v, \\
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**Envelope Analysis and Virtual Values**

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Envelope Analysis and Curves

Lemma

Revenue of any IC mechanism is $E_v[x_1(v) \cdot \phi_1(v) + x_{gb}(v) \cdot \phi_{gb}(v)]$

---

Diagram showing $v_1$, $V_1$, $v_{gb}$, and $V_{gb}$ axes with a graph illustrating the function $E_v[x_1(v) \cdot \phi_1(v) + x_{gb}(v) \cdot \phi_{gb}(v)]$.
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Revenue of any IC mechanism is $E_v[x_1(v) \cdot \phi_1(v) + x_{gb}(v) \cdot \phi_{gb}(v)]$

- $\phi_{gb}(v) = v_{gb} - \frac{1-F_{gb}(v_{gb})}{f_{gb}(v_{gb})}$
- $\phi_1(v) = V_1(v_{gb}) - V'_1(v_{gb}) \frac{1-F_{gb}(v_{gb})}{f_{gb}(v_{gb})}$

where $F_{gb}, f_{gb}$ are c.d.f and p.d.f of $v_{gb}$
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- If $r(v_{gb})$ nondecreasing then $r(v_{gb})\phi_{gb}(v_{gb}) \geq \phi_1(v_{gb})$
Envelope Analysis and Curves

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- If $r(v_{gb})$ nondecreasing then $r(v_{gb})\phi_{gb}(v_{gb}) \geq \phi_1(v_{gb})$
- If further $\phi_{gb}$ is increasing then $x^*$ is optimal
Beyond Regularity

If ratio $r$ increasing, then only “downward” IC constraints bind
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Generalized virtual value:

$$
\hat{\phi}(v) = v - \sum_{v': \text{IC from } v' \text{ to } v \text{ binds}} \lambda(v')(v' - v),
$$

Thus

$$
\hat{\phi}(v)_{gb} \geq \hat{\phi}_1, \quad \text{and} \quad v^*_1 = 0.
$$
Beyond Regularity

If ratio $r$ increasing, then only “downward” IC constraints bind

Generalized virtual value:

$$\hat{\phi}(v) = v - \sum_{v'} \lambda(v')(v' - v),$$

$v'$: IC from $v'$ to $v$ binds

Thus $r\hat{\phi}_{gb} \geq \hat{\phi}_1$, and $x_1^* = 0.$
Beyond Paths: Orthogonalization

Two paths $V_1, \hat{V}_1$ (both with monotone ratio), same marginal $F_{gb}$. Let $p^* = \max_p (1 - F_{gb}(p))$. PB with price $p^*$ is opt for each instance. Consider their mixture: Profit $\leq$ profit if seller "knows" the curve. So optimal to PB with price $p^*$.

$V_1 = \alpha \times + (1 - \alpha) \times \hat{V}_1$

Question: When can a distribution be decomposed?

1. to ratio-monotone curves
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![Diagram showing the mixture of $V_1$ and $\hat{V}_1$]
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$r$ stochastically nondecreasing in $v_{gb}$ ($Pr(r \geq \hat{r} \mid v_H) \uparrow$ in $v_{gb}$)
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Support of each $\mu_{\mid q}$ ratio-monotone

$q \in [0, 1]$ for $q = 1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}$

Strassen '65, Kamae et al. '77: generalization to higher dimensions
When Can a Distribution be Decomposed?

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Decompose distribution $\mu$ into $\{\mu \mid q\}_{q \in [0,1]}$

![Diagram showing contour lines for different values of $q$.]
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Decompose distribution \( \mu \) into \( \{\mu \mid q\}_{q \in [0,1]} \)

1. Support of each \( \mu \mid q \) ratio-monotone

2. \( q \) independent from \( v_{gb} \)

\[ q = \frac{1}{4} \]
\[ q = \frac{2}{4} \]
\[ q = \frac{3}{4} \]
\[ q = 1 \]
When Can a Distribution be Decomposed?

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Strassen '65, Kamae et al. '77: generalization to higher dimensions
Related Work

Technically:

- Orthogonalization: Eso, Szentes '07; Pavan et al. '14
- Wilson '93, Armstrong '96: fixed paths
- Carroll '16: virtual values, fixed paths

Bundling:

- Mostly additive values
  - Fang and Norman '06: Pure bundling vs. selling separately
  - Daskalakis et al. '17: PB optimal if values i.i.d $c, c+1$ for large $c$
  - Pavlov '11, Menicucci et al. '15: Other i.i.d distributions
- McAfee and McMillan '88, Manelli and Vincent '06: optimality of deterministic mechanisms
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 PB optimal if \( v_{gb} \) implies high “relative utility” \( \frac{v_1}{v_{gb}} \)
Main Result

PB optimal if high $v_{gb}$ implies high “relative utility” $v_1/v_{gb}$

Thanks!