Consumer Surplus and
Multi-Product Market Segmentation

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Abstract

We consider markets served by a multi-product seller who can engage in second and third degree price discrimination. We characterize markets for which the maximum consumer surplus across all possible segmentations equals the total surplus from the efficient allocation minus the profit for the seller in the unsegmented market. We show that this benchmark is achievable for all markets with a given set of consumer types if and only if the seller never finds it profitable to screen types by offering multiple products. The same condition also characterizes when the entire “surplus triangle” of Bergemann et al. (2015) is achievable.

1 Introduction

Market segmentation is the practice of using consumer data, such as geographic location, age, and purchase history, to treat different subsets of consumers as separate markets. This common practice affects both producer and consumer surplus. Relative to the unsegmented market, the

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surplus of a monopolistic seller who prices optimally in each market segment weakly increases (since he can offer the same prices in all segments). But this increase may be larger or smaller than the increase in total surplus resulting from the change in the allocation of products to consumers, so consumer surplus may increase or decrease. Because consumer surplus is always non-negative and the total surplus is at most the surplus from the efficient allocation, any consumer-producer surplus pair that arises from some segmentation lies in a “surplus triangle” (a term coined by Bergemann et al., 2015) like the one depicted in Figure 1. The top vertex corresponds to the seller obtaining all the surplus from the efficient allocation, and is achieved by first-degree price discrimination. The lower right vertex corresponds to the seller obtaining the same surplus as in the unsegmented market and consumers obtaining all the remaining surplus from the efficient allocation. We refer to this consumer surplus as “first best consumer surplus” because it is the highest consumer surplus in the surplus triangle. The lower left vertex corresponds to the seller obtaining the same surplus as in the unsegmented market and consumers obtaining surplus 0.

Bergemann et al. (2015) showed that, remarkably, any market with a single product can be segmented in ways that achieve the lower right vertex and the lower left vertex. Thus, the entire surplus triangle can be achieved by considering all possible segmentations of the market. In particular, for every market with a single product there exists some information about consumer preferences that, if provided to the seller who then uses it to segment the market and set the

Figure 1: The surplus triangle is the convex combination of three vertices.

[Diagram of a surplus triangle with vertices labeled as follows: Top vertex: producer surplus, Unsegmented market, First best consumer surplus, Consumer surplus, and highlighted triangles indicating the convex combination of vertices.
profit-maximizing price in each market segment, leads to first best consumer surplus. That is, the resulting allocation is efficient and consumers appropriate the entire surplus increase. A potentially important policy implication is that proper regulation and usage of consumer data can lead to first best consumer surplus without regulating the seller’s pricing strategy.

The goal of this paper is to investigate whether and to what extent these results hold when the seller can offer more than one product, as is often the case in practice. We consider a setting with multiple products in which each consumer’s (multi-dimensional) type is his valuations for the products. A market corresponds to a distribution over consumer types that specifies their proportions in the market, and a segmentation is a distribution over markets that averages to the original market. To facilitate the comparison to the single product case we assume that there is a “best product,” it is efficient to allocate this product to all consumers, and consumer types are ranked so that, for each product, the valuation of any consumer type is higher than the valuations of lower types. A leading example is digital goods, such as streaming services, where the best product corresponds to the “premium” or “full feature” version of the service.

The key difference relative to the single-product case is that in our setting the seller may optimally screen consumers by offering multiple products in a single market. By combining market segmentation and screening within market segments the seller engages in second and third degree price discrimination, and investigating this combination is part of the conceptual contribution of our analysis. From a technical perspective, the main obstacle to determining whether the surplus triangle, and first best consumer surplus in particular, can be achieved is that no general characterization of profit-maximizing mechanisms is known when there are multiple products. We develop an approach that sidesteps this difficulty.

Our first key result, Proposition 3, shows that if screening is profit-maximizing for a given market, then no segmentation of this market achieves first best consumer surplus (or the surplus triangle). Of course, screening implies inefficiency because not all consumers obtain the best product; the result shows that it is impossible to achieve efficiency via segmentation without the seller appropriating some of the gains. We also show that the optimality of screening in one market prevents achievability of first best consumer surplus and the surplus triangle in other markets with the same set of consumer types, even if screening is not optimal in those markets.

1We do not, however, assume that all consumers have the same ordinal ranking over products, only that they all prefer the best product to all other products.

2More precisely, offering only one product is not profit maximizing.
In fact, when there are only two consumer types, if screening is optimal for some market with these types, then first-best consumer surplus is not achievable for any market with these types except, trivially, for markets in which the seller optimally sells the best product to all consumers (“efficient markets”). When there are more than two consumer types, however, there may be some inefficient markets with a given set of types in which first best consumer surplus and the surplus triangle are achievable and other markets with this set of types in which they are not achievable.

Our second key result shows that first best consumer surplus is not achievable for any inefficient market with a given set of types if and only if for any market for which it is profit-maximizing to sell only the best product, there is only one price at which it is optimal to do so. This characterization has a geometric interpretation: the set of markets for which screening is profit maximizing separates the sets of markets for which screening is not profit maximizing (which differ by the price charged for the best product). A similar characterization holds for the achievability of the surplus triangle. On the other hand, if screening is not optimal for any market with a given set of types, then we are essentially back to the single product setting of Bergemann et al. (2015) and first best consumer surplus and the surplus triangle are achievable for every market.

Taken together, these results show that first best consumer surplus and the surplus triangle are achievable for all markets with a given set of types if and only if the seller optimally offers a single product in all markets. This can happen because only a single product is available, as in Bergemann et al. 2015 or because multiple products are available but it is profit-maximizing to offer only a single product. In all other cases, first best consumer surplus is not achievable for some or all inefficient markets, as determined by our second key result, and similarly for the surplus triangle.

These results relate the achievability of first best consumer surplus and the surplus triangle to the optimality of screening. However, whether screening is optimal for a given market is itself an intractable question because of the difficulty of characterizing optimal mechanisms. We are able to formulate equivalent results in terms of properties of the set of consumer types by building on the results of Haghpanah and Hartline (2021). Their results show that, roughly, screening is not optimal for any market with a given set of types if and only if the ratio between the value of each product and the value of the best product is higher for higher types. Intuitively, when the ratios are ranked in this way selling a less desirable product to some type reduces the seller’s revenue
from this type by more than the resulting decrease in information rents from higher types. For the converse, we extend the results of [Haghpanah and Hartline (2021)] to show that for a given set of types the set of markets for which screening is optimal separates the sets of markets for which screening is not optimal if and only if for every pair of types there is a product such that the ratio between the value of the product and the value of the best product is lower for the higher type. By our second key result, first best consumer surplus is then not achievable for any inefficient market, and similarly for the surplus triangle.

Returning to the implication of [Bergemann et al. (2015)]’s analysis that proper regulation and usage of consumer data in single-product markets can lead to first best consumer surplus without regulating the seller’s pricing strategy, our results show that this implication fails in multi-product markets for which screening is optimal, and we provide characterizations that identify such markets.

**Related work.** A growing part of the literature on third degree price discrimination studies surplus across all possible segmentations of a given market for a single product. The most closely related paper to ours is [Bergemann et al. (2015)], who show that the set of producer and consumer surplus pairs that result from all possible segmentations of a given market with a single product coincides with the surplus triangle. [Glode et al. (2018)] study optimal disclosure by an informed agent in a bilateral trade setting, and show that the optimal disclosure policy leads to socially efficient trade, even though information is revealed only partially. [Ichihashi (2020)] and [Hidir and Vellodi (2020)] consider maximum consumer surplus when a multi-product seller offers a single product in each market. [Ichihashi (2020)] considers a finite number of products and compares two regimes, one in which the seller may offer the same product at different prices to different segments, and another one in which the seller fixes the price in advance. [Hidir and Vellodi (2020)] characterize optimal segmentations with a continuum of products. [Braghieri (2021)] studies market segmentation with a continuum of firms, each producing a single differentiated product. In contrast, the seller in our setting may offer multiple products in a market in order to screen consumers.

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3We reiterate that we use the term screening to mean that there are at least two products in the seller’s menu. A mechanism that offers a single product at a high price and therefore excludes certain consumers is not a screening mechanism.
Parts of our analysis use results from the literature on information design and multi-dimensional screening. Haghpanah and Hartline (2021) identify conditions under which selling only the grand bundle of products is optimal. Under these conditions we can use a segmentation from Berge-mann et al. (2015) to achieve the surplus triangle. To show that these conditions are necessary, however, and to identify markets for which first best consumer surplus is unachievable, we develop a novel approach that does not rely on characterizing optimal mechanisms.

Structure of the paper. Section 2 presents the model, formalizes the main questions, and offers some preliminary observations. Section 3 provides results for markets with two consumer types. Section 4 studies markets with more than two types. Section 5 discusses and concludes. The appendix contains proofs not given in the text and a two-type, two-product example.

2 Model

There is a monopolistic seller, a mass 1 of consumers, and a set \( T = 1, \ldots, n \) of consumer types. There is a set \( A = 0, 1, \ldots, k \) of products, where \( k \geq 1 \) and product 0 is the outside option.\(^4\) A product can correspond to a particular quantity or quality of a good or a service or to a bundle of goods or services. For example, if a streaming service offers a movie subscription, a series subscription, and full-access subscription that combines both, then there are four products (including the outside option). The cost of production is 0. The valuation of type \( i \in T \) for a product \( a \in A \) is \( v^i_a \geq 0 \), with \( v^i_0 = 0 \). Assume without loss of generality that there are no redundant products, that is, for each pair of products \( a \neq a' \), there exists a type \( i \) such that \( v^i_a \neq v^i_a' \). We assume that some product \( \bar{a} \in A \) is the “best product” that all consumers prefer, that is, \( v^i_{\bar{a}} > v^i_a \) for all types \( i \) and products \( a \neq \bar{a} \). This facilitates the comparison to the single-product setting of Bergemann et al. (2015), which corresponds to \( k = 1 \).\(^5\) In the streaming setting the best product would be the full-access subscription. We place no restrictions on how products other than the best product are ranked by different types. We assume that types are

\(^4\)The finiteness assumption simplifies some of the notation and proofs. All of our results continue to hold when the sets of types and products are compact.

\(^5\)The assumption implies that it efficient to allocate the best product to all types, that if the seller optimally offers a single product in a market (formally defined below), he offers the best product, and if the seller is restricted to offering a single product in a market, he offers the best product. This also means that if the seller offers more than one product and engages in second degree price discrimination, the outcome is necessarily inefficient.
ranked so that a higher type has a higher valuation for any product, that is, $v_a^1 < v_a^2 < \ldots < v_a^n$ for any product $a \neq 0$. In the streaming setting, the higher the consumer’s type the more he likes to watch shows and movies, so the higher his valuation is for every kind of subscription. But some types of consumers may prefer a movie subscription to a series subscription while other types have the opposite preference.

An allocation $x \in X = \Delta(A)$ is a distribution over products, where $x_a$ denotes the probability of product $a$. An allocation $x$ is empty if $x_0 = 1$, and is non-empty otherwise. For each type the efficient allocation $x$ satisfies $x_a = 1$. The (expected) utility of a type $i$ consumer from an allocation $x$ and a payment $p$ is $v^i \cdot x - p = (\sum_a v^i_a x_a) - p$.

A mechanism consists of an allocation function $x : T \rightarrow X$ and a payment function $p : T \rightarrow R$. Mechanism $M = (x, p)$ is incentive compatible (IC) if for all types $i$ and $j$,

$$v^i \cdot x(i) - p(i) \geq v^j \cdot x(j) - p(j).$$

(1)

Mechanism $M$ is individually rational (IR) if for all types $i$,

$$v^i \cdot x(i) - p(i) \geq 0.$$

(2)

Henceforth, by “mechanism” will refer to an IC and IR mechanism, unless otherwise stated.

A market $f \in \Delta(T)$ is a distribution over types, where $f_i$ denotes the fraction of consumers with type $i$. The expected utility of consumers in market $f$ with mechanism $M = (x, p)$ is $EU(f, M) = E_{i \sim f}[v^i \cdot x(i) - p(i)]$. A mechanism $(x, p)$ is optimal for market $f$ if it maximizes revenue

$$E_{i \sim f}[p(i)]$$

across all mechanisms.$^6$ For a market $f$, let $ER(f)$ be the maximum expected revenue, $\mathcal{M}(f)$ be the set of optimal mechanisms, and $CS(f)$ be the highest consumer surplus (expected utility) across all optimal mechanisms,

$$CS(f) = \max_{M \in \mathcal{M}(f)} EU(f, M).$$

A segmentation $\mu \in \Delta(\Delta(T))$ of a market $f$ is a distribution over markets that average to $f$, that is, $E_{f' \sim \mu}[f'] = f$. We refer to a market $f'$ in the support of the segmentation $\mu$ as a market

$^6$An optimal mechanism exists: the revenue of any mechanism is at most $E_{i \sim f}[v^i_a]$, and the set of mechanisms is closed.
segment (or simply a segment). Let \( \text{SEG}(f) \) denote the set of segmentations of \( f \). Abusing notation, let \( CS(\mu) \) be the consumer surplus in segmentation \( \mu \),

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CS(\mu) = E_{f \sim \mu}[CS(f)].
\]

When discussing segmentations of a given market \( f \), we refer to \( f \) as the unsegmented market.

We often represent a mechanism indirectly by a menu of allocation-price pairs, where each type chooses a pair that maximizes its utility. If a type is indifferent between two allocation-price pairs, it chooses the one with a higher price. If, further, the prices are identical, then the tie breaking can be arbitrary since it does not affect consumer surplus or revenue. Unless stated otherwise, every menu includes the outside option at price 0.

We distinguish between screening and non-screening mechanisms. A mechanism \((x, p)\) is a non-screening mechanism if it can be represented by a menu with a single allocation-price pair, in addition to the outside option at price 0. Of particular interest is the set of non-screening mechanisms \( \{N^i\}_{i \in T} \), where mechanism \( N^i \) offers the best product \( \bar{a} \) at price \( v^i_{\bar{a}} \). The allocation and payment functions \((x, p)\) of mechanism \( N^i \) are as follows: \( x_0(j) = 1 \) and \( p(j) = 0 \) for all \( j < i \), and \( x_{\bar{a}}(j) = 1 \) and \( p(j) = v^i_{\bar{a}} \) for all \( j \geq i \). Among all non-screening mechanisms, \( N^i \) is optimal for some \( i \)\(^7\). A mechanism is a screening mechanism if it is not a non-screening mechanism, that is, every menu that represents it includes at least two positive allocation-price pairs.

Finally, we say that market \( f \) is a non-screening market if for some \( i \), mechanism \( N^i \) is optimal for \( f \). Otherwise, that is, if \( N^i \) is not optimal for any \( i \), we say \( f \) is a screening market. In this case, any optimal mechanism for market \( f \) is a screening mechanism.

### 2.1 Upper Bound on the Maximum Consumer Surplus

Given a market \( f \), we denote by \( MCS(f) = \max_{\mu \in \text{SEG}(f)} CS(\mu) \) the maximum consumer surplus across all segmentations of \( f \), and refer to a segmentation that achieves the maximum as a consumer-optimal segmentation. By definition, \( MCS(f) \geq CS(f) \). Also, \( MCS(f) \) is at most the expected surplus of an efficient allocation, \( E_{i \sim f}[v^i_{\bar{a}}] \), minus the maximum expected revenue,

\(^7\)Consider any non-screening mechanism \( M \) that offers a single allocation \( x \) at price \( p \). The mechanism that offers \( \bar{a} \) at price \( p \) obtains at least as much revenue. Further, a price \( p \) such that \( v^i_{\bar{a}} < p < v^i_{\bar{a}} \) for some type \( i \) cannot be optimal, since offering \( \bar{a} \) at price \( v^i_{\bar{a}} \) generates more revenue. Thus, among all non-screening mechanisms it is optimal to offer \( \bar{a} \) at price \( v^i_{\bar{a}} \) for some \( i \).
This is because for any segmentation, the sum of the expected revenue and the consumer surplus is at most $E_i \sim_f [v_i^a]$, and the expected revenue is at least $ER(f)$ (since the seller can offer a mechanism in $M(f)$ for all market segments). We refer to this upper bound $E_i \sim_f [v_i^a] - ER(f)$ on consumer surplus as first best consumer surplus (FBCS). The following lemma formalizes this discussion.

Lemma 1 For any market $f$, $CS(f) \leq MCS(f) \leq E_i \sim_f [v_i^a] - ER(f)$.

We study the conditions under which the upper bound is tight.

Definition 1 1. FBCS is achievable for a market $f$ if $MCS(f) = E_i \sim_f [v_i^a] - ER(f)$.

2. A segmentation $\mu$ of market $f$ achieves FBCS if $CS(\mu) = E_i \sim_f [v_i^a] - ER(f)$.

If a market $f$ has an optimal mechanism with an efficient allocation, then the single-segment segmentation achieves FBCS: the surplus generated is $E_i \sim_f [v_i^a]$ and the seller’s profit is $ER(f)$. Thus, $CS(f) = E_i \sim_f [v_i^a] - ER(f)$. We refer to such markets as efficient.

Definition 2 A market $f$ is efficient if $N_i(f)$ is an optimal mechanism for the market, where $i(f)$ is the lowest type in the support of $f$. Otherwise, the market is inefficient.

2.2 The Surplus Triangle

Given a market $f$, denote by $\Gamma(f)$ the set of consumer-producer surplus pairs resulting from all possible segmentations of $f$. Abusing notation, let $ER(\mu) = E_f \sim_\mu [ER(f)]$ be the producer surplus resulting from segmentation $\mu$, and consider a consumer-producer surplus pair $(CS(\mu), ER(\mu))$. Since $ER(\mu) \geq ER(f)$, $CS(\mu) \geq 0$, and $CS(\mu) + ER(\mu) \leq E_i \sim_f [v_i^a]$, the set $\Gamma(f)$ is a subset of the “surplus triangle”

$$\Delta(f) = \{(a, b) : b \geq ER(f), a \geq 0, a + b \leq E_i \sim_f [v_i^a]\},$$

which is illustrated in [Figure 1]. We study the conditions under which every pair in $\Delta(f)$ results from some segmentation of $f$.

Definition 3 The surplus triangle is achievable for a market $f$ if $\Gamma(f) = \Delta(f)$.
Bergemann et al. (2015) coined the term “surplus triangle” and showed that it is achievable for any market \( f \) with a single product. The surplus triangle is also obviously achievable for any “singleton market,” which consists only of consumers of some single type \( i \). In this case, the surplus triangle consists of the single pair \((0, v^i_a)\). For non-singleton markets, however, our results show that the surplus triangle is not always achievable when there are multiple products.

To proceed, observe that the surplus triangle is the convex hull of its vertices, and a convex combination of segmentations is a segmentation whose consumer-producer surplus pair is the same convex combination of the consumer-producer surplus pairs of the segmentations. Thus, to determine whether the surplus triangle is achievable it is enough to determine whether each of the three vertices of the surplus triangle is generated by some segmentation. The top vertex, \((0, E_{i \sim f}[v^i_a])\), is generated by first-degree price discrimination. The lower right vertex, \((ER(f), E_{i \sim f}[v^i_a]) - ER(f)\), is generated by segmentations that achieve first best consumer surplus. The lower left vertex, \((0, ER(f))\), generates the lowest possible total surplus of \( ER(f) \).

**Definition 4** A segmentation \( \mu \) of market \( f \) achieves the lowest possible total surplus if the resulting consumer-producer surplus pair is \((0, ER(f))\). If such a segmentation exists then the lowest possible total surplus is achievable for market \( f \).

The above discussion shows the following.

**Lemma 2** The surplus triangle is achievable for a market if and only if first best consumer surplus and the lowest possible total surplus are achievable for the market.

The rest of the paper investigates when FBCS is achievable for all markets, some markets, or no inefficient markets and when the surplus triangle is achievable for all markets, some markets, or no non-singleton markets (efficient or inefficient).

### 2.3 Conditions for Achieving FBCS

We start by specifying two conditions, which are together necessary and sufficient for a segmentation to achieve FBCS. First, because the resulting allocation is efficient, every segment must be efficient [Definition 2]. Second, the seller should not benefit from the segmentation, that is,
every optimal mechanism for the unsegmented market must be optimal for every segment. This is summarized by the following lemma.

**Lemma 3** For any segmentation $\mu$ of a market $f$, the following are equivalent:

1. $\mu$ achieves FBCS.
2. For some optimal mechanism $M$ of $f$ and every segment $f'$ of $\mu$, $f'$ is efficient and has an optimal mechanism $M$.
3. For every optimal mechanism $M$ of $f$ and every segment $f'$ of $\mu$, $f'$ is efficient and has an optimal mechanism $M$.

### 3 Two Types

We first consider markets with only two types of consumers, and identify each market by its fraction $q \in [0, 1]$ of type 2 consumers. The following lemma shows that the set of markets $[0, 1]$ can be qualitatively divided into at most three regions. The first region consists of markets in which the fraction of type 1 consumers is high, so they are non-screening markets in which mechanism $N^1$ is optimal. The second region consists of markets in which the fraction of type 2 consumers is high, so they are non-screening markets in which mechanism $N^2$ is optimal. The third region, which may be empty, consists of the remaining, intermediate markets. These markets are screening markets, that is, the allocations of the two types are different and non-empty. Moreover, the optimal mechanisms may vary across markets in this region. To formalize this, denote by $\mathcal{F}(M)$ the (possibly empty) set of markets for which a particular mechanism $M$ is optimal.

**Lemma 4** There exist thresholds $q_1$ and $q_2$, $0 \leq q_1 \leq q_2 \leq 1$, such that $\mathcal{F}(N^1) = [0, q_1], \mathcal{F}(N^2) = [q_2, 1]$, and $\mathcal{F}(M) \subseteq [q_1, q_2]$ for any mechanism $M \neq N^1, N^2$.

*Otherwise, there is a segment $f'$ such that the seller can benefit by offering in $f'$ an optimal mechanism for $f'$ and offering in all other segments an optimal mechanism for the unsegmented market. Conversely, if any optimal mechanism for the unsegmented market is also optimal in every segment and every segment is efficient, then the segmentation achieves FBCS.*
Figure 2: (a) \( q_1 = q_2 \). For any market, either \( N^1 \) or \( N^2 \) is optimal. (b) \( q_1 < q_2 \). Neither \( N^1 \) nor \( N^2 \) is optimal for markets in the interval \((q_1, q_2)\).

If \( q_1 = q_2 \), then all markets are non-screening markets. Since the seller offers only the best product in each market, the setting is equivalent to one with a single product. Bergemann et al. (2015)'s result then shows that the surplus triangle, and FBCS in particular, are achievable for all markets. It is less clear what can be said about the achievability of FBCS if \( q_1 < q_2 \). The proposition below shows that if \( q_1 < q_2 \), that is, there are some screening markets, then FBCS is unachievable for any inefficient market (the markets in \((q_1, 1)\)).\(^9\) As discussed in Section 2, a single-segment segmentation achieves FBCS for any efficient market. We thus obtain a characterization of the achievability of FBCS.

**Proposition 1** For any inefficient market \( q \), FBCS is achievable if and only if \( q_1 = q_2 \).

**Proof.** Suppose that \( q_1 = q_2 \). For completeness, we replicate Bergemann et al. (2015)'s result that FBCS is achievable for all markets. This is obviously true for the markets \([0, q_1] \cup \{1\}\) because they are efficient. Consider market \( q \in [q_2, 1] \), so mechanism \( N^2 \) is optimal for \( q \), and a segmentation of \( q \) into \( q' = 1 \) and \( q'' = q_1 = q_2 \).\(^{10}\) Both \( q' \) and \( q'' \) are efficient and have \( N^2 \) as an optimal mechanism, so the segmentation achieves FBCS by Lemma 3.

Now suppose that \( q_1 < q_2 \), and suppose that some segmentation \( \mu \) of a market \( q \) achieves FBCS. We show that \( q \) is efficient, that is, \( q \) is in \([0, q_1] \cup \{1\}\). By Lemma 3, every segment in \( \mu \) is efficient, and any optimal mechanism for \( q \) is optimal for every segment. The only optimal mechanism for market 1 is mechanism \( N^2 \). But since \( q_1 < q_2 \) and \( \mathcal{F}(N^2) = [q_2, 1] \), \( N^2 \) is not optimal for any market in \([0, q_1]\). Therefore, either every segment of \( \mu \) is equal to 1, in which case \( q = 1 \), or every segment is in \([0, q_1]\), in which case \( q \in [0, q_1] \). Therefore, \( q \) is efficient. \( \blacksquare \)

\(^9\)A market \( q < 1 \) is efficient if and only if \( N^1 \) is optimal for the market, and these are the markets \([0, q_1]\). The singleton market 1 is clearly efficient.

\(^{10}\)The segmentation assigns probability \( \alpha \) to \( q' \), and probability \( 1 - \alpha \) to \( q'' \), where \( \alpha = \frac{q - q_2}{1 - q_2} \).
We now turn to the achievability of the surplus triangle. Proposition 2 will show that if \( q_1 = q_2 \) then the surplus triangle is achievable for all markets, and if \( q_1 < q_2 \) then the surplus triangle is not achievable for any non-singleton market, that is, for any market in \((0,1)\). As discussed in \textbf{Section 2} the surplus triangle is a singleton and is achievable for any singleton market. We thus obtain a characterization of the achievability of the surplus triangle.

**Proposition 2** For any non-singleton market \( q \), the surplus triangle is achievable if and only if \( q_1 = q_2 \).

\textbf{Proposition 1} and \textbf{Proposition 2} provide a complete characterization of when FBCS and the surplus triangle are achievable when there are only two types (and any number of products). The characterization is in terms of the regions for which different mechanisms are optimal, and shows that the existence of screening markets prevents achievability for all inefficient or non-singleton markets, including non-screening markets. Haghpanah and Hartline \textbf{(2021)} characterize the two cases, \( q_1 = q_2 \) or \( q_1 < q_2 \), in terms of the valuations of the two types, which are a primitive of the model. The characterization shows that \( q_1 = q_2 \) if and only if for any product \( a \), type 2 has a higher ratio of valuations of product \( a \) to \( \bar{a} \), that is, \( r^1_a \leq r^2_a \), where \( r^i_a = v^i_a / v^i_{\bar{a}} \). Figure 3 illustrates this inequality for the case of two products.

\textbf{4 More than Two Types}

We now consider markets with any number of types and any number \( k \geq 1 \) of products. The logic of Bergemann \textit{et al.} \textbf{(2015)} shows that if for a given set of types all markets are non-screening markets, then the surplus triangle (and thus FBCS) is achievable for every market with this
set of types. We will show that this condition is in fact necessary by proving that FBCS (and thus the surplus triangle) is not achievable for any screening market. Of course, a screening mechanism is inefficient; What the result will show is that if a market is inefficient because it is a screening market, then it is impossible to achieve efficiency via segmentation without the seller appropriating some of the gains. This key result, Proposition 3, will also be useful in characterizing when FBCS and the surplus triangle are unachievable for every inefficient and non-singleton market, and not just screening markets, with a given set of types.

We begin with our key result.

**Proposition 3** If FBCS is achievable for market $f$, then $f$ is a non-screening market.

The proof of Proposition 3 is not a simple generalization of parts of the proof of Proposition 1. That proof relies on the set of markets being an interval, which implies that all inefficient markets, and thus all screening markets, are higher than the highest efficient market $q_1$. Since the only efficient market higher than $q_1$ is the singleton market 1 that consists of type 2 consumers, any segmentation of a screening market $f$ into efficient markets must include market 1 as a segment. But no screening mechanism is optimal for market 1, so by Lemma 3, the segmentation does not achieve FBCS. With more than two types, the set of segmentations is a higher-dimensional simplex, so the convex hull of the set of efficient markets for which a screening mechanism is also optimal may include screening markets. In Figure 4, this set is depicted in green and its convex hull is the shaded region, whose interior consists of screening markets. Such screening markets could thus conceivably be segmented in a way that achieves FBCS. The proof of Proposition 3 shows this is not the case.

To prove Proposition 3 consider a market $f$ with an optimal mechanism $M$, and suppose that first best consumer surplus is achievable for $f$. We will prove that $f$ is a non-screening market by showing that mechanism $N^j$ is also optimal for $f$, where $j$ is the lowest type that is not excluded in $M$ (a type is excluded if it gets an empty allocation). Consider a segmentation of market $f$ that achieves FBCS, and take any segment $f'$. By Lemma 3 $M$ is optimal for $f'$. Consider the lowest type $i(f')$ in the support of $f'$. We must have $i(f') \leq j$, otherwise every type in the support of $f'$ gets strictly positive utility in $M$, since it can mimic type $j$ and get strictly positive utility, so $M$ is not optimal for market $f'$. By Lemma 3, $f'$ is efficient so $N^i(f')$.

\[11\] With two types, this set is the singleton $\{q_1\}$.
Figure 4: The set of markets with three types and the screening and non-screening regions. The convex hull (shaded gray) of the set of efficient markets for which a screening mechanism is also optimal (in green) includes screening markets.

\[
\begin{array}{c|c|c|c|c}
\text{Types} & M & M' & (x, p) \\
1 & \text{outside option} & \text{outside option} & x' = \epsilon; \\
2 & \text{outside option} & p' = \epsilon v^i_{a} \\
i - 1 & \bullet & \bullet & p' = p - \epsilon(v^j_{a} - v^i_{a}) \\
i & \bullet & \bullet & \\
j - 1 & \bullet & \bullet & \\
j & \bullet & \bullet & \\
n & \bullet & \bullet & \\
\end{array}
\]

Figure 5: Construction of mechanism \(M'\) from mechanism \(M\) in the proof of Lemma 5.

Lemma 5 Consider a market \(f'\) and an optimal mechanism \(M = (x, p)\), and let \(j\) be the lowest type that gets a non-empty allocation in \(M\), that is, \(j = \min\{j' : x_0(j') < 1\}\). Suppose that for some \(i \leq j\), \(N^i\) is also optimal for \(f'\). Then, \(N^j\) is also optimal for \(f'\).

Proof. Assume without loss of generality that \(f'\) has full support on types 1 to \(n\). Assume for contradiction that mechanisms \(M\) and \(N^i\) are optimal for \(f\) but mechanism \(N^j\) is not. We construct a mechanism \(M'\) that has a higher revenue than \(M\). In \(M'\), types below \(i\) get an empty allocation (as they do in \(M\)). Types \(i\) to \(j - 1\) get product \(a\) with probability \(\epsilon > 0\) and the outside option with probability \(1 - \epsilon\) and pay \(\epsilon v^i_{a}\). Types \(j\) to \(n\) have the same allocation as in \(M\), but their payment is decreased by \(\epsilon(v^j_{a} - v^i_{a})\) relative to their payment in \(M\). See Figure 5.

Mechanism \(M'\) has a higher revenue than mechanism \(M\). Compared to \(M\), \(M'\) gains \(\epsilon v^i_{a}\) from every type \(i' \geq i\), and loses \(\epsilon v^j_{a}\) from every type \(i' \geq j\). The difference in revenue is \(\epsilon v^i_{a} \Pr[i' \geq i] - \epsilon v^j_{a} \Pr[i' \geq j]\), which is \(\epsilon\) times the difference between the revenue of mechanism
and the revenue of mechanism \( N^j \). This difference is strictly positive by the assumption that \( N^i \) is optimal but \( N^j \) is not. It remains to show that \( M' \) is IR and IC for small enough \( \epsilon > 0 \).

IR holds for types \( 1, \ldots, i - 1 \) because they are excluded in \( M' \). A type \( i' = i, \ldots, j - 1 \) has utility \( \epsilon v_{i'}^{j} - \epsilon v_{i}^{j} \geq 0 \), and a type \( i' \geq j \) has a higher utility in \( M' \) than in \( M \). Thus, IR holds for any \( \epsilon > 0 \).

For IC, observe that \( M' \) coincides with \( M \) in the limit as \( \epsilon \) goes to 0. Thus, if an IC constraint holds strictly in \( M \), then it is satisfied in \( M' \) for small enough \( \epsilon \). In mechanism \( M \) a type \( i' \) strictly prefers not to mimic another type \( i'' \) in two cases: (1) if \( i' > j \) and \( i'' < j \); (2) if \( i' < j \) and \( i'' \geq j \). In case (1), type \( i' \) has a strictly positive utility in \( M \) because it can mimic type \( j \). Thus \( i' \) strictly prefers not mimic type \( i'' \) (and get utility 0) in \( M \). In case (2), type \( i' \) gets a strictly negative utility from mimicking \( i'' \) because \( v^{i'} \cdot x(i'') - p(i'') < v^j \cdot x(i'') - p(i'') \leq 0 \), where the last inequality follows since the utility of type \( j \) is 0 and incentive compatibility of mechanism \( M \) implies that the utility of type \( j \) from mimicking type \( i'' \) cannot be positive.

We next verify the remaining IC constraints in mechanism \( M' \). Consider a type \( i' < j \). As discussed in case (2) above, such a type \( i' \) does not benefit from mimicking types \( i'' \geq j \). Type \( i' \) prefers the allocation of types \( 1, \ldots, i - 1 \) (the outside option) to the allocation of types \( i, \ldots, j - 1 \) if and only if \( \epsilon(v_{i'}^{j} - v_{i}^{j}) \leq 0 \), that is, \( i' \leq i \). Thus truthtelling maximizes the utility of a type \( i' < j \). For a type \( i' \geq j \), note that mimicking a type \( j, \ldots, n \) is not beneficial since \( M \) is IC and all such types get the same additional payment in \( M' \). From case (1) above, a type \( i' > j \) does not benefit from mimicking types \( 1, \ldots, j - 1 \). Finally, the utility of type \( j \) in \( M' \) is at least \( \epsilon(v_{j}^{j} - v_{a}^{j}) > 0 \), which is the utility it would get by mimicking types \( i, \ldots, j - 1 \) and is no lower than the utility of 0 it would get by mimicking types \( 1, \ldots, i - 1 \).

Lemma 5 shows that mechanism \( N^j \) is optimal for every segment \( f' \). The following result shows that \( N^j \) is also optimal for the original market \( f \), which completes the proof of Proposition 3.

Lemma 6 For any mechanism \( M \), the set \( \mathcal{F}(M) \) is convex.

Lemma 6 follows from the observation that for any market \( f \) and any segmentation of \( f \), the revenue from a mechanism \( M \) is the weighted average of the revenues in the segments. If \( M \) is optimal for the segments but not for \( f \), some other mechanism would give a strictly higher revenue for \( f \). The same must therefore be true for at least one of the segments, contradicting the optimality of \( M \) for the segments.
4.1 Achievability of FBCS and the Surplus Triangle

Proposition 3 and the logic of Bergemann et al. (2015) imply that FBCS and the surplus triangle are achievable for all markets with a given set of types $T$ if and only if all markets with that set of types are non-screening markets, that is, $\bigcup_i \mathcal{F}(N^i) = \Delta(T)$.

**Proposition 4** For any set of types $T$, the following are equivalent:

1. FBCS is achievable for every market.
2. The surplus triangle is achievable for every market.
3. Every market is a non-screening market.

**Proof.** $(2) \rightarrow (1)$: By definition.

$(3) \rightarrow (1)$ and $(2)$: If offering only $\bar{a}$ is optimal for all markets, the setting is equivalent to one with a single product $\bar{a}$. The results of Bergemann et al. (2015) then imply $(1)$ and $(2)$.

$(1) \rightarrow (3)$: Immediate from Proposition 3.

Proposition 4 characterizes the achievability of FBCS and the surplus triangle for all markets in terms of whether all markets are non-screening markets. Whether non-screening is optimal for a given market is in general difficult to ascertain. But Haghpanah and Hartline (2021) provide a simple characterization of the sets of types for which all markets are non-screening markets. For the characterization, let $r^i_a = v^i_a / v^i_{\bar{a}}$ be the ratio between type $i$’s valuations of products $a$ and $\bar{a}$.

**Proposition 5** (Haghpanah and Hartline, 2021) For any set of types $T$, the following are equivalent:

1. Every market is a non-screening market.
2. The ratio $r^i_a$ is non-decreasing in $i$ for all $a$.

From Proposition 4 and Proposition 5 we have the following result, which is illustrated in Figure 6.

**Theorem 1** For any set of types $T$, the following are equivalent:

1. FBCS is achievable for every market.
2. The surplus triangle is achievable for every market.

3. Every market is a non-screening market.

4. The ratio $r^i_a$ is non-decreasing in $i$ for all $a$.

Using the notation from Section 3, Theorem 1 states that with two types FBCS and the surplus triangle are achievable for every market if and only if $q_1 = q_2$. This generalizes parts of Proposition 1 and Proposition 2. However, Proposition 1 and Proposition 2 show that if $q_1 < q_2$, then first best consumer surplus is unachievable for all inefficient markets and the surplus triangle is unachievable for all non-singleton markets. Theorem 1 does not make such a statement, which is in fact not true with more than two types. The next subsection provides necessary and sufficient conditions for first best consumer surplus to be unachievable for all inefficient markets and for the surplus triangle to be unachievable for all non-singleton markets.

4.2 Unachievability of FBCS and the Surplus Triangle

In contrast to Proposition 1 and Proposition 2, with more than two types it may be that some markets are screening markets and yet FBCS and the surplus triangle are achievable for some inefficient and non-singleton non-screening markets with the same set of types. To identify the condition for unachievability for all inefficient and non-singleton markets, let us interpret the condition $q_1 < q_2$ in Proposition 1 and Proposition 2 as stating that the set of screening markets, $(q_1, q_2)$, separates the sets $[0, q_1]$ and $[q_2, 1]$ of non-screening markets. Our second main result shows that this is the correct condition for any number of types. It is illustrated in Figure 7.

**Theorem 2** For any set of types $T$, the following are equivalent:
1. **FBCS is unachievable for every inefficient market.**

2. **The surplus triangle is unachievable for every non-singleton market.**

3. **For every market, \( N^i \) is optimal for at most one \( i \).**

4. **For every pair of types \( i < j \), there exists some product \( a \) such that \( r^i_a > r^j_a \).**

To see why statement (3) implies statement (1) in Theorem 2, suppose that FBCS is achievable for some inefficient market \( f \) with full support. By Proposition 3, market \( f \) is a non-screening market, so for some \( i > 1 \) mechanism \( N^i \) is optimal for \( f \). Because \( f \) has full support, at least one segment in any segmentation must include consumers of type 1. By Lemma 3, both \( N^1 \) and \( N^i \) are optimal for that segment, so statement (3) does not hold. The formal proof also considers markets without full support.

To show that statement (4) implies statement (3) in Theorem 2, we cannot apply the results of Haghpanah and Hartline (2021) as we did in the proof of Theorem 1. Instead, we develop a new result that relates properties of type ratios to the set of non-screening mechanisms that may be optimal for any market. This is the content of the following lemma.

**Lemma 7** Consider a pair of types \( i < j \) such that \( r^i_a > r^j_a \) for some \( a \). Then, for any market \( f \), mechanisms \( N^i \) and \( N^j \) are not both optimal.

The proof of Lemma 7 shows that, given a pair of types \( i < j \) such that \( r^i_a > r^j_a \) for some product \( a \), if both \( N^i \) and \( N^j \) are assumed optimal, then there exists a mechanism that outperforms \( N^j \).

The remaining implications required to show Theorem 2 are proved in Appendix B.2.
4.3 The Remaining Case

With more than two types, Theorem 1 and Theorem 2 do not cover all possible cases: it is possible that some markets are non-screening markets but the set of non-screening markets does not separate the sets of screening markets. This remaining case is illustrated in Figure 8 and described by the following result, which is an immediate corollary of Theorem 1 and Theorem 2.

**Theorem 3** For any set of types \( T \), the following are equivalent:

1. FBCS is achievable for some but not all inefficient markets.
2. The surplus triangle is achievable for some but not all non-singleton markets.
3. There exists a screening market and there exists a market for which \( N^i \) and \( N^j \) are both optimal for some \( i \neq j \).
4. There exists a pair of types \( i < j \) such that \( r^i_a \leq r^j_a \) for all \( a \), and there exists a pair of types \( i' < j' \) such that \( r^i'_a > r^j'_a \) for some \( a \).

The set of inefficient markets for which FBBCS is achievable in Figure 8 is the shaded area in Figure 9. To see why, observe that by Proposition 3 this set is a subset of \( \mathcal{F}(N^2) \cup \mathcal{F}(N^3) \). For a segmentation of a market \( f \) in \( \mathcal{F}(N^2) \) to achieve FBBCS, every segment \( f' \) must be in \( \mathcal{F}(N^2) \) and have \( N_i(f') \) as an optimal mechanism. The set of such markets \( f' \) is shown in green in Figure 9, and \( f \) is in the convex hull of these markets. Similarly, the set of markets \( f \in \mathcal{F}(N^3) \) for which FBBCS is achievable is the convex hull of the set of markets that are shown in red in Figure 9.
Figure 9: The set of inefficient markets for which FBCS is achievable (in gray), and the efficient markets to which they can be segmented to achieve FBCS (in green and red)

5 Conclusion

We studied the achievability of FBCS and the surplus triangle in a multi-product environment. A key feature of our model is that the seller may find it profitable to screen consumers in a market segment by selling multiple products, thus combining second and third degree price discrimination. With two consumer types, we provided a complete characterization of when FBCS and the surplus triangle are achievable. With more than two types, we provided a characterization of when FBCS is achievable for all markets or no inefficient market, and when the surplus triangle is achievable for all markets or no non-singleton market. Our analysis shows that the seller’s ability to screen consumers when screening is profitable interferes with the achievability of FBCS and the surplus triangle.

Understanding when FBCS is achievable may be valuable because it indicates when proper regulation and usage of consumer data can lead to FBCS without regulating the seller’s pricing strategy. Our results show that FBCS is achievable for every market if and only if the ratio between the value of each product and value of the best product is higher for higher types. That is, when individuals who have higher absolute values have higher relative values for each product compared to the best product.

As an illustration, consider a movie streaming service and a newly-released movie for which the service can offer a rental option and a purchase option. Suppose that there are two types of consumers. The first type does not like movies that much, but prefers to purchase the movies he watches and rewatch parts of them from time to time. The second type likes to watch new movies, but does not much like to watch the same movie twice. This type’s willingness to pay for renting or buying the movie is higher than the first type’s, but he is willing to pay relatively
less to buy the movie (compared to renting it) than the first type. Then, regardless of the proportions of the two types in the market, it is optimal for the streaming service to offer only the purchase option, and FBCS and the surplus triangle are achievable. If, on the other hand, the type that likes movies more also likes to rewatch movies and the other type does not, then some markets are screening markets, so FBCS is not achievable for any inefficient market and the surplus triangle is not achievable for any non-singleton market.

We conclude by indicating several directions for future research. One possibility is to consider a more general environment with fewer restrictions on consumers’ preferences. A more general environment may make a direct comparison to Bergemann et al. (2015) more difficult, and characterizing achievability and unachievability of FBCS and the surplus triangle will likely require additional insights. Another direction is to consider a model with non-linear production costs instead of one with multiple products. Finally, it will be interesting to investigate the maximal consumer surplus when FBCS is not achievable. We do this for a two-type, two-product example in Appendix C.

References


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12 For example, when the “best product” differs across types, an efficient mechanism for a non-singleton market must offer multiple products.

13 A preliminary investigation with two types indicates that such a model may deliver similar results.


## A Proofs for Section 3

### A.1 Proof of Lemma 4

**Proof of Lemma 4.** We first show that for any mechanism $M$, $F(M)$ is a closed interval. Indeed, if $M$ is optimal for two markets $q, q'$, then it is also optimal for any convex combination $q''$ of these markets, because for any mechanism, the revenue in $q''$ is the same convex combination of the revenues in $q$ and in $q'$. And $F(M)$ is closed because the revenue from any mechanism is continuous in the market $q$. We now argue that $q_1 \leq q_2$ and for any $M \neq N^1, N^2$, we have $F(M) \subseteq [q_1, q_2]$. To see this, consider any two mechanisms $M, M'$ with payment rules $p \neq p'$. Then there is at most a single market $q$ where the two mechanisms have the same revenue, $qp(1) + (1-q)p(2) = qp'(1) + (1-q)p'(2)$. Therefore, the intersection of $F(M)$ and $F(M')$ is at most a single market. The claim now follows from observing that for any mechanism $M \neq N^1, N^2$, the payment rules of $M, N^1$, and $N^2$ are all different. ■

### A.2 Proof of Proposition 2

**Proof of Proposition 2.** Suppose that $q_1 = q_2$. As noted by Bergemann et al. (2015), the same segmentation that achieves FBCS also achieves the surplus triangle.

Now suppose that $q_1 < q_2$. Consider any inefficient market $q > q_1$. By Proposition 1, FBCS, and therefore the surplus triangle, is unachievable for such a market. Now consider an efficient market $q \leq q_1$ so mechanism $N^1$ is optimal for $q$. In this case the lowest possible total surplus is unachievable. This is because if the consumer surplus is 0 in some segment $q'$, mechanism $N^2$ must be optimal for $q'$. Then mechanism $N^1$ is not optimal for $q'$ and the segmentation increases producer surplus. ■
B Proofs for Section 4

B.1 Proof of Lemma 7

Proof of Lemma 7. Assume for contradiction that $r^i_a > r^j_a$ for some $i < j$ and $a$, and $N^i$ and $N^j$ are both optimal for a market $f$. Denote by $q_i$ the fraction of types $i$ and higher in market $f$, and by $q_j$ the fraction of types $j$ and higher in market $f$. For $N^i$ and $N^j$ to be both optimal, we must have $v^i_a q_i = v^j_a q_j$, that is $q_i = \frac{v^j_a q_j}{v^i_a}$. Thus we can write

\[ v^i_a q_i = v^i_a \left( \frac{v^j_a q_j}{v^i_a} \right) = \left( \frac{v^j_a v^i_a}{v^i_a} \right) q_j > v^j_a q_j, \quad (3) \]

where the inequality followed from the assumption that $r^i_a > r^j_a$ (that is, $v^i_a / v^i_a > v^j_a / v^i_a$).

Construct a mechanism $M$ that improves upon $N^j$ as follows. Types $i, \ldots, j-1$ get product $a$ with probability $\epsilon$ and pay $\epsilon v^i_a$. Types $j, \ldots, n$ get product $\bar{a}$ and pay $v^j_a - \epsilon(v^j_a - v^i_a)$. This is illustrated in Figure 10.

Let us compare the revenue of $M$ with the revenue of $N^j$. Types $i, \ldots, j-1$ pay $\epsilon v^i_a$ more in $M$ than in $N^j$. Types $j$ and higher pay $\epsilon(v^j_a - v^i_a)$ less in $M$ than in $N^j$. The difference in expected revenue is

\[ \epsilon v^i_a (q_i - q_j) - \epsilon (v^j_a - v^i_a) q_j = \epsilon (v^i_a q_i - v^j_a q_j) > 0, \]

where the inequality followed from inequality (3). So to complete the proof, we show that $M$ is IC and IR, which contradicts the assumption that $N^j$ is optimal.

Mechanism $M$ is IR. Types lower than $i$ are excluded. A type $i'$ from $i$ to $j-1$ has utility $\epsilon(v^i_a - v^i_a) \geq 0$. Types $j$ and higher have a higher utility in $M$ than in $N^j$.

For IC, observe similarly to the proof of Lemma 5 that if an incentive constraint holds strictly in $N^j$, then it is satisfied in $M$ for small enough $\epsilon > 0$. In particular, (1) a type $i' > j$ does not benefit from mimicking a type $i'' < j$, (2) a type $i' < j$ does not benefit from mimicking a type $i'' \geq j$.

Figure 10: Construction of the mechanism $M$ in the proof of Lemma 7
We now verify the remaining incentive constraints. A type \( i' < j \) prefers the allocation of types \( i, \ldots, j - 1 \) to the outside option if and only if \( \epsilon(v_a^{i'} - v_a^i) \geq 0 \), that is, \( i' \geq i \). Thus the incentive constraints are satisfied for types \( i' < j \). For types \( i' \geq j \), note that mimicking any type \( j, \ldots, n \) is not beneficial since all such types have the same allocation and payment. Finally, the utility of type \( j \) in \( M \) is \( \epsilon(v_j a - v_i a) \), which is the utility it would receive by mimicking types \( i, \ldots, j - 1 \), and is strictly more than the utility it would receive by mimicking types \( 1, \ldots, i - 1 \).

B.2 Proof of Theorem 2

Proof of Theorem 2.

(3) \( \rightarrow \) (1) and (2): To see that (3) implies (1), suppose for contradiction that (1) is violated, that is, some segmentation \( \mu \) of an inefficient market \( f \) achieves FBCS. From Proposition 3, \( N^i \) is optimal for market \( f \) for some \( i \). By Lemma 3, \( N^i \) is optimal for every segment \( f' \) of \( \mu \). The lowest type \( i(f') \) in the support of \( f' \) must satisfy \( i(f') \leq i \), otherwise all types in the support of \( f' \) get strictly positive utility in the optimal mechanism \( N^i \). Moreover, at least one segment \( f' \) must satisfy \( i(f') < i \). Otherwise, if \( i(f') = i \) for all segments \( f' \), then \( i(f) = i \) and \( f \) is efficient. Now consider a segment \( f' \) such that \( i(f') < i \). By Lemma 3, for market \( f' \), \( N^{i(f')} \) is optimal. That is, for \( i(f') < i \), \( N^i \) and \( N^{i(f')} \) are both optimal for some market \( f' \). Therefore, (3) is violated. We have thus shown that (3) implies (1). That (3) implies (1) also shows that if (3) holds, then the surplus triangle is not achievable for any inefficient market.

To see that (3) implies (2), it remains to show that the surplus triangle is not achievable for any non-singleton efficient market. Consider a non-singleton efficient market \( f \) and suppose that a segmentation \( \mu \) achieves the lowest possible total surplus (Definition 4). Consider a segment \( f' \) whose support includes type \( \tilde{i}(f) \), the highest type in the support of \( f \). Because \( \mu \) achieves the lowest possible total surplus, consumer surplus in \( f' \) is 0, so \( N^{\tilde{i}(f)} \) is optimal for \( f' \). And since \( f \) is efficient, \( N^{\tilde{i}(f)} \) is optimal for \( f \), where \( \tilde{i}(f) \) is the lowest type in the support of \( f \). So by Lemma 3, \( N^{\tilde{i}(f)} \) is also optimal for \( f' \).

(4) \( \rightarrow \) (3): Directly from Lemma 7.

(1) \( \rightarrow \) (4) and (2) \( \rightarrow \) (4): Suppose for contradiction that for some \( i < j \), \( r_a^i \leq r_a^j \) for all \( a \). By Proposition 5, either \( N^i \) or \( N^j \) is optimal for any market with support in \( \{i, j\} \). By Proposition 1 and Proposition 2, FBCS and the surplus triangle are achievable for every market with support.
C A Two Type Example

In this section we discuss a parametric example to highlight our results. We directly calculate the closed form expression for the maximum consumer surplus and compare it to FBCS. Even though the calculations are straightforward, they are not easily extendable beyond this example.

Suppose that there are two products and two types. A type 1 consumer has valuation \( v \in (0, 1) \) for one unit and valuation 1 for two units. A type 2 consumer has valuation 1 for one unit and valuation 2 for two units. The two types are illustrated in Figure 11, in which case (a) corresponds to \( v \leq 0.5 \) and case (b) corresponds to \( v \geq 0.5 \). A market \( q \) consists of a fraction \( 1 - q \) of type 1 consumers and a fraction \( q \) of type 2 consumers.

To identify maximum consumer surplus in different markets, it is useful to first identify the optimal mechanism in each market. Consider the following three mechanisms and their revenue in a market \( q \). Mechanism \( N^1 \) offers product 2 at price 1. Mechanism \( N^2 \) offers product 2 at price 2. Mechanism \( S \) screens; it offers each consumer a choice between buying product 1 at price \( v \) or product 2 at price \( v + 1 \). It can be shown that for any market \( q \), one of these three mechanisms is optimal, as illustrated in Figure 11. If \( v \leq 0.5 \), then mechanisms \( N^1 \) is optimal for markets in \([0, 0.5]\) and mechanism \( N^2 \) is optimal for markets in \([0.5, 1]\). If \( v \geq 0.5 \), then mechanism \( N^1 \) is optimal for markets in \([0, 1 - v]\), mechanism \( S \) is optimal for markets in \([1 - v, v]\), and mechanism \( N^2 \) is optimal for markets in \([v, 1]\).

Next, we compute the (average) consumer surplus in each market \( q \) generated by the optimal
mechanism for that market. Type 1 does not receive any information rents in any optimal mechanism. Thus, consumer surplus $CS(q)$ in market $q$ is $q$ times the utility of type 2 in the optimal mechanism for that market. Consumer surplus $CS(q)$ is illustrated in Figure 13.

A segmentation of market $q$ is a distribution $\mu$ over markets $[0, 1]$ such that $E_{q' \sim \mu}[q'] = q$. The maximum consumer surplus is $MCS(q) = \max_\mu E_{q' \sim \mu}[CS(q')]$, that is, the highest consumer surplus across all segmentations $\mu$. The maximum consumer surplus is obtained by concavifying the function $CS$. That is, $MCS(q) = \overline{CS}(q)$, where $\overline{CS}$ is the lowest concave function that is point-wise at least as high as $CS$.

The maximum consumer surplus $MCS(q)$ is at least $CS(q)$ and at most first best consumer surplus $FBCS(q)$, which is the surplus from the efficient allocation (that is, product 2 for each type) minus the seller’s revenue in market $q$. If the optimal mechanism for market $q$ implements the efficient allocation, then the two bounds are equal, that is, $CS(q) = FBCS(q)$, so $CS(q) = MCS(q) = FBCS(q)$. This is the case for a market $q$ for which mechanism $N^1$ is optimal and for market $q = 1$ which contains only type 2 consumers and for which mechanism $N^2$ is optimal. We refer to such markets as efficient, and otherwise as inefficient. If a market is efficient, then there is no scope for market segmentation to increase consumer surplus.

We can now address the possibility of achieving first best consumer surplus for all markets $q \in [0, 1]$. The relationship between maximum consumer surplus, $MCS$, and first best consumer surplus $FBCS$ is given by:

$$MCS(q) = \max_\mu E_{q' \sim \mu}[CS(q')]$$

This inequality holds for all $q \in [0, 1]$. When $MCS(q) = FBCS(q)$, the market is efficient, and when $MCS(q) > FBCS(q)$, the market is inefficient. If there is more than one optimal mechanism we choose the one with higher consumer surplus.

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If there is more than one optimal mechanism we choose the one with higher consumer surplus.
surplus, $FBCS$, is illustrated in Figure 14, and depends on the value of $v$. If $v$ is in $(0, 0.5)$, as in Figure 14 (b) and (c), then $FBCS$ is not achievable for any inefficient market. The only difference between cases (b) and (c) in Figure 14 is that in the former, $MCS(q)$ strictly exceeds $CS(q)$ for every inefficient market $q$ whereas in the latter $MCS(q) = CS(q)$ for market $q = v$.

If $v \in (0, 0.5]$, as in Figure 14 (a), then $FBCS$ is achievable for all markets. Equivalently, $FBCS$ is achievable for all markets if and only if for every market either mechanism $N_1$ or $N_2$ is optimal, that is, the seller does not find it profitable to screen consumers.

This example can also be used to illustrate how close $MCS$ is to $FBCS$ when $FBCS$ is not achievable. If $v$ is in $(0.5, 1)$, as in Figure 14 (b) and (c), then the ratio $FBCS/MCS$ increases in $q$ in the interval $(1 - v, v)$ and decreases in the interval $(v, 1)$. Consider the maximal point $q = v$. At this point, we have $FBCS = 1 - v$; If $v \in (0.5, \sqrt{5} - \frac{1}{2})$, then $MCS = \frac{(1-v)^2}{v}$ so $FBCS/MCS = \frac{v}{1-v}$, which increases in $v$; If $v \in [\sqrt{5} - \frac{1}{2}, 1)$, then $MCS = (1 - v)v$, so $FBCS/MCS = \frac{1}{v}$, which decreases in $v$.

What is the economic significance of $v$ being greater than or smaller than 0.5? For type 2 consumers, product 2 is twice as valuable as product 1. For type 1 consumers, whether $v$ is greater than or smaller than 0.5 determines whether product 2 is more than or less than twice as valuable as product 1. In other words, when $v \leq 0.5$ the second unit of the product is relatively more complementary to the first unit of the product for type 1 consumers than for type 2 consumers, and vice versa when $v > 0.5$.

Turning to the surplus triangle, it is trivially achievable for markets with a single type of consumer ($q = 0$ and $q = 1$). For all other markets, the same conditions that characterize achievability of $FBCS$ also characterize when the surplus triangle is achievable for every market.

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15See Haghpanah and Siegel (2021) for a detailed investigation of when $MCS$ strictly exceeds $CS$. 

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or no market (efficient or inefficient). Indeed, whenever FBCS is not achievable, the surplus triangle is clearly not achievable. And the results of Bergemann et al. (2015) show that when the seller does not find it optimal to screen, that is, in every market $q$ only offers two units as a bundle, the entire surplus triangle is achievable.