543: Game Theory **Week 4**
Dynamic Mechanisms

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Preliminary and incomplete.

Thanks to 534 Fall 2020 students for proofreading! (Remaining errors are mine.)

Source(s): Borgers chapter 11 [Börgers, 2015], [Courty and Hao, 2000].

In this lecture we first study a generalization of our screening problem of selling a single product where the buyer learns her values over time. In general, we see that it is optimal for the seller to "dynamically screen" the buyer's types, that is, extract information from the buyer as soon as she learns it. We will then formalize the following statement: "the buyer only earns information rents from the information she has when she walks into the mechanism, and not from any information she receives after that". Finally, we consider a setting where information is static but allocation is dynamic, and show that the seller does not benefit from dynamic allocation.

Dynamic Private Information

Consider selling a single good to a single buyer. The twist is that the buyer a priori doesn't completely know her value. Instead she learns it after having signed a contract with the seller.

- Valuation $\theta \in [\underline{\theta}, \bar{\theta}]$, buyer realizes θ after accepting or rejecting to participate in the mechanism
- What does the buyer know? The buyer receives a signal $\tau \in [\underline{\tau}, \bar{\tau}]$ with cdf $G(\tau)$ and pdf $g(\tau) > 0$
- τ, θ are correlated, so τ is informative of the buyers value θ. Let $F(\theta|\tau)$ denote the cdf of θ conditioned on τ, and assume that the joint density f satisfies $f(\theta, \tau) > 0$.
- We refer to τ as the ex ante type, and θ as the ex post type of the buyer

- We assume the following: $\partial F(\theta|\tau)/\partial\tau < 0$: high τ means high θ

Note that there are two dimensions of private information. We saw before that in a static setting, unless types are one-dimensional, then characterizing incentive compatibility is quite complex. We will see later that because of this complexity, identifying optimal mechanisms would be difficult. But as we will see today, in this dynamic setup, even though there are two dimensions of private information, we can characterize incentive compatibility and identify optimal mechanisms.

What does the class of all mechanisms look like?

We instead just define direct mechanisms.

Definition 0.1 *A (dynamic) "direct mechanism" is defined by a pair of functions* (q, t) *where* $q : [\underline{\tau}, \bar{\tau}] \times [\underline{\theta}, \bar{\theta}] \to [0, 1]$ *and* $t : [\underline{\tau}, \bar{\tau}] \times [\underline{\theta}, \bar{\theta}] \to R$.

A (not necessarily direct) mechanism defines a dynamic single agent decision problem (with initial uncertainty about θ).

- The buyer's optimal decision rule is a pair $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 : [\underline{\tau}, \bar{\tau}] \to [\underline{\tau}, \bar{\tau}]$ specifies the reported τ given the true τ, and $\sigma_2 : [\underline{\tau}, \bar{\tau}] \times [\underline{\theta}, \bar{\theta}] \times [\underline{\tau}, \bar{\tau}] \to [\underline{\theta}, \bar{\theta}]$ specifies the reported θ given the true τ and reported τ (in the first period), and the true θ.

Here is the revelation principle.

Proposition 0.1 *Consider any mechanism* Γ *with optimal strategy* σ. *Let* $q(\tau, \theta)$ *be the probability that the buyer receives the product in* Γ *when the buyer plays* σ, *and similarly* $t(\tau, \theta)$ *the expected transfer. Then there exists a direct mechanism* (q', t') *with optimal strategy* σ' *s.t.*

- $\sigma'_1(\tau) = \tau, \sigma'_2(\tau, \theta, \tau) = \theta$
- $(q, t)(\tau, \theta) = (q', t')(\tau, \theta)$.

Proof: Define the direct mechanism so that the second property is satisfied, $(q', t')(\tau, \theta) := (q, t)(\tau, \theta)$.

We now prove the first property, namely that σ' is optimal in the direct mechanism. By definition every type (τ, θ) obtains the same utility from following σ' in the direct mechanism (q', t') and from following σ in the indirect mechanism γ. If the type reports τ' in the first stage and then $\theta'(\theta, \tau')$ in the second stage, she gets the same utility in the direct mechanism as playing the strategy of τ' before learning θ, and strategy of $(\tau', \theta'(\theta, \tau'))$ after learning θ. So the deviation is profitable in Γ' if corresponding deviation is profitable in Γ. **QED**

Notice that the proposition claims $\sigma_2'(\tau, \theta, \tau) = \theta$ for all τ, θ which is weaker than claiming $\sigma_2'(\tau, \theta, \tau') = \theta$ for all τ, τ', θ. What does this mean?

We next define incentive compatibility. For this we define some notation. The ex post utility of the buyer with type (τ, θ) is $u(\tau, \theta) := \theta q(\tau, \theta) - t(\tau, \theta)$. Define $\hat{U}(\tau'|\tau) := \int_{\underline{\theta}}^{\bar{\theta}} u(\tau'|\hat{\theta}) f(\hat{\theta}|\tau) d\hat{\theta}$ and define $U(\tau) = \hat{U}(\tau|\tau)$.

Definition 0.2 *A direct mechanism (q, t) is incentive compatible (IC) if*

- $u(\tau, \theta) \geq \theta q(\tau, \theta') - t(\tau, \theta'), \forall \theta, \theta', \tau$. *This says that reporting θ truthfully is optimal on-path.*
- $U(\tau) \geq \int [\theta q(\tau', \theta^r(\hat{\theta})) - t(\tau', \theta^r(\hat{\theta}))] f(\theta|\tau) d\theta, \forall \tau, \tau', \theta^r : [\underline{\theta}, \bar{\theta}] \to [\underline{\theta}, \bar{\theta}]$. *This says that reporting τ truthfully is optimal.*

As discussed before, the first property only requires truth telling about θ on path. The stronger notion of truth telling off-path would require that $\theta q(\tau', \theta') - t(\tau', \theta') \geq \theta q(\tau', \theta) - t(\tau, \theta)$.

We next define individual rationality. Notice that the definition only requires that the expected utility of type τ is non-negative. It does not require the ex-post utility to be non-negative. Interpret?

Definition 0.3 *A direct mechanism (q, t) is individually rational (IR) if $U(\tau) \geq 0$.*

So notice that the first property also implies that being truthful is also optimal off-path. Suppose that τ reports τ'. Then the utility of reporting θ' is $\theta q(\tau', \theta') - t(\tau', \theta')$. But the IC constraint for type τ' tells us that $\theta q(\tau', \theta') - t(\tau', \theta') \geq \theta q(\tau', \theta) - t(\tau', \theta)$. What this says is that ex post utility only depends on θ and not τ. So if the buyer wants to be truthful about θ after reporting τ truthfully, then it is also optimal to be truthful about θ after any report about τ.

So the optimality of σ_1 simplifies to $U(\tau) \geq \hat{U}(\tau'|\tau)$.

Proposition 0.2 *A direct mechanism (q, t) is incentive compatible if and only if $u(\tau, \theta) \geq \theta q(\tau, \theta') - t(\tau, \theta')$ and $U(\tau) \geq \hat{U}(\tau'|\tau)$.*

Proof: The two conditions are necessary for incentive compatibility by definition. So we only need to show that they are sufficient. $u(\tau, \theta) \geq \theta q(\tau, \theta') - t(\tau, \theta')$ implies that $\theta q(\tau', \theta) - t(\tau', \theta) \geq \theta q(\tau', \theta') - t(\tau', \theta')$. So it is optimal to be truthful in the second stage,

$$\int \theta q(\tau', \theta) - t(\tau', \theta) f(\theta|\tau) d\theta$$

$$\geq \int \theta q(\tau', \theta^r(\theta)) - t(\tau', \theta^r(\theta)) f(\theta|\tau) d\theta$$

QED

Recall that in static settings, we could characterize incentive compatibility in terms of monotonicity of allocation and revenue equivalence. Can we extend static IC characterization to dynamic? We will see that the allocation need not be monotone in the ex ante type τ.

Proposition 0.3 *A direct mechanism (q, t) is "incentive compatible w.r.t θ" iff*

- *$q(\tau, \theta)$ increasing in θ (monotonicity in the second stage)*
- *$u(\tau, \theta)$ continuous in θ, $\partial u(\tau, \theta)/\partial\theta = q(\tau, \theta)$ (this implies revenue equivalence in the second stage, which is what follows)*
- *$t(\tau, \theta) = t(\tau, \underline{\theta}) + (\theta q(\tau, \theta) - \underline{\theta} q(\tau, \underline{\theta})) - \int^{\theta} q(\tau, \hat{\theta}) d\hat{\theta}$.*

What about monotonicity in τ? Isn't the set of ex ante types τ one-dimensional so that we get monotonicity in terms of τ? No. Why?

Notice that the revenue equivalence above is rather week: it says that if two mechanisms have the same allocation rule, and also the same transfer for the lowest ex post type for any ex ante type τ, then the two mechanisms have the same revenue. So there is too much degree of freedom.

We now develop revenue equivalence in the first stage. A key step to get revenue equivalence was to relate the derivative of the agent's utility with its allocation.

Lemma 0.4 *If (q, t) is IC, then $U(\tau)$ increasing and continuous.*

Proof: u, F continuous. Integration by parts:

$$\hat{U}(\tau'|\tau) = \int u(\tau', \theta) f(\theta|\tau) d\theta$$

$$= \int q(\tau', \theta)[1 - F(\theta|\tau)] d\theta.$$

$$U(\tau_2) - U(\tau_1) \geq \hat{U}(\tau_1|\tau_2) - \hat{U}(\tau_1|\tau_1)$$

$$= \int q(\tau_1, \theta)[F(\theta|\tau_1) - F(\theta|\tau_2)] \geq 0$$

Continuity won't do. **QED**

Now we can get an envelope theorem.

Proposition 0.5 *U is differentiable a.e.*

$$U'(\tau) = \frac{\partial \hat{U}(\tau|\tau)}{\partial \tau} = -\int q(\tau,\theta)\frac{\partial F(\theta|\tau)}{\partial \tau}d\theta.$$

and

$$\int t(\tau,\theta)f(\theta|\tau)d\theta = \int\left[\theta q(\tau,\theta)f(\theta|\tau) + (t(\underline{\tau},\theta) - \theta q(\underline{\tau},\theta))f(\theta|\underline{\tau}) + \int q(\hat{\tau},\theta)\frac{\partial F(\theta|\hat{\tau})}{\partial \tau}d\hat{\tau}\right]d\theta$$

Proof:

$$\begin{aligned}\frac{\partial \hat{U}(\tau'|\tau)}{\partial \tau} &= \frac{\partial}{\partial \tau}\int q(\tau',\theta)[1-F(\theta|\tau)]d\theta \\ &\quad - \int q(\tau',\theta)\frac{\partial F(\theta|\tau)}{\partial \tau}d\theta\end{aligned}$$

Now the envelope theorem part:

$$\begin{aligned}\frac{U(\tau)-U(\tau+\delta)}{\delta} &\leq \frac{\hat{U}(\tau|\tau)-\hat{U}(\tau|\tau+\delta)}{\delta} \\ \frac{U(\tau-\delta)-U(\tau)}{\delta} &\geq \frac{\hat{U}(\tau|\tau-\delta)-\hat{U}(\tau|\tau)}{\delta}\end{aligned}$$

so

$$U'(\tau) = \partial\hat{U}(\tau|\tau)/\partial\tau.$$

Second part:

$$\begin{aligned}U(\tau) &= U(\underline{\tau}) + \int^{\tau} U'(\tau')d\tau' \\ &= U(\underline{\tau}) - \int^{\tau}\int q(\tau',\theta)\frac{\partial F(\theta|\tau')}{\partial \tau}d\theta d\tau'\end{aligned}$$

QED

According to Proposition 0.3, $t(\tau,\theta)$ pinned down using q except for a degree of freedom $t(\tau,\underline{\theta}), \forall\tau$. This is from IC of ex post type. Proposition 0.5 also imposes constraints on transfer. What do they imply together? Given an allocation rule, the only degree of freedom is $t(\underline{\tau},\underline{\theta})$:

Proposition 0.6 *If (q,t) is IC, then "the transfer rule is pinned down given the allocation rule and $t(\underline{\tau},\underline{\theta})$". In particular,*

$$t(\tau,\theta) = t_0(\tau) + \theta q(\tau,\theta) - \int^{\theta} q(\tau,\hat{\theta})d\hat{\theta}$$

where

$$t_0(\tau) = t(\underline{\tau},\underline{\theta}) - \underline{\theta}q(\underline{\tau},\underline{\theta}) + \int\int q(\hat{\tau},\theta)\frac{\partial F(\theta|\hat{\tau})}{\partial \tau}d\hat{\tau}d\theta + \int\int^{\theta}[q(\tau,\theta')f(\theta|\tau) - q(\underline{\tau},\theta')f(\theta'|\underline{\tau})]d\theta' d\theta$$

Proof: Plug into 0.5 the definitions of $t(\tau,\theta)$ and $t(\underline{\tau},\theta)$ in 0.3, we have
$t(\tau,\underline{\theta}) - \underline{\theta}q(\tau,\underline{\theta})$

$$= t(\underline{\tau},\underline{\theta}) - \underline{\theta}q(\underline{\tau},\underline{\theta}) + \int\int q(\hat{\tau},\theta)\frac{\partial F(\theta|\hat{\tau})}{\partial\tau}d\hat{\tau}d\theta + \int\int^{\theta}[q(\tau,\theta')f(\theta|\tau) - q(\underline{\tau},\theta')f(\theta'|\underline{\tau})]d\theta'd\theta$$

Note that $t(\tau,\underline{\theta})$ and therefore $t(\tau,\theta)$ is pinned down given q and $t(\underline{\tau},\underline{\theta})$. **QED**

Recall from implementation discussion that if appropriate monotonicity properties hold (either weak monotonicity or cyclical monotonicity), then a decision rule is implementable. What about in the dynamic setting? Here is a partial answer. It is partial since it only provides sufficient but not necessary conditions for implementability.

Proposition 0.7 *If q increasing in τ,θ, then there exists t (of Proposition 0.6) s.t. (q,t) is incentive compatible.*

Proof: Define t using 0.6. By proposition 0.3, (q,t) IC w.r.t. θ. So only need to prove $U(\tau) - \hat{U}(\tau'|\tau) \geq 0$.

$$\begin{aligned}
&U(\tau) - \hat{U}(\tau'|\tau)\\
&= \hat{U}(\tau|\tau) - \hat{U}(\tau'|\tau') + \hat{U}(\tau'|\tau') - \hat{U}(\tau'|\tau)\\
&= \int_{\tau'}^{\tau} U'(\hat{\tau}) - \frac{\partial\hat{U}(\tau'|\hat{\tau})}{\partial\tau}d\hat{\tau}.
\end{aligned}$$

Recall from 0.5

$$U'(\tau) = -\int q(\tau,\theta)\frac{\partial F(\theta|\tau)}{\partial\tau}d\theta$$

Recall from proof of 0.5,

$$\frac{\partial\hat{U}(\tau'|\tau)}{\partial\tau} = -\int q(\tau',\theta)\frac{\partial F(\theta|\tau)}{\partial\tau}d\theta$$

substituting

$$\begin{aligned}
&U(\tau) - \hat{U}(\tau'|\tau)\\
&= \int_{\tau'}^{\tau}\int [q(\tau',\theta) - q(\hat{\tau},\theta)]\frac{\partial F(\theta|\hat{\tau})}{\partial\tau}d\theta d\hat{\tau}
\end{aligned}$$

By FOSD, $\frac{\partial F(\theta|\tau)}{\partial\tau} \geq 0$, and since q increasing, $U(\tau) - \hat{U}(\tau'|\tau) \geq 0$. **QED**

Again recall from our screening discussion that for IR, it was sufficient that the lowest type has non-negative utility. The same holds here:

Proposition 0.8 *A direct mechanism (q,t) is individually rational if and only if $U(\tau) \geq 0$.*

Characterize Optimal Mechanism

As in the static case, we can use revenue equivalence to write expected revenue only in terms of the allocation rule and the utility of the lowest type. In particular, by Proposition 0.5,

$$\int U(\tau)g(\tau)d\tau = -[(1-G(\tau))U(\tau)]|_{\underline{\tau}}^{\bar{\tau}} + \int (1-G(\tau))U'(\tau)d\tau$$
$$= U(\underline{\tau}) - \int\int (1-G(\tau))q(\tau,\theta)\frac{\partial F(\theta|\tau)}{\partial\tau}d\theta d\tau$$

So the expected revenue is

$$\int\int [\theta q(\tau,\theta) - u(\tau,\theta)]f(\theta|\tau)g(\tau)d\tau$$
$$= \int\int \theta q(\tau,\theta)f(\theta|\tau)g(\tau)d\theta d\tau - \int U(\tau)d\tau$$
$$= \int\int [\theta + \frac{1-G(\tau)}{g(\tau)}\frac{\partial F(\theta|\tau)/\partial\tau}{f(\theta|\tau)}]q(\tau,\theta)f(\theta|\tau)g(\tau)d\theta d\tau - U(\underline{\tau})$$

To optimize:

- Given IR, we set $U(\underline{\tau}) = 0$.
- Define $\phi(\tau,\theta) = \theta + \frac{1-G(\tau)}{g(\tau)}\frac{\partial F(\theta|\tau)}{f(\theta|\tau)}$.
- $q(\tau,\theta) = 1$ if $\phi(\tau,\theta) \geq 0$, and $= 0$ otherwise.
- So if given assumptions, q increasing, then we are done

Assumption 1. $\phi(\tau,\theta)$ increasing in τ and θ.

- hazard rate multiplied by $\frac{\partial F(\theta|\tau)}{f(\theta|\tau)}$ "the informativeness measure".

So there exists $p(\tau)$ such that $\phi(\tau,\theta) \geq 0$ and therefore $q(\tau,\theta) = 1$ if $\theta \geq p(\tau)$, and $\phi < 0$ and $q = 0$ otherwise.

Summary:

Proposition 0.9 *Given assumption 1, a direct mechanism (q,t) is optimal if and only if q maximizes virtual surplus, $t(\tau,\theta) = t_0(\tau) + p(\tau)$ if $\theta \geq p(\tau)$, $t(\tau,\theta) = t_0(\tau)$ otherwise, where $t_0(\tau)$ is given by 0.6 and $t(\underline{\tau},\underline{\theta})$ is such that $U(\underline{\tau}) = 0$.*

Proof: Given assumption 1, q that maximizes virtual surplus is monotone in (τ,θ). So by 0.7, to verify that (q,t) is IC we only need to verify that the transfer rule t above is same as 0.6, that is

$$t(\tau,\theta) = t_0(\tau) + \theta q(\tau,\theta) - \int^{\tau} q(\tau,\hat{\theta})d\hat{\theta}.$$

If $\theta < p(\tau)$, then $q(\tau,\theta') = 0$ for all $\theta' \leq \theta$, so $t(\tau,\theta) = t_0(\tau)$.

If $\theta \geq p(\tau)$, then $q(\tau,\theta') = 1$ for all $\theta' \in [p(\tau),\theta]$, so

$$t(\tau,\theta) = t_0(\tau) + \theta - \int_{p(\tau)}^{\theta} 1 d\theta' = t_0(\tau) + p(\tau)$$

QED

Recall that in the screening setting, we could implement the optimal (direct) mechanism by simply offering a take-it-or-leave-it price. What about here?

Role of Private Information

Optimal mechanism gives rents to the buyer, that is, some types of the buyer have positive utility. We will show here that the rents only come from ex ante private info. That is, the buyer gets rents only from the information she already has when she walks into the mechanism, and not from the information she realizes afterwards.

To get some intuition, consider the case of an uninformative τ, that is, $F(\theta|\tau) = F(\theta|\tau')$ for all τ, τ'. The optimal mechanism "extracts the full surplus" by setting $q(\tau,\theta) = 1$ and $t(\tau,\theta) = E[\theta]$. Since the buyer walks in with no information, she gets zero information rents. We now generalize and formalize this.

What does it mean to even talk about the "extra" information that the buyer receives after she walks into the mechanism?

To formalize rewrite so that payoff type is first signal plus independent signal.

- In period 2, observe $\gamma = F(\theta|\tau)$. Since $F(\theta|\tau)$ strictly increasing in θ, $\theta = F^{-1}(\gamma|\tau)$.
- So (τ,γ) uniquely pins down (τ,θ).
  - Example: θ is uniformly drawn from $[\tau, \tau+2]$.
  - Then learning (τ,θ) is equivalent to learning (τ,γ) where $\gamma = (\theta-\tau)/2$.
- So we can talk about the buyer learning (τ,γ), and that's isomorphic to the buyer learning (τ,θ).
- Nice thing about γ: it is independent of θ (in fact, uniformly distributed on $[0,1]$). We think of γ as the "extra information" that the buyer learns after walking into the mechanism.
- Now given a mechanism (q,t), we can redefine the mechanism using $\tilde{q}(\tau,\gamma), \tilde{t}(\tau,\gamma)$.

We will show that γ can be extracted at no cost! More formally, we compare the seller's revenue in the optimal mechanism to a situation where the seller has a superpower to observe γ. We show that the optimal mechanism is the same with or without this power.

We first model γ as private information (as we have been doing so far). Recall optimal direct mechanism in (τ, θ) space maximizes

$$\int\int \phi(\tau,\theta)q(\tau,\theta)f(\theta|\tau)g(\tau)d\theta d\tau - U(\underline{\tau}). \tag{1}$$

γ is uniformly distributed regardless of τ. So the expected revenue is

$$\int\int \tilde{\phi}(\tau,\gamma)\tilde{q}(\tau,\gamma)g(\tau)d\gamma d\tau - U(\underline{\tau}), \tag{2}$$

where $\tilde{\phi} = \phi(\tau, F^{-1}(\gamma|\tau))$ and $\tilde{q} = q(\tau, F^{-1}(\gamma|\tau))$. Formally, we change the variables of the integral from θ to γ and use $f(\theta|\tau)d\theta = d\gamma$,

Now go back to assuming γ is public information. What is optimal?

- A mechanism $\tilde{q}, \tilde{t}$ that induces buyer to truthfully reveal τ but not γ.
- Call this "IC with observable γ"

The IC condition on τ is

$$\begin{aligned} U(\tau) &:= \int [F^{-1}(\gamma|\tau)\tilde{q}(\tau,\gamma) - \tilde{t}(\tau,\gamma)]d\gamma \\ &\geq \int [F^{-1}(\gamma|\tau)\tilde{q}(\tau',\gamma) - \tilde{t}(\tau',\gamma)]d\gamma. \end{aligned}$$

Proposition 0.10 *If $(\tilde{q}, \tilde{t})$ is "IC with observable γ", then*

$$U(\tau) = U(\underline{\tau}) + \int\int \frac{\partial F^{-1}(\gamma|\hat{\tau})}{\partial \tau}\tilde{q}(\hat{\tau},\gamma)d\gamma d\hat{\tau}.$$

Proof:

$$\begin{aligned} &\lim_{\delta\to+0} \frac{U(\tau+\delta) - U(\tau)}{\delta} \\ &\geq \lim_{\delta\to+0} (\int \frac{F^{-1}(\gamma|\tau+\delta) - F^{-1}(\gamma|\tau)}{\delta}\tilde{q}(\tau,\gamma)d\gamma) \\ &= \int \frac{\partial F^{-1}(\gamma|\tau)}{\partial\tau}\tilde{q}(\tau,\gamma)d\gamma. \end{aligned}$$

Similarly,

$$\begin{aligned} &\lim_{\delta\to+0} \frac{U(\tau) - U(\tau-\delta)}{\delta} \\ &\leq \int \frac{\partial F^{-1}(\gamma|\tau)}{\partial\tau}\tilde{q}(\tau,\gamma)d\gamma. \end{aligned}$$

So

$$U'(\tau) = \int \frac{\partial F^{-1}(\gamma|\tau)}{\partial \tau} \tilde{q}(\tau, \gamma) d\gamma.$$

Next integrate U'. **QED**

Using Proposition 0.10, revenue with observable γ is

$$\int \int \tilde{\phi}(\tau, \gamma) \tilde{q}(\tau, \gamma) g(\tau) d\gamma d\tau - U(\underline{\tau}).$$

Proposition 0.11 *Suppose that Assumption 1 holds. If $\tilde{q}, \tilde{t}$ optimal when γ private, then it is also optimal when γ is publicly observable.*

An extreme already known: if buyer didn't have any private information at the contracting time, seller can extract all surplus.

Extension DOESN'T hold without assumption 1:

- If $\tilde{q}(\tau, \gamma)$ not monotone in γ, then by 0.3 it is not implementable with private γ.

Dynamic Allocations

In contrast to before, the mechanism will not use new information. The optimal mechanism would be static.

- One object, fixed private value θ
- periods $\tau = 1, \ldots, T$
- common discount factor δ
- agent utility $\sum_\tau \delta^{\tau-1}(\theta q_\tau - t_\tau)$
- Seller $\sum_\tau \delta^{\tau-1} t_\tau$

Maybe buyer sends a message, then seller learns from buyer behavior.

Definition 0.4 *Direct mechanism $q : [\underline{\theta}, \bar{\theta}] \to [0, 1]^T, t : [\underline{\theta}, \bar{\theta}] \to R^T$.*

Define type the utility θ from being truthful, $u(\theta) = \sum_\tau \delta^{\tau-1}[\theta q_\tau(\theta) - t_\tau(\theta)]$

Definition 0.5 *A direct mechanism* (q, t) *is IC if* $u(\theta) \geq \sum_\tau \delta^{\tau-1}[\theta q_\tau(\theta') - t_\tau(\theta')]$.

Definition 0.6 *A direct mechanism* (q, t) *is IR if* $u(\theta) \geq 0$.

If $T = 1$, post price p^* maximizing $p(1 - F(p))$, define (q^s, t^s).

Proposition 0.12 (q^*, t^*), $q^*_\tau = q^s, t^*_\tau = t^s$ *for all* τ *is optimal.*

Proof: Note that q^*, t^* is IC and IR. Revenue is $p^*(1-F(p^*))\sum_\tau \delta^{\tau-1} = p^*(1-F(p^*))\beta, \beta = (1-\delta^\tau)/(1-\delta)$

Assume exists $\hat{q}, \hat{t}$ such that revenue $> p^*(1 - F(p^*))\beta$.

Define static mechanism

- $\hat{q}^s(\theta) = \sum_\tau \delta^{\tau-1}\hat{q}_\tau(\theta)/\beta$
- $\hat{t}^s(\theta) = \sum_\tau \delta^{\tau-1}\hat{t}_\tau(\theta)/\beta$

Note $\hat{q}^s, \hat{t}^s$ is IC and IR. revenue is

$$\int \hat{t}^s(\theta) f(\theta) d\theta$$
$$= \int [\sum_\tau \delta^{\tau-1}\hat{t}_\tau(\theta)/\beta] f(\theta) d\theta$$
$$> p^*(1 - F(p^*)).$$

QED

The mechanism doesn't learn from buyer's behavior.

Commitment is important.

References

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[Courty and Hao, 2000] Courty, P. and Hao, L. (2000). Sequential screening. The Review of Economic Studies, 67(4):697–717.