543: Game Theory **Week 3**
VCG and Implementability

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Preliminary and incomplete.

Thanks to 534 Fall 2020 students for proofreading! (Remaining errors are mine.)

Source(s): Borgers chapter 5 and 7 [Börgers, 2015], [Rochet, 1985].

In this lecture we first study designing mechanisms that lead to efficient outcomes. We show that such mechanisms exists. We then study the more general question of characterizing what outcomes are implementable.

VCG

We start with the VCG mechanism, named after the people who came up with the key ideas, Vickrey, Clarke, and Groves. Before giving the mechanism, we introduce the (very general) setup.

Setup:

- Agents are $1, \ldots, N$
- We want to design a mechanism to choose an alternative $a \in A$. E.g., we may want to decide if the government should spends in money on building roads vs. schools. Then $A = \{\text{road}, \text{school}\}$. Or we want to allocate a single product to one of the N agents. Then $A = \{1, \ldots, N\}$.
- If alternative a is selected and agent i is asked to pay t_i, then her utility is $u_i(a, \theta_i) - t_i$.

Definition 0.1 *A "Direct mechanism" is (q, t) where $q : \Theta \to A$ is the decision rule and $t_i : \Theta \to R, \forall i$ is the transfer rule.*

Definition 0.2 *A direct mechanism (q, t) is "dominant strategy incentive compatible" (DSIC) if*

$$u_i(q(\theta), \theta_i) - t_i(\theta) \geq u_i(q(\theta'_i, \theta_{-i}), \theta_i) - t_i(\theta'_i, \theta_{-i}) \forall i, \theta'_i, \theta$$

Implementing Efficient Decision Rules

Definition 0.3 *q^* <u>efficient</u> if*

$$\sum_i u_i(q^*(\theta), \theta_i) \geq \sum_i u_i(a, \theta_i), \forall \theta, a$$

Why do we define efficiency the way we do?

Definition 0.4 (VCG mechanism) *A VCG mechanism is a direct mechanism (q, t) where q efficient and for all i, exists $\tau_i : \theta_{-i} \rightarrow R$ such that*

$$t_i(\theta) = -\left(\sum_{j \neq i} u_j(q(\theta), \theta_j)\right) + \tau_i(\theta_{-i})$$

Proposition 0.1 *VCG is DSIC.*

Proof: The utility of reporting θ_i' is

$$\begin{aligned}
&= u_i(q(\theta_i', \theta_{-i}), \theta_i) + \left(\sum_{j \neq i} u_j(q(\theta_i', \theta_{-i}), \theta_j)\right) - \tau_i(\theta_{-i}) \\
&= \left(\sum_j u_j(q(\theta_i', \theta_{-i}), \theta_j)\right) - \tau_i(\theta_{-i}) \\
&\leq \left(\sum_j u_j(q(\theta_i, \theta_{-i}), \theta_j)\right) - \tau_i(\theta_{-i}) \\
&= u_i(q(\theta_i, \theta_{-i}), \theta_i) - t_i(\theta)
\end{aligned}$$

QED

Alternatively, consider difference in payment form reporting θ_i and θ_i',

$$-(\sum_{j \neq i} u_j(q(\theta), \theta_j) - \sum_{j \neq i} u_j(q(\theta_i', \theta_{-i}), \theta_j))$$

So i pays the difference in the utilities of other agents.

A commonly used τ is as follows:

$$\tau_i(\theta_{-i}) = \max_a \sum_{j \neq i} u_j(a, \theta_j).$$

Then the payment of type i,

$$(\max_a \sum_{j \neq i} u_j(a, \theta_j)) - \sum_{j \neq i} u_j(q(\theta), \theta_j).$$

So i pays his "externality" on other players, i.e., the difference between the utilities of other agents from efficient allocation and their utility from an allocation that maximizes the utilities of all the other agents.

As an example, consider selling a single product. What is the VCG mechanism?

- $A = \{0, \dots, N\}$.
- $u_i(a, \theta_i) = \theta_i$ if $a = i$, $u_i(a, \theta_i) = 0$ otherwise
- Efficient $\max_a \sum_i u_i(a, \theta_i)$: give to highest
- $t_i(\theta) = \tau_i(\theta_{-i})$ if i highest, $-\max_j \theta_j + \tau_i(\theta_{-i})$ if i not highest
- e.g., $\tau_i(\theta_{-i})$ = highest of others
- then $t_i(\theta)$ = highest of others if i wins, 0 otherwise
- Second price auction!

Incentive Compatibility

Now that we know that there always exists a mechanism with an efficient allocation rule, we ask a more general question: Given an allocation rule, when can we "implement" it? That is, when can we construct a transfer rule so that the resulting mechanism is DSIC? For simplicity we go back to a single agent, where DSIC becomes just IC, i.e., truth telling is an optimal strategy for the single agent. (Why don't we talk about IR?)

The setup is the same as before, except with a single agent:

- Choose alternative $a \in A$
- Utility $u(a, \theta) - t$

Definition 0.5 *"Direct" $q : \Theta \to A$ (decision rule), $t : \Theta \to R$ (transfer rule).*

Definition 0.6 *(q, t) IC if*

$$u(q(\theta), \theta) - t(\theta) \geq u(q(\theta'), \theta) - t(\theta'). \tag{1}$$

Weak Monotonicity

Definition 0.7 *We say that q <u>implementable</u> if exists t s.t. (q, t) IC.*

We show below that a necessary condition for implementability is weak monotonicity.

Definition 0.8 *q <u>weakly monotone</u> if for all θ, θ'*

$$u(q(\theta), \theta) - u(q(\theta'), \theta) \geq u(q(\theta), \theta') - u(q(\theta'), \theta')$$

As an example, consider our screening setup from week 1 (selling a single indivisible product to a single buyer):

- Screening: $a \in [0,1]$, $\theta \in R^+$, $u(a,\theta) = a\theta$.
- condition: $q(\theta)\theta - q(\theta')\theta \geq q(\theta)\theta' - q(\theta')\theta'$.
- So: q increasing. We did prove that in fact monotonicity is necessary for incentive compatibility in our screening setup. We show below that this holds generally.

Proposition 0.2 *If q implementable, then q weakly monotone.*

Proof: IC:

$$u(q(\theta),\theta) - t(\theta) \geq u(q(\theta'),\theta) - t(\theta')$$
$$\Leftrightarrow u(q(\theta),\theta) - u(q(\theta'),\theta) \geq t(\theta) - t(\theta').$$

Similarly,

$$u(q(\theta'),\theta') - u(q(\theta),\theta') \geq t(\theta') - t(\theta).$$

Summing up gives weak monotonicity. **QED**

Example: Weak monotonicity is NOT sufficient for IC

	θ^1	θ^2	θ^3
a	0	-1	1
b	1	0	-1
c	-1	1	0

Consider $q(\theta^1) = a, q(\theta^2) = b, q(\theta^3) = c$.

- This allocation rule is weakly monotone. Check.
- BUT, there is no transfer rule to make IC. Without loss of generality let $t(\theta^1) = 0$.

$$\text{IC1: } 0 \geq 1 - t(\theta^2), 0 \geq -1 - t(\theta^3)$$
$$\text{IC2: } -t(\theta^2) \geq 1, 1 - t(\theta^3)$$
$$\text{IC3: } -t(\theta^3) \geq 1, -1 - t(\theta^2).$$

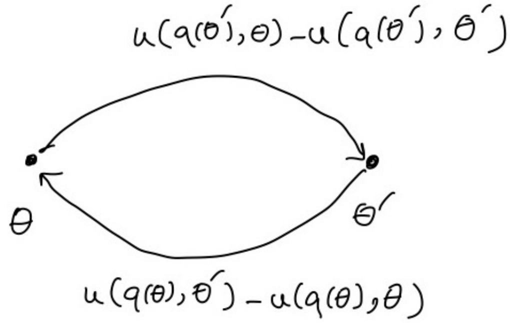
So $t(\theta^2) = 1, t(\theta^3) = -1$, $t(\theta^2) - t(\theta^3) = -1$. Contradiction.

Issue: Weak monotonicity ensures that for every two type, exists transfer. We want a different order of quantifiers: transfers such that for every two types, their IC constraints are satisfied.

Cyclical Monotonicity

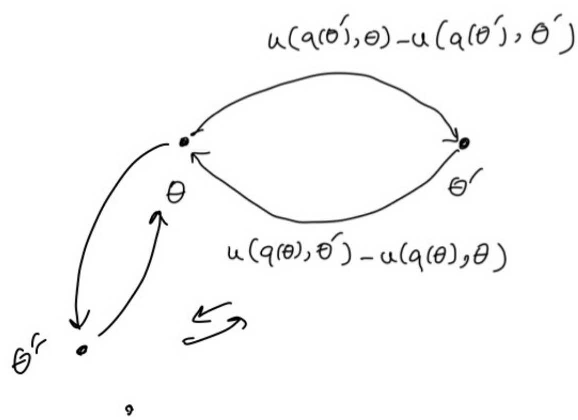
Rewrite weak monotonicity:

$$(u(q(\theta), \theta') - u(q(\theta), \theta)) + (u(q(\theta'), \theta) - u(q(\theta'), \theta')) \leq 0.$$



Definition. q cyclically monotone if for any sequence $\theta^1, \ldots, \theta^k (= \theta^1)$,

$$\sum_i u(q(\theta^i), \theta^{i+1}) - u(q(\theta^i), \theta^i) \leq 0$$



Verify in the example before that cyclical monotonicity is violated.

Proposition 0.3 *q implementable iff cyclically monotone.*

Proof: First $\Rightarrow$:

$$\begin{aligned} & u(q(\theta^i), \theta^{i+1}) - t(\theta^i) \leq u(q(\theta^{i+1}), \theta^{i+1}) - t(\theta^{i+1}) \\ \Leftrightarrow & u(q(\theta^i), \theta^{i+1}) - u(q(\theta^{i+1}), \theta^{i+1}) \leq t(\theta^i) - t(\theta^{i+1}). \end{aligned}$$

Sum over i,

$$\sum_{i}^{k-1} \big(u(q(\theta^i), \theta^{i+1}) - u(q(\theta^{i+1}), \theta^{i+1})\big) \leq 0,$$
$$\sum_i u(q(\theta^i), \theta^{i+1}) - \sum_i u(q(\theta^{i+1}), \theta^{i+1}) \leq 0,$$
$$\sum_i u(q(\theta^i), \theta^{i+1}) - \sum_i u(q(\theta^i), \theta^i) \leq 0,$$
$$\sum_i \big(u(q(\theta^i), \theta^{i+1}) - u(q(\theta^i), \theta^i)\big) \leq 0$$

Now $\Leftarrow$.

Consider arbitrary $\tilde{\theta}$.

Define $V : \Theta \to R$,

$$V(\theta) := \sup_{\theta^1 = \tilde{\theta}, \theta^2, \ldots, \theta^k = \theta} \sum_{i=1}^{k} (u(q(\theta^i), \theta^{i+1}) - u(q(\theta^i), \theta^i))$$

V well defined: if there exists a cycle, it can be removed. Therefore the sup is over finitely many paths.

Now use V to construct payments. Let V be utility, that is

$$t(\theta) = u(q(\theta), \theta) - V(\theta).$$

Check IC:

$$u(q(\theta), \theta) - t(\theta) \geq u(q(\theta'), \theta) - t(\theta')$$
$$\Leftrightarrow u(q(\theta), \theta) - (u(q(\theta), \theta) - V(\theta))$$
$$\geq u(q(\theta'), \theta) - (u(q(\theta'), \theta') - V(\theta'))$$
$$\Leftrightarrow V(\theta) \geq V(\theta') + u(q(\theta'), \theta) - u(q(\theta'), \theta').$$

Last inequality holds since you can go from $\tilde{\theta}$ to θ' and then θ.

QED

One Dimensional Type Space

This is when weak monotonicity is sufficient for implementability.

Setup: It has structure on alternatives and preferences.

- R a complete and transitive order on A.

- P derived strict order: $aPb \Leftrightarrow aRb, \text{not } bRa$.
- I derived indifference: $aIb \Leftrightarrow aRb \& bRa$.

Now order types

- $\theta \succ_R \theta'$ if
  1. $u(a,\theta) - u(a',\theta) > u(a,\theta') - u(a',\theta'), \forall aPa'$
  2. $u(a,\theta) - u(a',\theta) = u(a,\theta') - u(a',\theta') = 0, \forall aIa'$

"θ has stronger preference for high alternatives than θ'".

Definition 0.9 *q monotone w.r.t R if $\theta \succ_R \theta' \Rightarrow q(\theta)Rq(\theta')$.*

Proposition 0.4 *IF q is weakly monotone, then q monotone w.r.t. any complete and transitive R.*

Proof: Suppose q weakly monotone. Take $\theta \succ_R \theta'$. Need to show $q(\theta)Rq(\theta')$.

weak monotonicity: $u(q(\theta),\theta) - u(q(\theta'),\theta) \geq u(q(\theta),\theta') - u(q(\theta'),\theta')$.

Suppose $q(\theta)Pq(\theta')$: since $\theta \succ_R \theta'$:

$$u(q(\theta'),\theta) - u(q(\theta),\theta) > u(q(\theta'),\theta') - u(q(\theta),\theta')$$
$$u(q(\theta),\theta) - u(q(\theta'),\theta) < u(q(\theta),\theta') - u(q(\theta'),\theta').$$

Contradicts weak monotonicity. **QED**

Converse can't hold since $\succ_R$ may be incomplete. Exercise: construct example.

Definition 0.10 *Take R. Θ "one-dimensional with respect to R" if for all $\theta \neq \theta'$, either $\theta \succ_R \theta'$ or $\theta' \succ_R \theta$.*

- *Take $a'Pa$. Assign θ real $u(a',\theta) - u(a,\theta)$. This mapping is invertible. Also order induced by mapping the same for all a, a'.*

Proposition 0.5 *Suppose that Θ is one dimensional w.r.t. R. Then q implementable if and only if q is weakly monotone if and only if q is monotone w.r.t R.*

Proof: First weakly monotone $\Leftrightarrow$ monotone w.r.t. R. weak monotone $\Rightarrow$ monotone w.r.t. R already done.

Show monotonicity w.r.t R implies weak monotonicity.

Take θ, θ'. Θ one dimensional: $\theta \succ_R \theta'$.

Since q monotone: $q(\theta) R q(\theta')$.

Since $\theta \succ_R \theta'$: $u(q(\theta), \theta) - u(q(\theta'), \theta) > u(q(\theta), \theta') - u(q(\theta'), \theta')$ if aPa', with equality if aIa'. So q weakly monotone.

Now show q monotone w.r.t. $R \Leftrightarrow q$ implementable.

If q implementable, then weak monotone, then monotone w.r.t. R.

Now show if q monotone w.r.t R, then q implementable.

Assume for simplicity: A finite. $-c < u(a', \theta) - u(a, \theta) < c$.

Let $\{a^1, \ldots, a^n\}$ be the range of q, $\{q(\theta) | \theta \in \Theta\}$. WLOG $a^n R a^{n-1} \ldots R a^1$.

Let $\Theta^k := \{\theta | q(\theta) = a^k\}$.

Monotonicity and one-dimensionality: $k' > k, \theta^k \in \Theta^k, \theta^{k'} \in \Theta^{k'}$ then $\theta^{k'} \succ_R \theta^k$.

Define $\tau^k := \inf\{u(a^k, \theta) - u(a^{k-1}, \theta) | \theta \in \Theta^k\}$.

Define transfer: $t(\theta) = 0$ if $\theta \in \Theta^1$, $t(\theta) = \sum_{k \geq 2} \tau^k$.

Now verify IC. Say $\theta \in \Theta^k$ reports in $\Theta^{k'}$ where $k' > k$. Utility compared to truthtelling is

$$
\begin{aligned}
& u(a^{k'}, \theta) - u(a^k, \theta) - \sum_{\ell=k+1}^{k'} \tau^\ell \\
&= \sum_{\ell=k+1}^{k'} \left(u(a^\ell, \theta) - u(a^{\ell-1}, \theta) \right) - \sum_{\ell=k+1}^{k'} \tau^\ell \\
&\leq \sum_{\ell=k+1}^{k'} u(a^\ell, \theta) - u(a^{\ell-1}, \theta) - \sum_{\ell=k+1}^{k'} u(a^\ell, \theta) - u(a^{\ell-1}, \theta) \\
&= 0.
\end{aligned}
$$

QED

We report one more very useful property of one dimensional type spaces. It says that only local incentive constraints are sufficient to imply all incentive constraints. The proof will be left as an exercise.

Proposition 0.6 *Suppose that Θ is one dimensional w.r.t. some R, and suppose $\theta^{i+1} \succ_R \theta^i$ for all i. Then a direct mechanism (q, t) is incentive compatible if it is locally incentive compatible, that is, for each i,*

$$
\begin{aligned}
u(q(\theta^i), \theta^i) - t(\theta^i) &\geq u(q(\theta^{i+1}), \theta^i) - t(\theta^{i+1}), \\
u(q(\theta^{i+1}), \theta^{i+1}) - t(\theta^{i+1}) &\geq u(q(\theta^i), \theta^{i+1}) - t(\theta^i).
\end{aligned}
$$

References

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[Rochet, 1985] Rochet, J. (1985). The taxation principle and multi-time hamilton-jacobi equations. Journal of Mathematical Economics, 14(2):113 – 128.