543: Mechanism Design **Auctions**	**Week 2**

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Preliminary and incomplete.

Thanks to 534 Fall 2020 students for proofreading! (Remaining errors are mine.)

Source(s): Borgers chapter 3 [Börgers, 2015], Krishna chapter 6 [Krishna, 2009], [Myerson, 1981], and [Crémer and McLean, 1985].

In this lecture we study Bayesian incentive compatibility. We apply the concept to the design of auctions and public projects in the independent private values setting. We then study correlated and interdependent values.

We start with some examples.

Examples

The setting There is a single product for sale. The seller has no value for the product, so wants to maximize the expected revenue. There are two buyers with values θ_1 and θ_2. Each buyer only knows her own value, but not the value of the other buyer. The auctioneer knows neither. The values are drawn independently from the uniform distribution on $[0, 1]$:

- $f(\theta_1) = 1$, $F(\theta_1) = \theta_1$, same for 2

What are some possible ways to sell the product?

- First price auction: Each buyer submits a bid. The highest bid wins and pays her bid.
- Second price auction: Each buyer submits a bid. The highest bid wins and pays the bid of the other buyer.

Which auction has more revenue?

On one hand, in the first price auction the winner pays the highest bid, compared to the second highest bid in the second price auction. So one might think that the first price auction has higher revenue. But on the other hand, buyers bid lower than their value in the first price auctions, whereas as we will see, in the second price auction it is optimal for buyers to bid their value. Which force is stronger? We will see that the answer is neither: the revenue of the first price auction equals the revenue of the second price auction. Let's study the two auctions.

First price auction An equilibrium is for each buyer to bid half of their value, $\sigma_i(\theta_i) = \theta_i/2$. Below we will check that this is indeed an equilibrium. (How did we come up with the strategies? Good question. We will see later a way to find potential equilibria via the Revenue equivalence.) That is, assume that player 2 bids half of her value. We will verify that it is optimal for player one to bid also half of her value. So suppose that her value is θ_1 and she bids θ_1'.

- First of all, there is no reason to bid more than 0.5. By bidding $\theta_1' \geq 0.5$ buyer 1 always wins. So it is better to bid 0.5 instead of $\theta_1' > 0.5$.
- What is the best $\theta_1' \leq 0.5$? Buyer 1 wins if the bid of player 2, which is $\theta_2/2$, is less than θ_1'. This happens with probability $2\theta_1'$. The utility conditioned on winning is $\theta_1 - \theta_1'$. So player 1 maximizes
$$(\theta_1 - \theta_1')2\theta_1'$$
The maximizer is $\theta_1' = \theta_1/2$. Verify.
- What is the revenue? In equilibrium θ_1 wins with probability θ_1 and pays $\theta_1/2$ if she wins. So expected revenue from player 1 is $E[\theta_1 \cdot \theta_1/2] = 1/6$. The expected revenue from both players is therefore 1/3.

Second price auction Now let analyze the second price auction. It is a dominant strategy of each player to be truthful. Dominant strategy means that truthfullness is a best strategy even if the other buyer is not truthful.

- What is the revenue? Since both players are truthful, the highest bid is the highest value. Therefore θ_1 wins with probability θ_1. Conditioned on winning, she pays $\theta_1/2$. So expected revenue from player 1 is $E[\theta_1 \cdot \theta_1/2]$ which is the same as in the first price auction. Interesting, no?! We will explain later.

But before we explain why the revenue of the two mechanisms are equal, lets think about the optimal auction. That is, the revenue maximizing mechanism among all possible games between the two players. Sounds hard to find. But we will show later that we can actually find the optimal auction using the revelation principle. It is a second price with reserve p. Meaning: both players bid. Here is what happens:

- If both bids are below p: no winner.
- Only one bid above p: that player wins, pays p.
- Both bids above p: highest bid wins, pays the other bid.
- The auction is truthful. Each player is facing a price, which is the maximum of the other players bid and p.
- We will show that the second price auction with appropriately chosen reserve p is optimal. Why? We will see.
- What is the optimal p? Does it depend on the number of players?

  commented before class No! Why? Because the reserve price matters only if there is a single bid above the reserve. So the best reserve is the best price to sell the product to a single buyer, which is 0.5. This was just the intuition, and we show later than $p = 0.5$ is indeed optimal.

1 Single unit auction

"Independent private values" setup There is a single product. $N \geq 2$ buyers. The value of each buyer i is θ_i. The values are drawn independently from a distribution with probability density function f and cumulative density function F. Supports are $[\underline{\theta}_i, \bar{\theta}_i]$. Let $\theta = (\theta_1, \ldots, \theta_N)$ be the profile of all values, the Θ the set of all value profiles.

What is a mechanism? An arbitrary extensive form game tree (with the N players and nature), where each terminal node specifies who gets the product and who pays what, $\{0, 1, \ldots, N\} \times R^N$.

Mechanism defines game of incomplete information. Our solution concept is Bayes Nash equilibrium (BNE). A mechanism may have multiple equilibria. We will assume that the seller can choose an equilibrium in that case.

- Think about multiplicity of equilibria. What does it mean when we say the seller can choose an equilibrium? What if the seller can't choose the equilibrium? How does that affect the design of optimal auctions?
- What if we want to consider other solution concepts, such as PBE? Wha if we want dominant strategy incentive compatibility?

Each player can decide to not participate and get utility zero. So we will design mechanisms in which each player gets utility of at least zero. We will call such mechanisms individually rational (IR).

Let Δ be the set of probability distributions over I and not selling $\Delta = \{(p_1, \ldots, p_N)|0 \leq p_i \leq 1, \sum_i p_i \leq 1\}$.

Definition 1.1 *A direct mechanism is specified by an allocation rule $p : \Theta \to \Delta$ and transfer rules $t_i : \Theta \to R$, for each buyer i. $p_i(\theta)$ is the probability that player i receives the product ($p_0(\theta)$ is the probability that the seller keeps the product). $t_i(\theta)$ is the payment of buyer i.*

The revelation principle says that for any mechanism and any of its BNE, there exists a direct mechanism in which truthtelling is BNE. So to find optimal mechanisms, we can focus on direct mechanisms with truthtelling BNE.

Lemma 1.1 *(Revelation principle) For every mechanism Γ and BNE σ of Γ with allocation probability p and transfer t, there exists direct (p', t') with BNE σ' such that*

- $\forall i, \theta_i, \sigma'_i(\theta_i) = \theta_i$.
- *For all θ, $p(\theta) = p'(\theta)$, $t(\theta) = t'(\theta)$.*

Proof: Proof is the same as for a single agent. **QED**

As practice, apply the revelation principle to the first price auction in the 2 buyer uniform $[0, 1]$ example discussed at the beginning of the lecture.

We say that a direct mechanism is incentive compatible (IC) if truthtelling is the BNE of the direct mechanism. To formalize, let $P_i(\theta_i)$ be the probability of winning of type θ_i in the mechanism, and $T_i(\theta_i)$ the expecetd payment of type θ_i. We call P_i the "interim allocation rule", and T_i the "interim transfer rule" of the mechanism. Formally,

- $P_i(\theta_i) = E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$, where θ_{-i} denotes vector of types of players other than i, and $(\theta_i, \theta_{-i})]$ is the vector of types of all players
- $T_i(\theta_i) = E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})]$

Definition 1.2 *A direct mechanism is Bayesian incentive compatible (BIC) if*

$$\theta_i P_i(\theta_i) - T_i(\theta_i) \geq \theta_i P_i(\theta'_i) - T_i(\theta'_i)$$

for all θ_i and θ'_i.

For example, in the second price auction and when the distributions are uniform on $[0, 1]$, $P_i(\theta_i) = \theta_i$ and $T_i(\theta_i) = \theta_i^2/2$.

Notice that $\theta_i P_i(\theta'_i) - T_i(\theta'_i)$ is the same as $E_{\theta_{-i}}[\theta_i p_i(\theta'_i, \theta_{-i}) - t_i(\theta'_i, \theta_{-i})]$. So IC is really just demanding that truth telling is BNE. The formulation we gave makes it easier to characterize IC.

Here is why: Suppose that we offer buyer i a direct screening mechanism (P_i, T_i) in the sense defined in the first week. I.e., buyer i reports a type θ'_i, receives the product with probability $P_i(\theta'_i)$ and pays $T_i(\theta'_i)$. BIC is really just saying that this screening mechanism is IC. So we should be able to characterize BIC since we know how to characterize IC for a single agent, as we do below.

1.1 Characterizing incentive compatibility

We now characterize incentive compatibility. Let $U_i(\theta_i) = \theta_i P_i(\theta) - t_i(\theta_i)$ be the "indirect utility" function for player i.

Lemma 1.2 *Direct mechanism is IC if and only if*

- P_i *is increasing*
- U_i *is increasing and convex, and* $P_i(\theta_i) = U_i'(\theta_i)$.

As we saw last time, integrating $P_i(\theta_i) = U_i'(\theta_i)$ gives $T_i(\theta_i) = \theta_i P_i(\theta_i) - \int^{\theta_i} P_i(x)dx - U_i(\underline{\theta}_i)$. We refer to the second property in the lemma as "revenue equivalence". It says that two direct mechansms have the same revenue if they have the same interim allocation rules (and the same utility for lowest types).

The revenue equivalence is actually more general. It allows us to compare the revenue of even indirect mechanisms. All we need to do is to calculate the probability of winning in the BNE of a given game. If they are equal for two mechanisms, then the two mechanisms have the same revenue. For example, in the case of the uniform distributions at the beginning of this lecture, we saw that the probability of winning is

- First price: $P_i(\theta_i) = \theta_i$
- Second price: $P_i(\theta_i) = \theta_i$

Therefore, by revenue equivalence, the two auctions have the same revenue.

1.2 Revenue Maximization

How to we find optimal auctions? Like the case of a single buyer, we can express the revenue of an auction in terms of "virtual values".

Recall: For a single agent, revenue of direct mechanism with allocation probability q is

$$E[q(\theta)(\theta - \frac{1-F(\theta))}{f(\theta)}] - u(\underline{\theta}).$$

Here we have the same: revenue of direct mechanism with allocation rules P_i is

$$E[\sum_i P_i(\theta)(\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)})] - \sum_i U_i(\underline{\theta}_i).$$

We refer to $\phi_i(\theta_i) = \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$ as the "virtual value" of type θ_i. To maximize revenue, we should find an allocation rule that maximizes the expression above, the "expected virtual surplus", over all mechanisms with increasing P_i that satisfy IR. Since U_i is increasing, it is optimal to set $U_i(\underline{\theta}_i) = 0$.

One difficulty is that the expression above is written in terms of the interim allocation rule P_i, whereas we describe a mechanism with its allocation rule p_i (and payment rules). We can rewrite the problem in terms of p_i:

$$E[\sum_i P_i(\theta)\phi_i(\theta_i)] = E_\theta[\sum_i p_i(\theta)\phi_i(\theta_i)]$$

So now the problem is to maximize the above expression over all allocation rules p_i such that the interim allocation rule P_i. is increasing. As before, lets relax monotonicity. The problem is to choose p_i for each i to maximize

$$E_\theta[\sum_i p_i(\theta)\phi_i(\theta_i)]$$

subject to $p_i(\theta) \geq 0$ and $\sum_i p_i(\theta) \leq 1$ for all θ and i.

An optimal allocation rule is

$$p_i(\theta) = \begin{cases} 1/M & \text{if } \phi_i(\theta_i) = \max_j \phi_j(\theta_j) > 0. \\ 0 & \text{o.w.} \end{cases}$$

where $M := |\{i : \phi_i(\theta_i) = \max_j \phi_j(\theta_j)\}|$. Notice that if ϕ_i increasing, then so is P_i.

Theorem 1.3 *If ϕ_i increasing for each i, then above p_i is optimal.*

If ϕ_i is increasing, then in fact any optimal mechanism must maximize the expected virtual surplus. Is that unique? What can we say about any optimal mechanism?

Example: For the uniform distribution, $\theta_i - \frac{1-F(\theta_i)}{f(\theta_i)} = 2\theta_i - 1$

- Notice that $2\theta_1 - 1 \geq 2\theta_2 - 1$ if $\theta_1 \geq \theta_2$. That is, the highest virtual value is the highest value.
- Also notice that $2\theta_1 - 1 \geq 0$ if $\theta_1 \geq 0.5$.
- So to maximize revenue, we should give the product to the buyer with higher value, if that value is above 0.5. Otherwise we should not give the product to anyone. More formally, here is the allocation rule:

$$p_1(\theta) = \begin{cases} 1 & \text{if } \theta_1 > \theta_2 \& \theta_1 > 0.5 \\ 0 & \text{if } \theta_1 < 0.5 \end{cases}$$

Is this incentive compatible? That is, can we find a transfer rule that together with this allocation rule constitute an IC mechanism? Yes. Here is the payment rule.

$$t_1(\theta) = \begin{cases} \max(\theta_2, 0.5) & \text{if } \theta_1 > \theta_2 \& \theta_1 > 0.5 \\ 0 & \text{if } \theta_1 < 0.5 \end{cases}$$

And similarly for buyer 2. The auction is the second price auction with reserve 0.5.

What about general distributions?

If distributions are symmetric and ϕ increasing

- $\phi_i(\theta_i) > \phi_j(\theta_j) \Leftrightarrow \theta_i > \theta_j$.
- So highest value above reserve $\phi^{-1}(0)$ gets it
- A simple mechanism with such an allocation rule?

1.2.1 Ironing

What if virtual values are not increasing? We define a slightly different virtual value for each type, which we call "ironed" virtual value, and then show that it is optimal for the allocation rule of the mechanism to maximize ironed virtual values.

Assume for simplicity that $\underline{\theta}_i = 0$ for all i. Assume $f_i > 0$ over support.

Optimize over mechanisms where $u_i(0) = 0$ for all i.

We transform the description of mechanisms to the "quantile space":

For type θ_i, $q_i := 1 - F_i(\theta_i)$ be its "quantile", inverse $\theta_i = F_i^{-1}(1 - q_i)$. So each type is uniquely identified by its quantile.

So quantiles are uniformly distributed on $[0, 1]$.

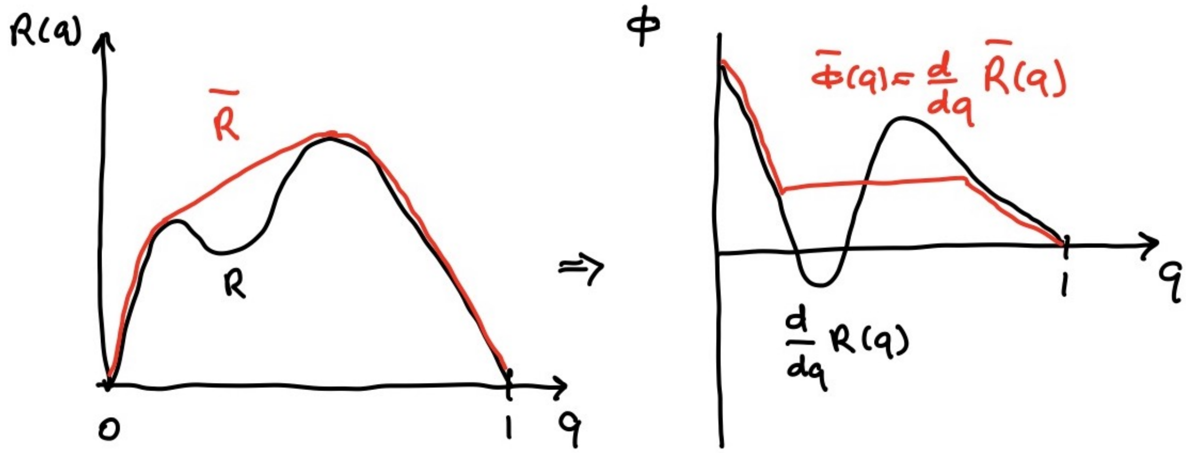
Def. Define "revenue curve" $R_i(q_i) = q_i F_i^{-1}(1 - q_i)$. Let $\bar{R}_i$ be its concave hull. $\bar{\phi}_i(q_i) = \frac{d}{dq}\bar{R}_i(q_i)$. Define the "ironed virtual value" $\bar{\phi}_i(\theta_i) = \bar{\phi}_i(1 - F_i(\theta_i))$.

(Note: the right hand side picture seems to suggest that $\frac{d}{dq}R(q)$ is equal to zero at $q = 1$. This need not be the case. In fact, in the left picture, $\frac{d}{dq}R(1) \leq 0$.)

Theorem 1.4 *The following mechanism is optimal*

$$p_i^*(\theta) = \begin{cases} 1/M & \text{if } \bar{\phi}_i(\theta_i) = \max_j \bar{\phi}_j(\theta_j) > 0. \\ 0 & o.w. \end{cases}$$

$M := |\{i : \bar{\phi}_i(\theta_i) = \max_j \bar{\phi}_j(\theta_j)\}|$.



To prove the theorem, first check that P_i increasing: this is because $\bar{\phi}_i(\theta_i)$ is increasing.

We next show that mechanism maximizes revenue.

Let $P_i(q_i) = P_i(F_i^{-1}(1-q_i))$ be the interim allocation of a type with quantile q_i.

Lemma 1.5 *Revenue of any direct IC mechanism with allocation probability p_i is* <u>*at most*</u> $\sum_i E_{q_i}[P_i(q_i)\bar{\phi}_i(q_i)]$, *with equality if for all i, $P_i'(q_i) = 0$ wherever $\bar{R}_i(q_i) > R_i(q_i)$.*

We will prove the lemma later. Before we do, note that the lemma implies that p^* is optimal: optimal revenue is at most $\max_{p:P_i \text{increasing} \forall i} \sum_i E_{q_i}[P_i(q_i)\bar{\phi}_i(q_i)] = \sum_i E_{q_i}[P_i^*(q_i)\bar{\phi}_i(q_i)] =$ revenue of p^*.

To prove Lemma 1.5, we first prove the following lemma. Define $\phi_i(q_i) = \frac{d}{dq}R_i(q_i)$.

Lemma 1.6 *Revenue of any direct IC mechanism with allocation probability p_i from i is $E_{q_i}[P_i(q_i)\phi_i(q_i)]$.*

Proof: Recall that expected revenue is $E_{\theta_i}[P_i(\theta_i)\phi_i(\theta_i)]$. By definition, if q_i is the quantile of type θ_i, i.e., $q_i = 1 - F_i(\theta_i)$, then $P_i(q_i) = P_i(F_i^{-1}(1-q_i)) = P_i(\theta_i)$. So we only need to show that $\phi_i(q_i) = \phi_i(\theta_i)$. By definition,

$$\begin{aligned}\phi_i(q_i) &= \frac{d}{dq} R_i(q_i) \\ &= \frac{d}{dq}(q_i F^{-1}(1-q_i)) \\ &= F^{-1}(1-q_i) - \frac{q_i}{f_i(F_i^{-1}(1-q_i))} \\ &= \theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)} \\ &= \phi_i(\theta_i).\end{aligned}$$

QED

Now use the above lemma to prove Lemma 1.5. Consider any p with increasing P_i. Revenue from i is

$$\begin{aligned}E_{q_i}[P_i(q_i)\phi_i(q_i)] &= \int_0^1 P_i(q_i) R_i'(q_i) dq_i \\ &= -\int_0^1 P_i'(q_i) R_i(q_i) dq_i \text{ (by integration by parts)} \\ &= -E_{q_i}[P_i'(q_i) R_i(q_i)] \\ &\leq -E_{q_i}[P_i'(q_i) \bar{R}_i(q_i)] \\ &= E_{q_i}[P_i(q_i)\bar{\phi}_i(q_i)],\end{aligned}$$

with equality if $P_i'(q_i) = 0$ whenever $\bar{R}_i(q_i) > R_i(q_i)$.

(Note quantile distribution is uniform. $Pr[q_i \leq z] = Pr[1 - F_i(\theta_i) \leq z] = Pr[\theta_i \geq F_i^{-1}(1 - z)] = 1 - F_i(F_i^{-1}(1-z)) = z$.)

Our analysis extends easily if the seller has a value c for keeping the product. For profit, just need to make sure $\max_j \bar{\phi}_j(\theta_j) > c$.

2 Beyond IPV

We now extend the independent private values setting in two important directions. First, we allow for correlated values. Second, we allow for interdependence.

2.1 Correlated values

Suppose that each bidder i has a private value θ_i for the product. But now the profile of values θ is jointly drawn from a distribution. Here is an example:

- There are two bidders. Each bidder's value is 1 or 2. So there are 4 possible value profiles. The probabilities of these profiles are as follows:
- $Pr[1,1] = Pr[2,2] = 1/3$, $Pr[1,2] = Pr[2,1] = 1/6$
- Notice that $Pr[2|2] = Pr[1|1] = 2/3$

To start, consider the second price auction with random tie-breaking. Notice that this auction is dominant strategy incentive compatible. So it is also Bayesian incentive compatible. What is the expected utility of each type?

- $U(1) = 1$. If value is 1, the bidder either loses, or wins (if the other bid is 1) and pays 1.
- $U(2) = 1/3$. With probability 1/3, the other bid is 1 in which case this bidder pays 1.

Notice that the second price auction is efficient. The highest value gets the product. But some bidder types get positive information rents. The seller wants to extract this rent. In fact, ideally, the seller wants to have efficient allocation and give zero rent to each type. We call this "full surplus extraction". In this example, full surplus extraction gives a revenue of 5/3 to the seller (i.e., revenue is 2 unless both values are 1, in which case revenue is 1), as if the seller knows the types of the buyers. We now show that this is possible!

Consider the second price auction, but with additional transfers that for each player, depend only on the bid of the other player $t(1) = -1/3, t(2) = 2/3$. Since these transfer for each player does not depend on what that player reports.

- $E(t|1) = \frac{2}{3}(-1/3) + 1/3(2/3) = 0$
- $E[t|2] = \frac{1}{3}(-1/3) + (2/3)(2/3) = 1/3$

Generalize, second price plus transfer $t_i(\theta_{-i})$ s.t.

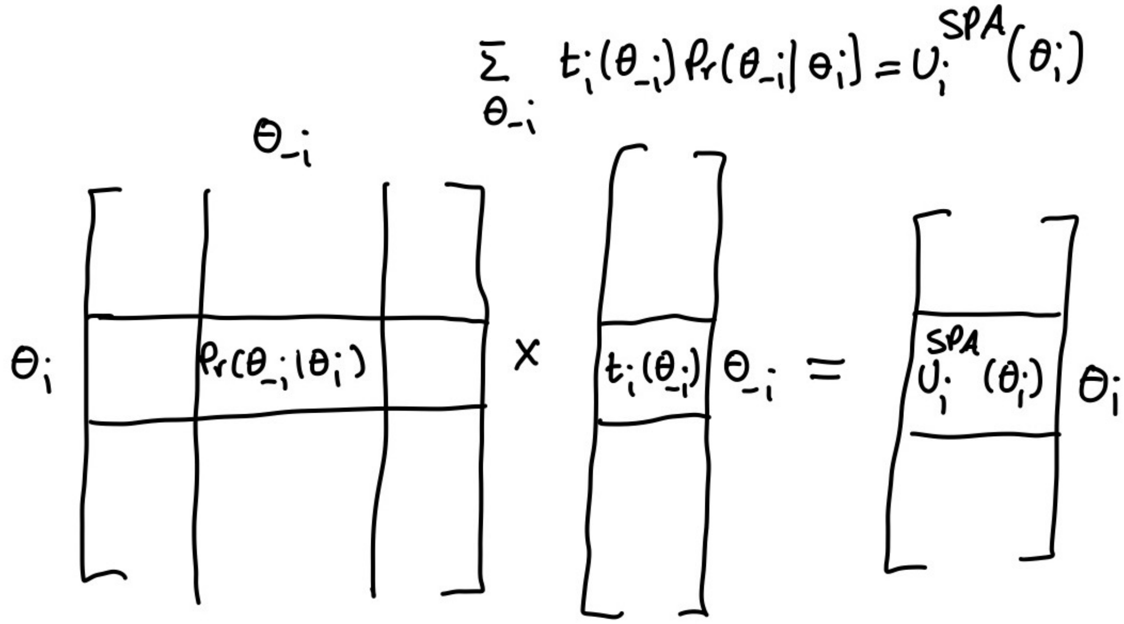
- $E[t_i(\theta_{-i})|\theta_i] = U_i^{SPA}(\theta_i)$

Theorem 2.1 *(Cremer-McLean). If the rows of the conditional probability matrix are linearly independent, there exists DSIC interim IR mechanism extracting full surplus.*

(linear independence: not $\exists \lambda : V_i \to R$ not all zero, s.t. $\sum_{t_i} \lambda(t_i) Pr[v_{-i}|t_i] = 0, \forall t_{-i}$)

The condition that the rows of the matrix are linearly independent is a quite weak property.

In a generalized setting, full surplus extraction is possible if and only if the matrix is linearly independent.



2.2 Correlated values: Characterizing BIC

We will not cover this in class.

Cremer and McLean also have a characterization of when full surplus extraction is possible with BIC mechanisms. CM2: f satisfies CM2 if for all i, s_i, there does not exist $\lambda_i : S_i \backslash s_i \to R^+$ such that

$$\sum_{s'_i \neq s_i} f(s_{-i}|s'_i)\lambda(s'_i) = f(s_{-i}|s_i)$$

Thm. If CM2 is satisfied, then for any (q, t) exists BIC (q, t') s.t. $T_i(s_i) = T'_i(s_i), \forall i, s_i$.

Proof: The proof uses the Farkas lemma. The Farkas lemmas say that for all n by m matrix A and $b \in R^n$, exactly one of the following two statements hold. (1) There exists $x \geq 0$ such that $Ax = b$. (2) There exists y such that $y^T A \leq 0, y^T b > 0$.

To apply the Farkas lemma, fix i and s_i. Let A be a matrix whose rows are indexed by s_{-i}, columns are indexed by $s'_i \neq s_i$, and the entry corresponding to s_{-i} and s'_i is $f(s_{-i}|s'_i)$. Let x be a column matrix where the entry corresponding to $s'_i \neq s_i$ is $\lambda(s'_i)$. Let b be a column matrix where the entry corresponding to s_{-i} is $f(s_{-i}|s_i)$. Now $\sum_{s'_i \neq s_i} f(s_{-i}|s'_i)\lambda(s'_i) = f(s_{-i}|s_i)$ is equivalent to $Ax = b$.

Farkas Lemma says that since there exist no $x, x \geq 0$ such that $Ax = b$, there must y such that $y^T A \leq 0, y^T b > 0$. Letting $g(s_{-i}, s_i)$ equal to $y(s_{-i})$, this can be written as

$$\sum_{s_{-i}} f(s_{-i}|s_i) g_i(s_{-i}, s_i) => 0,$$

$$\sum_{s_{-i}} f(s_{-i}|s'_i) g_i(s_{-i}, s_i) \leq 0, \forall s'_i.$$

Define $\epsilon = \sum_{s_{-i}} f(s_{-i}|s_i) g_i(s_{-i}, s_i)$. Let $h_i(s) = g_i(s) - \epsilon$. We have

$$\sum_{s_{-i}} f(s_{-i}|s_i) h_i(s_{-i}, s_i) = 0,$$

$$\sum_{s_{-i}} f(s_{-i}|s'_i) h_i(s_{-i}, s_i) < 0, \forall s'_i.$$

Let $t_i(s) = K h_i(s)$ for some very large K. Then $T_i(s_i) = 0$, and interim transfer of misreporting is a very large negative number. **QED**

2.3 Interdependent values

Setup:

- Players are $1, \ldots, N$
- Each player has a set of signals $X_i \in [0, w_i]$
- The value of player i is $v_i(X_1, \ldots, X_N)$, increasing in all coordinates, strictly in X_i
- (A special case is private values. That is when buyer i's value depends only on her own signal $v_i(X) = X_i$)
- (Another special case is common value. That is when all values are equal $v_i(X) = v(X)$)

2.3.1 Winner's Curse

Consider the case of common value. Suppose that players are symmetric (will be formalized below).

Given $X_1 = x$, estimate $E[V|X_1 = x]$ (V the random variable that denotes the common value)

- Now suppose that player 1 wins in first price auction
- Suppose that all buyers use the same strategy that is increasing in their private information. If player 1 wins, then must be that others' signals below x

- New estimate $E[V|X_1 = x, X_2, \ldots, X_N < x] \leq E[V|X_1 = x]$. When player 1 bids in the auction, her estimate of her value should not be $E[V|X_1 = x]$. It should be $E[V|X_1 = x, X_2, \ldots, X_N < x]$.
- Failure to foresee this is winner's curse

Example:

- $X_i = V + \epsilon_i$, ϵ_i i.i.d with $E[\epsilon_i] = 0$.
- Then $E[X_i|V = v] = v$
- BUT, $E[maxX_i|V = v] > maxE[X_i|V = v] = v$ (since max is convex). That is, buyer i should think that if she wins, her X_i is the maximum over all signals, and so the expected value of the product is less than her private signal.

2.3.2 Second Price Auction

Assume symmetry:

- $v_i(X) = u(X_i, X_{-i})$, u symmetric in X_{-i}, e.g. $u(x, y, z) = u(x, z, y)$
- Also joint density f of signals symmetric.
- Define $v(x, y) = E[V_1|X_1 = x, \text{highest of others } = y]$

Proposition 2.2 *A symmetric equilibrium in a second price auction is* $\sigma(x) = v(x, x)$.

Proof: Win if $b \geq \sigma(y)$, in which case payoff is $v(x, y) - \sigma(y)$. Expected payoff of bidding b if signal is x is

$$\int_0^{\sigma^{-1}(b)} (v(x, y) - \sigma(y))g(y|x)dy$$

$$= \int_0^{\sigma^{-1}(b)} (v(x, y) - v(y, y))g(y|x)dy$$

Note that $v(x, y) > v(y, y)$ if $x > y$, and $v(x, y) < v(y, y)$ if $x < y$. So maximized if $\sigma^{-1}(b) = x$, $b = \sigma(x)$.

QED

Example (will not be covered in class):

- 3 bidders, common value V uniform $[0,1]$
- Given $V = v$, signals are i.i.d and uniform in $[0, 2v]$.
- Define $Z = \max X_1, X_2, X_3$.
- $f(X_1, X_2, X_3|v) = 1/8v^3$ if $Z \leq 2v$, and $f = 0$ otherwise

$$f(X_1, X_2, X_3) = \int_{Z/2}^{1} \frac{1}{8v^3} dv = \frac{4 - z^2}{16z^2}$$

$$f(v|X = x) = f(v|Z = z) = \frac{1}{8v^3} \frac{16z^2}{4 - z^2}$$

on the interval $[z/2, 1]$.

$$E[V|X = x] = \int_{z/2}^{1} v f(v|X = x) dv = \frac{2z}{2 + z}$$

$$v(x, y) = E[V|X_1 = x, Y_1 = y] = E[V|Z = \max(x, y)]$$

$$= \frac{2 \max(x, y)}{2 + \max(x, y)}$$

Thus, equlibrium is

$$\sigma(x) = 2x/(x + 2).$$

References

[Börgers, 2015] Börgers, T. (2015). An introduction to the theory of mechanism design. Oxford University Press, USA.

[Crémer and McLean, 1985] Crémer, J. and McLean, R. P. (1985). Optimal selling strategies under uncertainty for a discriminating monopolist when demands are interdependent. Econometrica, 53(2):345–61.

[Krishna, 2009] Krishna, V. (2009). Auction theory. Academic press.

[Myerson, 1981] Myerson, R. (1981). Optimal auction design. Mathematics of Operations Research, 6(1):pp. 58–73.