K-Homology, Assembly and Rigidity Theorems for Relative Eta Invariants

Nigel Higson

Department of Mathematics
Pennsylvania State University

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K-homology, elliptic operators and C*-algebras.
Geometric K-homology and index theory.
The assembly map.
Relative eta invariants.

Joint work with John Roe.
K-Homology and elliptic operators

- $M$ = a smooth, closed manifold.
- $D$ = a linear elliptic operator on $M$ (partial differential, first order).

Atiyah’s observation: $D$ determines a class in the $K$-homology group $K_0(M)$ via the index map

\[ \text{Index}: K^0(M) \to \mathbb{Z} \]

\[ [E] \mapsto \text{Index}(D_E) \]

... and the families index map

\[ \text{Index}: K^0(M \times X) \to K^0(X). \]
The point: K-homology conceptualizes many constructions in index theory.

Example: The index is induced from the map $\varepsilon : M \to \mathrm{pt}$.

Example: The index of a boundary operator is zero:

$$
\cdots \longrightarrow K_1(W, \partial W) \longrightarrow K_0(\partial W) \longrightarrow K_0(W) \longrightarrow \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \longrightarrow K_1(\mathrm{pt}, \mathrm{pt}) \longrightarrow K_0(\mathrm{pt}) \cong K_0(\mathrm{pt}) \longrightarrow \cdots 
$$

Example: From $\alpha \in K_0(\mathbb{R}^2)$ and $\beta \in K^0(\mathbb{R}^2)$ with $\alpha \cap \beta = 1$ one obtains Bott periodicity.
Atiyah’s question: Can $K$-homology be defined using elliptic operators?

Note: if $D$ is an operator on $M$ and $f: M \rightarrow N$, then we obtain

$$f_*[D] \in K_0(N)$$

A good definition should naturally account for this . . . which the definition

$$K_0(M) = K^0(T^*M)$$

does not.
Atiyah’s suggestion: An analytic cycle for $K_0(X)$ is a bounded Fredholm operator $F$ on a Hilbert space $H$ that is equipped with an action of $C(X)$ . . . such that $F$ is pseudolocal: $\varphi_1 F \varphi_2$ is a compact operator, if $\varphi_1, \varphi_2$ have disjoint supports.

The point: (1) For such an $(F, H)$, the families index map

$$\text{Index} : K^0(X \times Y) \to K^0(Y).$$

is still defined.

(2) $F = D(I + D^* D)^{-1/2} = \text{order zero } \psi DO$ is an example.
Analytic cycles for the odd $K$-homology group
An analytic cycle for $K_1(X)$ is given by a Fredholm operator $F$, as before, which is now self-adjoint.

Compare: \[ \{ \text{Self-adjoint Fredholms} \} \sim \Omega(\text{Bott Spectrum}) . \]

Definition

$K_0^{\text{analytic}}(X) = \text{Grothendieck group of homotopy classes of cycles.}$

Theorem (Brown, Douglas and Fillmore; Kasparov)

The index mappings identify $K_0^{\text{analytic}}(X)$ with the $K$-homology group $K_0(X)$. 
**C*-algebra K-theory and duality**

**Definition (for later use . . . )**
\[ \mathcal{D}(X) = \text{commutant modulo compact operators of } C(X) \text{ (acting on a single fixed Hilbert space } H). \]

Now let \( C(X) = \text{compact operators on } H. \)

**Theorem (for later use . . . )**
\[ K_n^{\text{analytic}}(X) \cong K_{n-1}(\mathcal{D}(X)/C(X)). \]

**Explanation (n odd)**
If \( P \) is a projection element in \( \mathcal{D}(X)/C(X) \) then the operator
\[ F = 2P - I \]
lifts to an analytic cycle for \( K_1^{\text{analytic}}(X). \)
Geometric K-homology

Geometric cycle for $K_n(X)$: $(M, E, f)$

- $M = \text{spin}^c$-manifold ($n \equiv \dim(M)$).
- $E = \text{complex vector bundle on } M$.
- $f : M \to X$.

We can form $f_*(\lbrack D \rbrack \cap \lbrack E \rbrack) \in K_n(X)$.

$\Rightarrow (M, E, f)$ determines a $K_n(X)$ class.

Equivalence relation on geometric cycles.

- Direct sum/disjoint union and Bordism.
- Bundle Modification. $P = \text{principal } G\text{-bundle over } M$.
  $(S^{2k}, F, \varepsilon) = G\text{-equivariant cycle over pt, Index } = 1$.

$(M, E, f) \sim (M, E, f) \times_P (S^{2k}, F, \varepsilon)$
The index theorem

\[ K_{0}^{\text{geom}}(X) \xrightarrow{(M,E,f) \mapsto f_{\ast}[D_{E}]} K_{0}^{\text{analytic}}(X) \]

\[ h_{\ast}^{\text{geom}} \downarrow \]

\[ K_{0}^{\text{geom}}(Y) \xrightarrow{(M,E,f) \mapsto f_{\ast}[D_{E}]} K_{0}^{\text{analytic}}(Y) \]

\[ h: X \to Y \]

\[ h_{\ast}^{\text{analytic}} \downarrow \]

\[ \varepsilon: X \to \text{pt} \]

\[ \varepsilon_{\ast}^{\text{geom}} = \text{topological index} \]

\[ \varepsilon_{\ast}^{\text{analytic}} = \text{analytic index} \]
Poincaré bundle

- $\pi$ = (free) abelian group.
- $\hat{\pi}$ = Pontrjagin dual.
- $f : X \to B\pi$.

The Poincaré line bundle over $X \times \hat{\pi}$ is

$$P = (\hat{X} \times \hat{\pi} \times \mathbb{C}) / \pi,$$

where the action of $\pi$ is

$$g \cdot (x, y, z) = (gx, y, \langle g, y \rangle z).$$

We obtain a so-called dualizing class $[P] \in K^0(X \times \hat{\pi})$ and

$$K_n(X) \otimes [P] \to K^n(\hat{\pi}).$$
Mishchenko bundle

- \( \pi = \) any (torsion-free) group.
- \( C^*\pi = \) group \( C^*\)-algebra.
- \( f: X \to B\pi. \)

The **Mishchenko line bundle** over \( X \) is the flat line bundle

\[
M = (\hat{X} \times C^*\pi)/\pi.
\]

where the action of \( \pi \) is

\[
g \cdot (x, y) = (gx, gy).
\]

Its fibers are rank-one free right modules over \( C^*\pi. \)

We obtain a class \([M] \in K_0(C(X, C^*\pi))\) \((C^*\)-algebra \( K\)-theory\) and a “dualizing map”

\[
K_n(X) \xrightarrow{\otimes[M]} K_n(C^*\pi).
\]
Kasparov’s *assembly map* is defined to be

\[ K_n(B^\pi) \xrightarrow{\mu = \otimes [M]} K_n(C^{\ast} \pi). \]

**Key properties:**

- If \( D \) = Dirac operator on \( M \) and \( f: M \to B^\pi \), and if \( M \) has positive scalar curvature, then

  \[ \text{Lichnerowicz} \implies \mu(f_*[D]) = 0. \]

- If \( D \) = signature operator on \( M \) and \( f: M \to B^\pi \), then

  \[ \mu(f_*[D]) = \text{oriented homotopy invariant}. \]
The strong Novikov conjecture

**Strong Novikov conjecture**

*The Kasparov assembly map*

\[ K_n(B\pi) \xrightarrow{\mu} K_n(C^{\ast}_\pi) \]

*is always rationally injective.*

**Consequences**

- Gromov-Lawson-type results . . . for example if \( M \) is aspherical and spin, it does not admit a p.s.c. metric.
- The Novikov conjecture.
Integral strong Novikov conjecture
If $\pi$ is torsion-free, then the Kasparov assembly map is split injective.

(Generally false) isomorphism conjecture
If $\pi$ is torsion-free, then the Kasparov assembly map is an isomorphism.

Baum-Connes isomorphism conjecture
If $\pi$ is torsion-free, then the modified Kasparov assembly map

$$K_n(B\pi) \xrightarrow{\mu} K_n(C^*_\lambda \pi).$$

is an isomorphism.
A consequence of Baum-Connes

Baum-Connes for $\pi$
(torsion-free) $\Rightarrow$ Spectrum($x$) is connected,
for every $x \in C_\lambda^*(\pi)$

$\Rightarrow$ Spectra of elliptic operators on
$\pi$-covering spaces have discrete
component structures

Why? (1) Components in spectra determine projection matrices
over $C_\lambda^*(\pi)$ (functional calculus).
(2) Integrality of the Fredholm index.
The fiber of the assembly map

One can construct “structure groups” $S_n(\pi) \ldots$

\[ \cdots \rightarrow S_1(\pi) \rightarrow K_1(B\pi) \xrightarrow{\mu} K_1(C^*\pi) \rightarrow S_0(\pi) \rightarrow \cdots \]

- Close connections to surgery exact sequence.
- Obviously $BC \iff S_\ast(\pi) \equiv 0$.
- Some (limited) use in strong Novikov.
- Vanishing theorems in $K_n(C^*\pi)$ made concrete by constructions of elements in $S_n(\pi)$. 

Nigel Higson | K-Homology and Relative Eta Invariants
Recall we previously defined:

- \( H = L^2(X) \), say.
- \( \mathcal{D}(X) = C^*\)-algebra of “pseudolocal operators” on \( H \).

Now define, given \( f : X \to B_\pi \):

- \( H_\pi = L^2(X, M) \)
  - = sections of the Mishchenko line bundle
  - = a Hilbert \( C^*(\pi) \)-module
  - \( \cong L^2(U) \otimes C^*(\pi) \), locally.

- \( \mathcal{D}_\pi(X) = C^*\)-algebra of operators on \( H_\pi \) that are locally \( T \otimes I \), with \( T \in \mathcal{D}(X) \), modulo compact operators.

We get:

\[
0 \longrightarrow \mathcal{C}_\pi(X) \longrightarrow \mathcal{D}_\pi(X) \longrightarrow \mathcal{D}(X)/\mathcal{C}(X) \longrightarrow 0
\]
The analytic surgery exact sequence, continued

From

\[ 0 \to C_\pi(X) \to D_\pi(X) \to D(X)/C(X) \to 0 \]

we get:

\[ \to K_0(D_\pi(X)) \to K_0(D(X)/C(X)) \to K_1(C_\pi(X)) \to \]

\[ \overset{\text{def}}{\Downarrow} \quad \overset{\text{def}}{\Downarrow} \quad \underset{\mu}{\Downarrow} \]

\[ \to S_1(\pi) \to K_1(X) \to K_1(C^*(\pi)) \to \]
Example of an analytic structure

**Analytic structure** = Element of $S_n(\pi)$.

- $M = \text{spin-manifold (odd-dimensional) with p.s.c. metric } g$
- $f: M \rightarrow B_\pi$.
- $D_\pi = \text{Dirac coupled to the (flat) Mishchenko line bundle.}$

Lichnerowicz $\Rightarrow D^2_\pi \gg 0$

$\Rightarrow \text{Spectrum}(D_\pi) \text{ has gap at } \lambda = 0.$

$\Rightarrow \text{positive spectral projection } P_\pi \text{ of } D_\pi \text{ lies in } \mathcal{D}^*_\pi(X)$

**Definition**

$[M, g] = [P_\pi] \in S_1(\pi)$
**Eta invariants**

- $D = \text{self-adjoint elliptic operator on } M \text{ (odd-dimensional)}$.
- The *eta-function* of $D$ is

$$\eta_D(s) = \sum_{\text{Spectrum of } D} \text{sign}(\lambda)|\lambda|^{-s}$$

$$= \Gamma\left(\frac{s+1}{2}\right)^{-1} \int_0^\infty \text{Trace}(D e^{-tD^2}) \; t^{\frac{s-1}{2}} \; dt.$$

**Theorem**

*It is meromorphic in $s \in \mathbb{C}$ and regular at $s = 0$.*

**Definition**

$$\eta(D) = \eta_D(0) = \text{“Trace(sign(D))”}.$$
Relative eta invariants

- $D$ = self-adjoint elliptic operator on $M$.
- $f : M \to B_\pi$
- $\sigma_1, \sigma_2 : \pi \to U(N)$.

We can form twisted operators $D_1$ and $D_2$ using $\sigma_1$ and $\sigma_2$.

**Definition (Slightly simplified)**

$$\rho(D, f, \sigma_1, \sigma_2) = \frac{1}{2} \text{Tr}(\text{sign}(D_1)) - \frac{1}{2} \text{Tr}(\text{sign}(D_2))$$

$$= \frac{1}{2} \eta(D_1) - \frac{1}{2} \eta(D_2)$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{2}\right)^{-1} \int_0^\infty \left(\text{Tr}(D_1 e^{-tD_1^2}) - \text{Tr}(D_2 e^{-tD_2^2})\right) t^{-\frac{1}{2}} dt$$

**Notation**

With $\sigma_1$ and $\sigma_2$ fixed, we’ll write $\rho(D, f)$. 
Theorem (Atiyah-Patodi-Singer)

If $(M, D)$ is the boundary of $(W, Q)$, then

$$
\text{APS-Index}(Q) = \int_W \text{local expression} - \frac{1}{2} \eta(D).
$$

Corollary

If $(M, D, f)$ is a boundary, then $\rho(D, f) \in \mathbb{Z}$. 
Rigidity theorems for the relative eta invariant

- $M =$ closed, odd-dimensional spin-manifold with p.s.c. metric.
- $D =$ Dirac operator.
- $f: M \to B\pi$ and $\sigma_1, \sigma_2: \pi \to U(N)$.

Theorem (Keswani)

*If $\pi$ is torsion-free and the Kasparov assembly map is an isomorphism, then the relative eta invariants $\rho(D, f)$ vanish. Similarly, the relative eta invariants for the signature operators on homotopy equivalent oriented manifolds are equal.*

Remark

Related results have been investigated by Mathai, Weinberger, Chang, Schick, Piazza, ...
A theorem of Weinberger

Here is a warm-up theorem, interesting in its own right.

- \( M \) = closed, odd-dimensional spin-manifold with p.s.c. metric.
- \( D = \) Dirac operator.
- \( f : M \to B\pi \) and \( \sigma_1, \sigma_2 : \pi \to U(N) \).

Theorem (Weinberger)

*The relative eta invariants for Dirac operators on p.s.c. spin manifolds are *rational* numbers.*

Remark

There is a similar theorem for signature operators.
Proof using geometric K-homology

(1) For a geometric cycle \((M, E, f)\) define 
\[ \rho(M, E, f) = \rho(D_E, f), \]
for some choice of Dirac operator.

(2) By the APS theorem, different choices of Dirac, or indeed bordant cycles, will give the same relative eta invariant, modulo an integer.

(3) By an explicit calculation cycles equivalent via bundle modification have the same relative eta invariant (for suitable choices of Dirac operators).

(4) We obtain therefore an \(\mathbb{R}/\mathbb{Z}\)-index map

\[
\text{Index}_{\sigma_1, \sigma_2} : K_1(B\pi) \longrightarrow \mathbb{R}/\mathbb{Z}.
\]
(5) If the assembly map

$$\mu : K_1(B\pi) \longrightarrow K_1(C^*(\pi))$$

was rationally injective, then it would follow that $[D]$ is torsion in the group $K_1(B\pi)$ — and hence that $\text{Index}_{\sigma_1,\sigma_2}[D]$ is torsion in the group $\mathbb{R}/\mathbb{Z}$.

(6) But we can replace $\pi$ by any quotient group through which $\sigma_1$ and $\sigma_2$ factor — for instance by a linear group.

(7) The assembly map is rationally injective for a linear group.

QED
Recall the surgery exact sequence is constructed from:

\[ 0 \rightarrow C_\pi(X) \rightarrow D_\pi(X) \rightarrow D(X)/C(X) \rightarrow 0. \]

We want to complete the following diagram:

\[ \cdots \rightarrow K_0(C^*_\pi) \rightarrow S_1(\pi) \rightarrow K_1(B_\pi) \rightarrow \cdots \]

\[ \text{Tr}_{\sigma_1} - \text{Tr}_{\sigma_2} \downarrow \quad ? \quad ? \quad \downarrow \text{Index}_{\sigma_1,\sigma_2} \]

\[ 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0. \]

\[ \text{Tr}_{\sigma_1,\sigma_2}: S_1(\pi) \rightarrow \mathbb{R} \]
The relative trace map via analysis

- \( S_1(\pi) = K_0(D_\pi(X)) \)
- \( T \in D_\pi(X) \iff T = S \otimes I + K \) locally
- In more detail:
  1. \( T : L^2(X, M) \to L^2(X, M) \)
  2. \( L^2(U, M) \cong L^2(U) \otimes C^*(\pi) \) for small \( U \subset X \)
  3. \( \phi S \psi \) compact if \( \text{Supp}(\phi) \cap \text{Supp}(\psi) = \emptyset \).
    Also, \( K \) is compact.

**Lemma**

Replacing “compact” with “trace class” we obtain a dense subalgebra \( D^{\text{tracial}}_\pi(X) \) on which the formula

\[
\text{Tr}_{\sigma_1, \sigma_2}(T) = \text{Tr}_{\sigma_1}(K) - \text{Tr}_{\sigma_2}(K)
\]

defines a trace, and hence a map

\[
\text{Tr}_{\sigma_1, \sigma_2} : K_0(D^{\text{tracial}}_\pi(X)) \to \mathbb{C}.
\]
After (unfortunately considerable) additional work, we obtain from

\[ \text{Tr}_{\sigma_1, \sigma_2} : K_0(D_{\pi}^{\text{tracial}}(X)) \rightarrow \mathbb{C} \]

the required relative trace map on \( K_0(D_{\pi}(X)) \):

\[
\cdots \rightarrow K_0(C^*_\pi) \xrightarrow{\text{Tr}_{\sigma_1, \sigma_2}} S_1(\pi) \xrightarrow{\text{Tr}_{\sigma_1, \sigma_2}} K_1(B_{\pi}) \xrightarrow{\text{Index}_{\sigma_1, \sigma_2}} \cdots
\]

\[
\begin{array}{ccc}
0 & \rightarrow & \mathbb{Z} \\
\text{Tr}_{\sigma_1} - \text{Tr}_{\sigma_2} & \downarrow & \text{Tr}_{\sigma_1, \sigma_2} \\
 & & \downarrow \text{Index}_{\sigma_1, \sigma_2} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{R} / \mathbb{Z} & \rightarrow & 0
\end{array}
\]
Lemma

Applied to the structure class of a p.s.c. spin manifold we recover the relative eta invariant of the Dirac operator:

\[
\text{Tr}_{\sigma_1,\sigma_2}([M, g]) = \rho(D, f).
\]

If \( P_\pi \) = positive spectral projection of \( D_\pi \), then

\[
P_\pi = \frac{1}{2} (\text{sign}(D_\pi) + I).
\]

Therefore

\[
\text{Tr}_{\sigma_1,\sigma_2}(P) = \text{Tr}_{\sigma_1}(P) - \text{Tr}_{\sigma_1}(P)
= \frac{1}{2} \text{Tr}_{\sigma_1}(D_\pi) - \frac{1}{2} \text{Tr}_{\sigma_2}(D_\pi) = \rho(D, f).
\]

as required.
Proof of the main theorem, completed

- \( D = \) Dirac on p.s.c. spin-manifold.
- \( f : M \to B_\pi \).

**Theorem (Keswani)**

*If the Kasparov assembly map is an isomorphism, then the relative eta invariants of \( D \) are zero.*

For if the assembly map is an isomorphism, then the structure group \( S_1(\pi) \) is zero.