K-Theory and Noncommutative Geometry

Lecture 1
Elliptic Operators and Index Problems

Nigel Higson
Penn State University

July, 2002

http://www.math.psu.edu/higson/trieste
Noncommutative Geometry

Alain Connes


Connes-Moscovici Index Theorems

**Theorem.** Let \((A, H, D)\) be a spectral triple with simple dimension spectrum. Let \(\tau\) be the associated residue trace. The formula

\[
\phi_p(a^0, \ldots, a^p) = \sum_{k \geq 0} c_{p,k}\tau \left( e a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} |D|^{-p-2|k|} \right),
\]

where

\[
c_{p,k} = \frac{(-1)^{|k|}}{k_1! \cdots k_p!} \cdot \frac{\Gamma(|k| + \frac{n}{2})}{(k_1 + 1)(k_2 + 2) \cdots (k_p + p)}
\]

defines an index cocycle for \((A, H, D)\) in the cyclic \((b, B)\)-bicomplex.

**Theorem.** If \((A, H, D)\) is the spectral triple associated to a foliation then the cyclic class of the above index cocycle is in the range of the characteristic map

\[
H^*(\text{WSO}(n)) \longrightarrow \text{HC}^*(A).
\]
In the winter of 1900-1901 the Swedish mathematician Holmgren reported in Hilbert’s seminar on Fredholm’s first publications on integral equations, and it seems that Hilbert caught fire at once . . .

Hermann Weyl

*David Hilbert and his mathematical work*
Helmholtz Equation

Hilbert saw two things: (1) after having constructed Green’s function $K$ for a given region $\Omega$ and for the potential equation $\Delta u = 0$ . . ., the equation

$$\Delta \phi - \lambda \phi = 0$$

for the oscillating membrane changes into a homogeneous integral equation

$$\phi(s) - \lambda \int K(s, t) \phi(t) \, dt = 0$$

with the symmetric $K$, $K(t, s) = K(s, t)$ . . .; (2) the problem of ascertaining the “eigen values” $\lambda$ and “eigen functions” $\phi(s)$ of this integral equation is the analogue for integrals of the transformation of a quadratic form of $n$ variables onto principal axes.

Hermann Weyl

*David Hilbert and his mathematical work*
Helmholtz Equation, Continued

\[ \Delta u = \lambda u \]
\[ u|_{\partial \Omega} = 0 \]

Green's function:
\[ \begin{cases} 
\Delta_z K(z, w) = \delta_w, \\
K(z, w) = 0 & \text{on } \partial \Omega.
\end{cases} \]

If \( u|_{\partial \Omega} = 0 \) then
\[ \begin{cases} 
u(z) = \int_{\Omega} K(z, w) \Delta u(w) \, dw. \\
\Delta u = \lambda u & \iff u = \lambda K u.
\end{cases} \]

\[ K(z, w) = \frac{1}{2\pi} \log \left| \frac{1 - g(z)g(w)}{g(z) - g(w)} \right|. \]
Problem of H.A. Lorentz

... there is a mathematical problem which will perhaps arouse the interest of mathematicians ... In an enclosure with a perfectly reflecting surface there can form standing electromagnetic waves analogous to tones of an organ pipe ... there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between $\nu$ and $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its area.

H.A. Lorentz

Wolfskehl Lecture, 1910
Reformulation

If $\Omega$ is a bounded domain in $\mathbb{R}^2$ (with reasonable boundary $\partial \Omega$), and if $N(\lambda)$ is the number of eigenvalues $\lambda_n$ of $\Delta$ less than or equal to $\lambda$, then show that

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{\text{constant}}.$$ 

This is equivalent to the asymptotic relation

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{\text{constant}}{\text{Area}(\Omega)}.$$ 

**Note:** This is OK for rectangles: $\text{constant} = 4\pi$. 

\[ \Delta u_n = \lambda_n u_n \]
\[ u_n |_{\partial \Omega} = 0 \]
Here $m, n > 0$. It follows that

$$N(\lambda) = \# \left\{ (n, m) \middle| m > 0, n > 0 \right\} \quad \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \leq \frac{\lambda}{\pi^2} \right)$$

$$\sim \frac{1}{4} \left( \text{Area of Ellipse} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq \frac{\lambda}{\pi^2} \right)$$

Thus

$$\frac{N(\lambda)}{\lambda} \sim \frac{\text{Area}(\Omega)}{4\pi},$$

as required.
The idea was one of many, as they probably come to every young person preoccupied with science but while others soon burst like soap bubbles, this one soon led, as a short inspection showed, to the goal. I was myself rather taken aback by it as I had not believed myself capable of anything like it. Added to that was the fact that the result, although conjectured by physicists some time ago, appeared to most mathematicians as something whose proof was still far in the future.

Hermann Weyl

*Gibbs Lecture, 1948*
Self-Adjoint Operators

$T : H \to H$ densely defined Hilbert space operator.

$T$ is *self-adjoint* if

- $\langle Tv, w \rangle = \langle v, Tw \rangle$
- $T \pm iI$ maps $\text{Domain}(T)$ *one to one and onto* $H$.

**Note.** $\| (T \pm iI)v \|_2^2 = \|Tv\|_2^2 + \|v\|_2^2$.

**Spectral Theorem.** *There is a unique, $\ast$-homomorphism*

$$\left\{ \text{Bounded continuous functions on } \mathbb{R} \right\} \to \mathcal{B}(H)$$

such that $(x \pm i)^{-1} \mapsto (T \pm iI)^{-1}$.

**Definition.** The *spectrum* of $T$ is the complement of the largest open set $U$ in $\mathbb{R}$ such that

$$\text{supp}(f) \subseteq U \quad \Rightarrow \quad f(T) = 0.$$ 

**Example.** If $\langle Tv, v \rangle \geq 0$ for all $v$ (i.e., if $T$ is *positive*) then the spectrum of $T$ is contained within $[0, \infty)$. 
Self-Adjoint Extensions

**Definition.** \( T_1 \) is an *extension* of an unbounded operator \( T_0 \) if

- \( \text{Domain}(T_0) \subseteq \text{Domain}(T_1) \)
- \( T_1|_{\text{Domain}(T_0)} = T_0 \).

**Example (Friedrichs extension).**

\[
\begin{align*}
\Delta_0 &= \Delta \quad \text{on } C_c^\infty(\Omega) \\
\nabla_0 &= \nabla \quad \text{on } C_c^\infty(\Omega)
\end{align*}
\]

Denote by \( H_0^1(\Omega) \subseteq L^2(\Omega) \) the completion of \( C_c(\Omega) \) in the norm \( ||u||^2 = |u|^2 + |\nabla u|^2 \). Extend \( \nabla_0 \) to \( \nabla_1 \) on \( H_0^1(\Omega) \) by continuity. Then define \( \Delta_1 \) by

\[
\text{Graph}(\Delta_1) = \\
\{(u, v) \in H_0^1 \times L^2 \mid \langle \nabla_1 u, \nabla_1 w \rangle = \langle v, w \rangle \quad \forall w \in H_0^1(\Omega)\}.
\]

**Proposition.** \( \Delta_1 \) is a positive self-adjoint extension of the operator \( \Delta_0 \). \( \square \)
Compact Operators

Definition. A bounded linear operator on a Hilbert space is *compact* (or *completely continuous*, in old-fashioned terms) if it maps the closed unit ball of Hilbert space to a (pre)compact set.

Example. If $T$ is a norm-limit of finite-rank operators then $T$ is compact.

Elementary calculus $\Rightarrow$ the maximum value of the function $\varphi(v) = \|Tv\|^2$ on the closed unit ball of $H$ is an eigenvalue for $T^*T$.

Theorem. If $T$ is a compact and selfadjoint operator then there is an orthonormal basis for $H$ consisting of eigenvectors for $T$. The eigenvalues converge to zero.

Rellich Lemma. The inclusion of $H^1_0(\Omega)$ into $L^2(\Omega)$ is a compact operator.

Theorem. $(I + \Delta)^{-1}$ (defined using the Friedrichs extension) is a compact operator.

Remark. A convenient improvement: $\Delta$ is invertible, and $\Delta^{-1}$ is a compact operator.
Spectral Theory for the Laplacian

\[ \Delta = \nabla^* \nabla \]
Friedrichs extension of the Laplace operator.

**Theorem.** There is an orthonormal basis for \( L^2(\Omega) \) consisting of functions \( f_n \in H^1_0(\Omega) \) for which

\[ \Delta f_n = \lambda_n f_n \]

in the distributional sense. The eigenvalues \( \lambda_n \) are positive and converge to infinity.

**Remark.** In fact one can show that \( f_n \in C^\infty(\overline{\Omega}) \) and \( f_n|_{\partial \Omega} = 0 \). This follows from elliptic regularity, Sobolev theory. More on this in the next lecture.
Singular Values

**Definition.** The *singular values* $\mu_1(T), \mu_2(T), \ldots$ of a bounded operator $T$ are scalars

$$\mu_n(T) = \inf_{\text{dim}(V)=n-1} \sup_{v \perp V} \frac{\|Tv\|}{\|v\|}.$$

Observe that $\mu_1(T) \geq \mu_2(T) \geq \ldots$ and that

$$T \text{ is compact } \iff \lim_{n \to \infty} \mu_n(T) = 0.$$

Now let $T$ be compact, self-adjoint, and positive (meaning $\langle Tv, v \rangle \geq 0$). List the eigenvalues $\lambda_n(T)$ in decreasing order, and with multiplicity.

**Theorem.** If $T$ is compact, self-adjoint, and positive then $\mu_n(T) = \lambda_n(T)$.

**Proof.** $T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \\ & & \ddots \end{pmatrix}$
**Comparison of Singular Values**

**Lemma.** If \( K_0 \) and \( K_1 \) are operators on a Hilbert space \( H \), and if

\[
\langle K_0 v, v \rangle \geq \langle K_1 v, v \rangle \geq 0,
\]

for all \( v \in H \), then \( \lambda_n(K_0) \geq \lambda_n(K_1) \), for all \( n \).

We’re now ready begin the proof of Weyl’s theorem.

**Domain Dependence Inequality.** If \( \Omega_0 \subseteq \Omega_1 \) then \( N_{\Omega_0}(\lambda) \leq N_{\Omega_1}(\lambda) \), for all \( \lambda \).

**Proof.** If \( \Omega_0 \subseteq \Omega_1 \) then \( \text{domain}(\nabla_0) \subseteq \text{domain}(\nabla_1) \) and \( \nabla_1|_{\text{domain}(\nabla_0)} = \nabla_0 \).

\[
\nabla_1|_{\text{domain}(\nabla_0)} = \nabla_0
\]

It follows after a brief computation that \( \Delta_0^{-1} \leq \Delta_1^{-1} \).
Lemma. If \( w_0 \in \text{Domain}(\Delta_0) \) and \( w_1 \in \text{Domain}(\Delta_1) \), and if \( \Delta_0 w_0 = \Delta_1 w_1 \) on \( \Omega_0 \), then \( \| \nabla w_0 \|^2 \leq \| \nabla w_1 \|^2 \).

Proof. The hypotheses imply

\[ \langle w_0, \Delta_0 w_0 \rangle = \langle w_0, \Delta_1 w_1 \rangle \]

or in other words

\[ \langle \nabla w_0, \nabla w_0 \rangle = \langle \nabla w_0, \nabla w_1 \rangle. \]

Now apply Cauchy-Schwarz. \( \square \)

Theorem. \( \langle \Delta_0^{-1} v, v \rangle \leq \langle \Delta_1^{-1} v, v \rangle. \)

Note: We view \( \Delta_0^{-1} \) as an operator on \( L^2(\Omega_1) \) which is zero on the complement of \( L^2(\Omega_0) \).

Proof. Let \( v_1 \in L^2(\Omega_1) \) and let \( v_0 \in L^2(\Omega_0) \) be the restriction of \( v_1 \) to \( \Omega_0 \). Write \( v_0 = \Delta_0 w_0 \) and also \( v_1 = \Delta_1 w_1 \). The required inequality is then

\[ \langle w_0, \Delta_0 w_0 \rangle \leq \langle w_1, \Delta_1 w_1 \rangle, \]

or in other words \( \| \nabla w_0 \|^2 \leq \| \nabla w_1 \|^2. \) \( \square \)
Theorem. If $\Omega$ is any bounded open set then

$$\frac{\text{Area}(\Omega)}{4\pi} \leq \lim \inf_{\lambda \to \infty} \frac{N_\Omega(\lambda)}{\lambda}.$$ 

Proof. Let $I$ be a finite disjoint union of open squares $I_k$, as indicated.

Then $N_I(\lambda) \leq N_\Omega(\lambda)$, by the domain dependence inequality. But since $I$ is a disjoint union,

$$N_I(\lambda) = \sum N_{I_k}(\lambda).$$

Moreover for each square $I_k$ we have

$$\lim_{\lambda \to \infty} \frac{N_{I_k}(\lambda)}{\lambda} = \frac{\text{Area}(I_k)}{4\pi}$$

by direct computation. $\square$
Weyl’s Theorem. If $\Omega$ is a smooth, bounded domain in the plane, then

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{\text{Area}(\Omega)}{4\pi}.$$ 

Proof. Consider both $\Omega$ and $\Omega' = R \setminus \overline{\Omega}$ in a rectangle $R$:

\[
\frac{\text{Area}(\Omega)}{4\pi} + \frac{\text{Area}(\Omega')}{4\pi} \leq \liminf \frac{N_{\Omega}(\lambda)}{\lambda} + \liminf \frac{N_{\Omega'}(\lambda)}{\lambda} \\
\leq \limsup \frac{N_{\Omega}(\lambda)}{\lambda} + \liminf \frac{N_{\Omega'}(\lambda)}{\lambda} \\
\leq \limsup \frac{N_{R}(\lambda)}{\lambda} \\
\leq \frac{\text{Area}(R)}{4\pi}.
\]

$\square$
Trace Class Operators

Lemma.

\[ \mu_n(T_1 + T_2) \leq \mu_n(T_1) + \mu_n(T_2) \leq \mu_{2n}(T_1 + T_2). \]

\[ \mu_n(ST), \mu_n(TS) \leq \|S\| \mu_n(T) \quad \square \]

Definition. The trace ideal in \( \mathcal{B}(H) \) is

\[ \mathcal{L}^1(H) = \{ T | \sum \mu_n(T) < \infty \} \].

From the definition of \( \mu_n(T) \) it follows that if \( \{v_1, \ldots, v_N\} \) is any orthonormal set then

\[ \sum_{n=1}^{N} |\langle v_n, Tv_n \rangle| \leq \sum_{n=1}^{N} \mu_n(T). \]

Definition. If \( T \in \mathcal{L}^1(H) \) then

\[ \text{Trace}(T) = \sum_{j=1}^{\infty} \langle v_j, Tv_j \rangle \]

(the sum is over an orthonormal basis).
Simple algebra (reinforced by the guarantee of absolute convergence of the series) shows that $\text{Trace}(T)$ does not depend on the choice of orthonormal basis, and that

$$S \in \mathcal{B}(H), \ T \in \mathcal{L}^1(H) \ \Rightarrow \ \text{Trace}(ST) = \text{Trace}(TS).$$

**Example.** If $k$ is smooth on $\Omega \times \Omega$ and if

$$Kf(x) = \int_{\Omega} k(x, y)f(y) \, dy,$$

then $K$ is a trace-class operator, and

$$\text{Trace}(K) = \int_{\Omega} k(x, x) \, dx.$$
Zeta Functions

**Theorem.** If \( s > 1 \) then \( \Delta^{-s} \) is a trace-class operator.

This is not difficult; it follows from a refinement of the Rellich Lemma.

**Abelian-Tauberian Theorem.** Let \( T \) be a positive, invertible operator and assume that \( T^{-s} \) is trace-class for all \( s > 1 \). Then

\[
\lim_{s \downarrow 1} \left( (s-1) \text{Trace}(T^{-s}) \right) = C \quad \iff \quad \lim_{\lambda \to \infty} \frac{N_T(\lambda)}{\lambda} = C.
\]

See Hardy, *Divergent Series*. The theorem says

\[
\lim_{s \downarrow 1} \left( (s - 1) \sum_n \lambda_n^{-s} \right) = C \quad \iff \quad \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{\lambda_n \leq \lambda} 1 = C.
\]
Meromorphic Continuation Theorem (Minakshisundaram and Pleijel). Let $\Delta$ be the Laplace operator for a smooth, bounded domain, or for a closed surface. The zeta function

$$\zeta(s) = \text{Trace}(\Delta^{-s})$$

extends to a meromorphic function on $\mathbb{C}$ with only simple poles.

Proof. Uses the formula

$$\Gamma(s) \zeta(s) = \int_0^\infty \text{Trace}(e^{-t\Delta}) t^s \frac{dt}{t}$$

and asymptotics of the heat operators $e^{-t\Delta}$
Analysis of this type produces corrections to Weyl’s formula. For example:

**Theorem (Kac, McKean-Singer).** Let $\Delta$ be the Laplace operator for $\Omega$. Then

$$\text{Trace}(e^{-t\Delta}) = \frac{\text{Area}(\Omega)}{4\pi} t^{-1} - \frac{\text{Length}(\partial\Omega)}{8\sqrt{\pi}} t^{-\frac{1}{2}}$$

$$+ \frac{1}{6} \chi(\Omega) + o(1).$$

Note that while the initial terms are geometric, the constant term is topological.

I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures on what a complete analysis . . . should aim at . . .

Hermann Weyl

*Gibbs Lecture, 1948*
Pseudodifferential Operators

Let us now work on a closed manifold $M$ (the absence of any boundary simplifies things).

Roughly speaking the pseudodifferential operator algebra $\Psi(M)$ is obtained from the algebra of differential operators $\text{Diff}(M)$ by inverting $\Delta^{1/2}$, modulo lower order operators.

$$T \in \Psi_{n}(M) \iff T = D(I + \Delta)^{-\frac{m+n}{2}} + R,$$

where $D \in \text{Diff}_{m}(M)$ and $R$ has very low (even negative) order.

**Theorem (Seeley).** For every $T \in \Psi(M)$ the function $\zeta(s) = \text{Trace}(T(I + \Delta)^{-s})$ is meromorphic, with simple poles.
Analytic Order

Let $\Delta$ be an unbounded, positive, self-adjoint operator on $\mathcal{H}$. The complex powers $(I + \Delta)^s$ may be defined using spectral theory:

- If $\text{Re}(s) \leq 0$ then $(1 + \chi)^s$ is bounded and continuous on $\text{Spec}(\Delta)$ so we can form $(I + \Delta)^s$ directly. This operator is bounded, injective.

- If $\text{Re}(s) \geq 0$ we define $(I + \Delta)^s = ((I + \Delta)^{-s})^{-1}$. This is unbounded, and self-adjoint for $s$ real.

**Definition.** For $s \geq 0$ let $\mathcal{H}^s = \text{Domain}((I + \Delta)^{\frac{s}{2}})$ and let $\mathcal{H}^\infty = \cap_{s \geq 0} \mathcal{H}^s$.

**Definition.** Let $k \in \mathbb{R}$. An operator $T: \mathcal{H}^\infty \to \mathcal{H}^\infty$ has $\Delta$-order $-k$ or less if for every $s \geq 0$ it extends to a bounded operator $T: \mathcal{H}^s \to \mathcal{H}^{s+k}$.

Thus $\text{order}_\Delta(T)$ if and only if

$$\|(I + \Delta)^sTv\| \leq C_s\|(I + \Delta)^{s+k}v\|.$$
The Residue Trace

We shall (eventually) prove the following result:

**Theorem (Wodzicki, Guillemin).** The functional

\[ \tau(T) = \frac{1}{2} \text{Res}_{s=0} \left( \text{Trace}(T(I + \Delta)^{-s}) \right) \]

**is a trace on the algebra** \( \Psi(M) \).

In fact it is the **unique** trace on \( \Psi(M) \) (if \( M \) is connected and \( \dim(M) > 1 \)).

Moreover \( \tau \) is in principle **computable** from the coefficients of \( T \) (i.e. the coefficients of \( D \) in \( T = D(I + \Delta)^{-\frac{m+n}{2}} + R \)). This is because \( \tau \) **vanishes on operators of order** \( < -d \).

In contrast, the ordinary trace is **not computable** in this sense and certainly does not vanish on operators of very small order. (Moreover it is not even defined on operators of order \( > -d \).)
Elliptic Operators

An operator \( T \in \Psi_n(M) \) is \textit{elliptic} if it has an inverse in \( \Psi_{-n}(M) \), modulo operators of very low order:

\[
ST = I + R_1 \quad \text{and} \quad TS = I + R_2,
\]

where \( \text{order}(R_1), \text{order}(R_2) \ll 0 \).

Elliptic operators are \textit{Fredholm}. Roughly speaking, our objective is to solve the following problem:

**Residue Index Problem** Compute \( \text{Index}(T) \) in terms of the residue trace \( \tau \).

**Remark.** In terms of the ordinary trace one has

\[
\text{Index}(T) = \text{Trace}(TS - ST).
\]

The coefficients of \( S \) are computable in terms of those for \( T \). But unfortunately, since \( \text{Trace} \) is not computable in terms of the coefficients of operators, the above formula is not so useful.
We are going to carry out the whole program (pseudodifferential operators, residue trace, formulation of the index problem, solution) in the context not of manifolds and Riemannian geometry but of the ‘noncommutative spaces’ and noncommutative geometry of Alain Connes. The steps:

- Operators and residues in NCG
- Cyclic cohomology
- Formal solution of the index problem
- Tools for actual computation
- Example: transverse index theory
K-Theory and Noncommutative Geometry

Lecture 2
Pseudodifferential Operators

Nigel Higson
Penn State University

July, 2002
References for Today

The first two papers deal with the *residue trace* that we shall construct today.


We shall be looking at the residue trace using tools developed in Appendices A and B of the following paper, already cited in the previous lecture.

Review

\[ \Delta = \nabla^* \nabla \]
\[ H^s = \text{domain}(\Delta^s) \]

**Definition.** Let \( H^\infty = \bigcap_{s \geq 0} H^s \) and \( T: H^\infty \to H^\infty \).

\[ \text{order}_\Delta(T) \leq n \iff T[H^{s+n}] \subseteq H^s, \quad \forall s \geq 0. \]

**Definition.** \( \Psi_n(M, \Delta) = \) operators \( T \) on \( H^\infty \) which for every \( k \in \mathbb{N} \) have the form

\[ T = D(I + \Delta)^{-m} + R, \]

where \( D \) is differential, \( \text{order}(D) \leq m + n \), and \( \text{order}_\Delta(R) \leq -k. \)

**Theorem.** The functional

\[ \tau(T) = \frac{1}{2} \text{Res}_{s=0} \left( \text{Trace}(T(I + \Delta)^{-s}) \right) \]

is a trace on the algebra \( \Psi(M, \Delta) = \bigcup_n \Psi_n(M, \Delta). \)
Abstract Pseudodifferential Operators

Let $p$ be a positive integer. Let $\Delta$ be a positive, self-adjoint operator on $H$. Let $^1 H^s = \text{domain}(\Delta^s)$ and let $H^\infty = \bigcap_{s \geq 0} H^s$.

Let $\mathcal{D}$ be an algebra of operators on $H^\infty$, and assume that $D \in \mathcal{D} \Rightarrow \Delta D, \Delta D \in \mathcal{D}$. Assume that $\mathcal{D}$ is filtered, and write

$$\text{order}_\mathcal{D}(D) \leq n \quad \Leftrightarrow \quad D \in \mathcal{D}_n.\]

We shall say $\Delta$ is of \textit{Laplace type} for $\mathcal{D}$ if

$$\text{order}_\Delta(D) \leq \text{order}_\mathcal{D}(D)$$

$$\text{order}_\mathcal{D}(\Delta D - D\Delta) \leq \text{order}_\mathcal{D}(D) + p - 1.$$

Let $\Psi_n(\mathcal{D}, \Delta)$ be the set of operators $T$ on $H^\infty$ which, for every $k \in \mathbb{N}$, have the form

$$T = D(I + \Delta)^{-m/p} + R,$$

where $D \in \mathcal{D}$, $\text{order}_\mathcal{D}(D) \leq m + n$, $\text{order}_\Delta(R) \leq -k$.

$^1$Think of $\Delta$ as an order $p$ operator. The case $p = 2$ is most common.
Basic Elliptic Estimate

- An order \( n \) differential operator \( D \) gives bounded operators \( D : W^{s+n}(M) \to W^s(M) \).

- \( D \sim \sum_{|\alpha| \leq n} f_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha} \) is **elliptic of order** \( n \) if the function \( \sum_{|\alpha| = n} f_\alpha(x) \xi^\alpha \) is bounded below by a multiple of \( |\xi|^n \).

- The **basic elliptic estimate** asserts that if \( D \) is elliptic of order \( n \) then

\[
||Du||_s + ||u||_0 \geq C||u||_{s+n}
\]

for all smooth \( u \).

- This implies that for \( \Delta = \nabla^* \nabla \) (which is elliptic) the domain of \( \Delta^\frac{s}{2} \) is \( W^s(M) \).
Residue Trace

We shall prove these theorems:

**Theorem.** $\forall (\mathcal{D}, \Delta) = \cup_n \forall (\mathcal{D}, \Delta)$ is an algebra.

Let $\forall^c(\mathcal{D}, \Delta)$ be a subalgebra of $\forall(\mathcal{D}, \Delta)$, closed under $T \mapsto \Delta T - T \Delta$.

**Theorem.** Assume that for all $T \in \forall^c(\mathcal{D}, \Delta)$ the operator $T(I + \Delta)^{-s}$ is trace-class when $s \gg 0$, and that the function

$$\zeta(s) = \text{Trace}(T(I + \Delta)^{-s})$$

extends to a meromorphic function on $\mathbb{C}$ with simple poles. Then the functional

$$\tau(T) = \frac{1}{p} \text{Res}_{s=0}(\text{Trace}(T(I + \Delta)^{-s}))$$

is a trace on $\forall^c(\mathcal{D}, \Delta)$, meaning that $\tau(ST) = \tau(TS)$.

**Remark.** We shall discuss analytic continuation a bit later. Here, for now, we just assume it.
Binomial Expansion

Notation. If $T$ is an operator on $\mathcal{H}^\infty$ then write

$$T^{(1)} = [\Delta, T] \quad \text{and} \quad T^{(k)} = [\Delta, T^{(k-1)}].$$

Note that $\text{order}_D(D^{(k)}) \leq \text{order}_D(D) + k(p - 1)$.

Theorem. Let $D \in \mathcal{D}_n$ and let $s \in \mathbb{C}$. Then for every $k \in \mathbb{N}$,

$$[(I + \Delta)^s, D] = \binom{s}{1} D^{(1)}(I + \Delta)^{s-1} + \binom{s}{2} D^{(2)}(I + \Delta)^{s-2}$$

$$+ \cdots + \binom{s}{k} D^{(k)}(I + \Delta)^{s-k} + R_{k,s},$$

where

- $\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$
- $\text{order}_\Delta(R_{k,s}) \leq n + sp - k - 1$
- $R_{k,s}$ is holomorphic in $s$.

Remark. Holomorphic means holomorphic as a map from half spaces $\text{Re}(s) < \sigma$ into $\mathcal{B}(\mathcal{H}, \mathcal{H}^{1+k-p\sigma-n})$. 
**Proof of the Binomial Theorem.** Write $\Delta_1 = I + \Delta$. The idea is to use Cauchy’s formula:

$$\binom{s}{n} \Delta_1^{s-n} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-n-1} \, d\lambda$$

We’ll also use the standard *resolvent identity*

$$[X^{-1}, Y] = X^{-1}[Y, X]X^{-1}$$

We get

$$[\Delta_1^s, D] = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s [(\lambda - \Delta_1)^{-1}, D] \, d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-1}D^{(1)}(\lambda - \Delta_1)^{-1} \, d\lambda$$

$$= D^{(1)} \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-2} \, d\lambda$$

$$+ \frac{1}{2\pi i} \int_{\Gamma} \lambda^s [(\lambda - \Delta_1)^{-1}, D^{(1)}](\lambda - \Delta_1)^{-1} \, d\lambda$$

$$= \binom{s}{1} D^{(1)} \Delta_1^{s-1} + R_{1,s}.$$
Note that

\[ R_{1,s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-1} D^{(2)} (\lambda - \Delta_1)^{-2} \, d\lambda. \]

The next step in the iteration is

\[
R_{1,s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-1} D^{(2)} (\lambda - \Delta_1)^{-2} \, d\lambda
= D^{(2)} \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-3} \, d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma} \lambda^s [(\lambda - \Delta_1)^{-1}, D^{(2)}] (\lambda - \Delta_1)^{-2} \, d\lambda \\
= \binom{s}{2} D^{(2)} \Delta_1^{s-2} + R_{2,s},
\]

where

\[ R_{2,s} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^s (\lambda - \Delta_1)^{-1} D^{(3)} (\lambda - \Delta_1)^{-3} \, d\lambda. \]

All of these manipulations are valid for \( \text{Re}(s) < 0 \).
To handle the general case, prove the theorem first for integral $s$ (this is algebra) then write

$$\Delta^s_1 = \Delta^N_1 \Delta^{s-N}_1$$

and combine the two cases.

**Remark.** Having proved the binomial theorem for $D \in \mathcal{D}$, exactly the same result now follows for $T \in \Psi(\mathcal{D}, \Delta)$:

$$[(I + \Delta)^s, T] = \binom{s}{1} T^{(1)} (I + \Delta)^{s-1} + \binom{s}{2} T^{(2)} (I + \Delta)^{s-2} + \cdots + \binom{s}{k} T^{(k)} (I + \Delta)^{s-k} + R_{k,s},$$

where

- $\text{order}_\Delta(R_{k,s}) \leq sp + n - k - 1$
- $R_{k,s}$ is holomorphic in $s$.

This follows immediately from the presentation

$$T = D(I + \Delta)^{-\frac{m}{p}} + \text{low } \Delta\text{-order operator.}$$
Proof that $\Psi(\mathcal{D}, \Delta)$ is an algebra. If $T_1, T_2 \in \Psi(\mathcal{D}, \Delta)$ then

$$T_1T_2 = D_1(I + \Delta)^{-s_1}D_2(I + \Delta)^{-s_2} + \text{low order operator}$$

Now expand

$$(I + \Delta)^{-s_2}D_2 = D_2(I + \Delta)^{-s_2} + \left(-s_2 \right)D_2^{(1)}(I + \Delta)^{-s_2-1} + \cdots \quad \square$$

Proof that the residue is a trace. We have

$$\tau(ST) - \tau(TS) = \frac{1}{p} \Res_{s=0} \left( \Trace (T(I + \Delta)^{-s}S) - \Trace (TS(I + \Delta)^{-s}) \right).$$

Expand $(I + \Delta)^{-s}S$ to get

$$\tau(ST) - \tau(TS) = \frac{1}{p} \Res_{s=0} \left( (-s) \Trace (TS^{(1)}(I + \Delta)^{-s-1}) + \cdots \right). \quad \square$$
Meromorphic Continuation

How to show that $\text{Trace}(T\Delta_i^{-s})$ is meromorphic?

*Heat kernel expansion*

*Seeley’s Method*

*Guillemin’s method.*

**Definition.** Let $s \in \mathbb{C}$. Denote by $\Psi_s(\mathcal{D}, \Delta)$ the set of operators $T$ on $H^\infty$ which, for every $k$, may be written

$$T = D(I + \Delta)^{s-m/p} + R,$$

where $D \in \mathcal{D}_m$ and $\text{order}_\Delta(R) \leq -k$.

**Theorem.** $\Psi_{s_1}(\mathcal{D}, \Delta) \cdot \Psi_{s_2}(\mathcal{D}, \Delta) \subseteq \Psi_{s_1+s_2}(\mathcal{D}, \Delta)$. □

**Remark.** Note that $\Psi_s(\mathcal{D}, \Delta) = \Psi_0(\mathcal{D}, \Delta) \cdot (I + \Delta)^{\frac{s}{2}}$.

Fix $d \in \mathbb{R}$ and assume that every $T \in \Psi_s^c(\mathcal{D}, \Delta)$ is trace-class if $\text{Re}(s) < -d$. 
**Guillemin Lemma.** Suppose that for every holomorphic family\(^2\) of operators \(T(s) \in \Psi_s^c(\mathcal{D}, \Delta)\) there are \(U_i \in \Psi_1(\mathcal{D}, \Delta), \; V_i(s) \in \Psi_s^c(\mathcal{D}, \Delta),\) and a holomorphic family of operators \(R(s) \in \Psi_s^c(\mathcal{D}, \Delta)\) such that

\[
(d + s)T(s) = \sum_i [U_i, V_i(s)] + R(s - 1).
\]

Then \(\text{Trace}(T(s))\) is meromorphic, with simple poles.

**Proof.** If \(\text{Re}(s) \ll 0\) then

\[
\text{Trace}((d + s)T(s)) = \text{Trace}(\sum_i [U_i, V_i(s)]) + \text{Trace}(R(s - 1)) = \text{Trace}(R(s - 1))
\]

Hence \(\text{Trace}(T(s)) = (d + s)^{-1}\text{Trace}(R(s - 1))\).

**Remark.** The poles of \(\text{Trace}(T(-s))\) are located at \(d, d - 1, d - 2, \ldots\)

\(^2\)Various definitions of holomorphic are possible. Whichever is used, it should imply that \(\text{Trace}(T(s))\) is holomorphic when \(\text{Re}(s) < -d\).
Continuation, Step By Step

Domain of $\text{Trace}(T(s))$

Domain of $\text{Trace}(R(s - 1))$
The Manifold Case

\[ \Delta = \nabla^* \nabla \]
\[ \mathcal{D} = \text{Diff}(M) \]

**Lemma.** If \( \mathcal{D} \in \text{Diff}_n(M) \), and if \( \mathcal{D} \) is supported in a coordinate chart, then

\[
\sum_{i=1}^{d} [\mathcal{D}, x_i] \frac{\partial}{\partial x_i} = n\mathcal{D} - R,
\]

where \( R \in \text{Diff}_{n-1}(M) \). \qed

**Lemma.** If \( \mathcal{D} \in \text{Diff}_n(M) \), and if \( \mathcal{D} \) is supported in a coordinate chart, then

\[
(d + n)\mathcal{D} = \sum_{i=1}^{d} [\mathcal{D}, x_i] \frac{\partial}{\partial x_i} - \sum_{i=1}^{d} \left[ x_i \mathcal{D}, \frac{\partial}{\partial x_i} \right] + R
\]

where \( R \in \text{Diff}_{n-1}(M) \). \qed
**Theorem.** If \( T_s \in \Psi_s(M, \Delta) \) is holomorphic in \( s \), and if \( T_s \) is supported in a coordinate chart, then

\[
(d + s)T_s = \sum_{i=1}^{d} \left[ T_s, x_i \frac{\partial}{\partial x_i} \right] - \sum_{i=1}^{d} \left[ x_i T_s, \frac{\partial}{\partial x_i} \right] + R_{s-1}
\]

where \( R_s \in \Psi_s(M, \Delta) \) is holomorphic in \( s \).

**Proof.** Apply the binomial theorem:

\[
\sum_{i=1}^{d} \left[ \Delta^s_1, x_i \right] \frac{\partial}{\partial x_i} = \sum_{i=1}^{d} s[\Delta_1, x_i] \Delta^{s-1}_i \frac{\partial}{\partial x_i} + \text{lower terms}
\]

\[
= \sum_{i=1}^{d} s[\Delta_1, x_i] \frac{\partial}{\partial x_i} \Delta^{s-1}_i + \text{lower terms}
\]

\[
= 2s \Delta_1 \Delta^{s-1}_i + \text{lower terms}
\]

\[
= 2s \Delta^s_1 + \text{lower terms}
\]

Now use

\[
T \in \Psi_s(M, \Delta) \quad \Rightarrow \quad \begin{cases} T = D \Delta^{s+n/2} + \text{lower terms} \\ D \in \text{Diff}_n(M). \end{cases}
\]
**Computation of the Residue Trace**

The symbol of \( T \in \Psi_{-d}(M, \Delta) \) is an order \(-d\) homogeneous function on \( T^*M \). Integrating over the unit sphere bundle we obtain the quantity

\[
\tau'(T) = \int_{S^*M} \text{Symbol}(T) \, d\text{vol}.
\]

This is independent of the choice of metric on \( M^d \).

**Theorem (Guillemin, Wodzicki).** On \( \Psi_{-n}(M, \Delta) \) the residue trace \( \tau \) is a constant multiple of \( \tau' \).

**Proof.** Assume \( M^d \) is connected and \( d > 1 \). The de Rham cohomology of \( S^*M \) may be computed from the complex of differential forms which are polynomial in each fiber. If \( \tau'(T) = 0 \) then \( \text{Symbol}(T) \, d\text{vol} \) is exact and so \( \text{Symbol}(T) \) is a sum of partial derivatives. Hence \( \text{Symbol}(T) \) is a sum of Poisson brackets, and so \( T \) is a sum of commutators (up to a lower order operator). Hence \( \tau(T) = 0 \). \( \square \)

**Exercise.** The constant depends only on \( n \).

**Weyl’s Theorem.** \( \tau(D_{-\frac{d}{2}}) = \text{constant} \cdot \text{Vol}(M) \). \( \square \)
We’ll use this in a later lecture:

**Theorem.** If $\Lambda' : H^\infty \to H^\infty$, if $\Lambda'$ is positive, and if $\Lambda - \Lambda' \in \mathcal{D}_1$ then

1. $\Lambda'$ is of Laplace type for $\mathcal{D}$.
2. $\Psi(\mathcal{D}, \Lambda') = \Psi(\mathcal{D}, \Lambda)$.
3. If $T \in \Psi(\mathcal{D}, \Lambda)$ then $\text{Trace}(T(I + \Lambda')^{-s})$ is meromorphic, with simple poles, assuming this is so for all $\text{Trace}(T(I + \Lambda)^{-s})$, and the residue traces are equal.

**Proof.** The key observation is that if $\Lambda = \Lambda' - \Lambda$ then

\[
(I + \Lambda)^s - (I + \Lambda')^s = \int_{\Gamma} \lambda^s (\lambda - \Lambda_1)^{-1} A(\lambda - \Lambda_1)^{-1} d\lambda
\]

\[
- \int_{\Gamma} \lambda^s (\lambda - \Lambda_1)^{-1} A(\lambda - \Lambda_1)^{-1} A(\lambda - \Lambda_1)^{-1} d\lambda
\]

\[
+ \cdots
\]

The trace of each integral may be expanded in zeta functions $\zeta_{\mathcal{D}}(s - j) = \text{Trace}(\mathcal{D} \Lambda_1^{s-j})$. $\square$
Elliptic Operators

**Definition.** An (abstract) pseudodifferential operator \( T \in \Psi_n(\mathcal{D}, \Delta) \) is *elliptic* if there is a pseudodifferential operator \( S \in \Psi_{-n}(\mathcal{D}, \Delta) \) such that

\[
ST = I + R_1 \quad \text{and} \quad TS = I + R_2,
\]

where \( R_1, R_2 \in \Psi_{-1}(\mathcal{D}, \Delta) \).

**Remark.** If \( T \) is elliptic then for every \( k \) we can choose \( S \) so that \( R_1, R_2 \in \Psi_{-k}(\mathcal{D}, \Delta) \).

**Example.** If \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is a regular spectral triple and if \( P \in \mathcal{A} \) is an idempotent then \( PDP \) is an elliptic operator on \( PH \) (belonging to \( \Psi_{1}(P\mathcal{D}P, P\Delta P) \)).

**Example.** In the manifold case, differential operators

\[
D \in \text{Diff}(\mathcal{M}) \subseteq \Psi(\mathcal{M}, \Delta)
\]

which are elliptic in the above sense are elliptic in the standard sense.
Definition of Spectral Triple

**Definition.** A *spectral triple* is a triple \((A, H, D)\) consisting of a separable Hilbert space \(H\), an algebra \(A\) of bounded operators on \(H\), and a (typically unbounded) selfadjoint operator \(D\) on \(H\), for which:

- the operator \(a(I + D^2)^{-1}\) is compact, and

- if \(a \in A\) then the commutator \([D, a] = Da - ad\) is defined on \(\text{domain}(D)\) and extends to a bounded operator on \(H\).

Connes proposes that spectral triples will provide an extension of the notion of Riemannian geometric space which is broadly applicable to problems in fundamental physics, number theory, . . . .

**Our Objective:** Develop index theory for such a ‘noncommutative geometric space’.
The Standard Example

The basic idea behind spectral triples is that we organize operator theory not around $\Delta$ but around a ‘square root’ $D$, so that $D^2 = \Delta$.

The theory of Dirac-type operators in geometry provides examples in the context of Riemannian manifolds.

- $D = d + d^*$  
- $D^2 = \nabla^*\nabla$ on forms
- $D = \text{Dirac Operator}$  
- $D^2 = \nabla^*\nabla$ on spinors

Thus on a closed spin manifold $M$ we can take

- $A = C^\infty(M)$
- $D = \text{Dirac Operator}$ (its self-adjoint extension)
- $H = L^2(M, S)$
A Non-Standard Example

Let

\[
\begin{cases}
G = \text{simply connected nilpotent group} \\
\Gamma = \text{Lattice in } G \\
V = \Lambda^* \text{Lie}(G)
\end{cases}
\]

For example, let \( \Gamma \) be the integer Heisenberg group in the real Heisenberg group. Now set

- \( A = \mathbb{C}[\Gamma] \)
- \( H = \ell^2(\Gamma, V) \)
- \( (Df)(\gamma) = \log(\gamma) \Lambda f(\gamma) + \log(\gamma) \nabla f(\gamma) \)

**Remark.** If \( G = \mathbb{R}^n \) and \( \Gamma = \mathbb{Z}^n \) then this triple is isomorphic (via the Fourier transform) to the de Rham operator triple for the torus dual to \( \mathbb{Z}^n \).
Regularity

**Definition.** A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is *regular* if there is an algebra \(\mathcal{B}\) of bounded operators on \(\mathcal{H}\) such that

- \(\mathcal{A} \subseteq \mathcal{B}\) and \([D, \mathcal{A}] \subseteq \mathcal{B}\).

- For each \(T \in \mathcal{B}\) the commutator \([\Delta^{\frac{1}{2}}, T]\) is defined on \(\text{domain}(\Delta^{\frac{1}{2}})\) and extends to a bounded operator on \(\mathcal{H}\) which is again a member of \(\mathcal{B}\). Here \(\Delta = D^2\).

**Definition.** Let \((\mathcal{A}, \mathcal{H}, D)\) be a spectral triple, and assume that \(\mathcal{A} \cdot \mathcal{H}^\infty \subseteq \mathcal{H}^\infty\). Denote by \(\text{Diff}(\mathcal{A}, D)\) the algebra of operators generated by \(\mathcal{A}\) and \(D\).

**Filtration:** \[
\begin{aligned}
A, [D, A] &\in \text{Diff}_0(\mathcal{A}, D), \quad D \in \text{Diff}_1(\mathcal{A}, D) \\
\text{Diff}_i(\mathcal{A}, D) \cdot \text{Diff}_j(\mathcal{A}, D) &\subseteq \text{Diff}_{i+j}(\mathcal{A}, D) \\
[\Delta, \text{Diff}_k(\mathcal{A}, D)] &\subseteq \text{Diff}_{k+1}(\mathcal{A}, D)
\end{aligned}
\]
Theorem. A spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular if and only if

- each operator $a \in \mathcal{A}$ maps $\mathcal{H}^\infty$ into itself, and
- $\Delta = \mathcal{D}^2$ is of Laplace type for $\text{Diff}(\mathcal{A}, \mathcal{D})$.

Proof. If $\Delta$ is of Laplace type we can form the pseudodifferential operator algebra $\Psi(\mathcal{A}, \mathcal{D})$. Let $\mathcal{B} = \Psi_0(\mathcal{A}, \mathcal{D})$ (as in the definition of regularity).

Conversely, if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is regular then

$$\text{order}_\Delta \leq \text{order}_\mathcal{D}$$

by induction on the value of $\text{order}_\mathcal{D}$. Hence $\Delta$ is of Laplace type. $\square$
Definition. A spectral triple \((A, \mathcal{H}, D)\) is even if the Hilbert space \(\mathcal{H}\) is provided with a \(\mathbb{Z}/2\)-grading, if the algebra \(A\) acts as grading-preserving operators, and if the operator \(D\) is grading-reversing.

\[
\alpha = \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D_1 \\ D_0 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

Lemma. If \(P \in \Psi_0(A, D)\) is an idempotent operator and if \(T \in \Psi(A, D)\) is elliptic of positive order and grading-reversing, then the operator

\[
P_{TP}: P\mathcal{H}_0 \to P\mathcal{H}_1
\]

is Fredholm.

We obtain, for a ring \(R \subseteq \Psi_0(A, D)\), a map

\[
\text{Index}_T: K_0(R) \to \mathbb{Z}.
\]

Index Problem. Compute this, especially in the case \(R = A\) and \(T = D\).
The Dixmier Trace

**Review:** If $T$ is a compact operator and if $\mu_1(T), \mu_2(T), \ldots$ are its singular values then there orthonormal sets $\{v_n\}$ and $\{w_n\}$ in $H$ such that

$$Tv = \sum_n \mu_n(T) \langle v_n, v \rangle w_n$$

$$T^*Tv = \sum_n \mu_n(T)^2 \langle v_n, v \rangle v_n.$$ 

As a result compact operator theory shares much in common with the analysis of sequences.

**Inequalities:**

$$\begin{align*}
&\bullet \quad \mu_n(S + T) \leq \mu_n(S) + \mu_n(T) \\
&\bullet \quad \mu_n(S) + \mu_n(T) \leq \mu_{2n}(S + T) \\
&\bullet \quad \mu_n(ST) \leq \|S\|\mu_n(T) \\
&\bullet \quad \mu_n(TS) \leq \mu_n(T)\|S\| 
\end{align*}$$

**Example.** $\mathcal{L}^1(H) = \{ T \text{ compact} | \sum \mu_n(T) < \infty \}.$
Definition. Denote by $\mathcal{L}^{1,\infty}(H)$ the space of compact operators on $H$ for which

$$\sup_n n \cdot \mu_n(T) < \infty.$$  

Observe that $\mathcal{L}^{1,\infty}(H)$ is an ideal in $B(H)$.

**Abelian-Tauberian Theorem.** Suppose $T$ is compact.

$$\sup_n n \cdot \mu_n(T) = C \iff \lim_{N \to \infty} \frac{1}{\log(N)} \sum_{n \leq N} \mu_n(T) = C \iff \lim_{s \downarrow 1} \frac{1}{s-1} \sum_{n=1}^{\infty} \mu_n(T)^s = C. \quad \Box$$

Definition. If $T \in \mathcal{L}^1(H)$ is positive and $\text{LIM}_\omega$ is a Banach limit then define

$$\text{Tr}_\omega(T) = \text{LIM}_\omega \frac{1}{\log(N)} \sum_{n \leq N} \mu_n(T).$$

**Theorem (Dixmier).** If $\text{LIM}_\omega$ has the property

$$\text{LIM}_\omega(\sigma_1, \sigma_2, \sigma_3, \ldots) = \text{LIM}_\omega(\sigma_1, \sigma_1, \sigma_2, \sigma_2, \ldots)$$

then $\text{Tr}_\omega(T_1 + T_2) = \text{Tr}_\omega(T_1) + \text{Tr}_\omega(T_2). \quad \Box$
Fix a limit $\text{LIM}_\omega$ as in the theorem.

The theorem states that $\text{Tr}_\omega$ is additive on the cone of positive operators in $\mathcal{L}^{1,\infty}(H)$: It therefore extends to a linear functional on $\mathcal{L}^{1,\infty}(H)$. It is automatically a trace on $\mathcal{L}^{1,\infty}(H)$. In fact

$$S \in \mathcal{B}(H), \, T \in \mathcal{L}^{1,\infty}(H) \implies \text{Tr}_\omega(ST) = \text{Tr}_\omega(TS).$$

In general $\text{Tr}_\omega$ depends on the choice of $\text{LIM}_\omega$.

**Theorem (Connes).** Suppose that $\Psi(\mathcal{D}, \Delta)$ is a pseudodifferential operator algebra for which the residue trace is defined. If $d \in \mathbb{N}$ and if the zeta function $\text{Trace}(\Delta_1^{-s})$ is holomorphic for $\text{Re}(s) > d$ then $\Psi_{-d}(\mathcal{D}, \Delta)$ is contained in $\mathcal{L}^{1,\infty}(H)$ and

$$\tau(T) = \text{Tr}_\omega(T),$$

for all $T \in \Psi_{-d}(\mathcal{D}, \Delta)$ and all Dixmier traces $\text{Tr}_\omega$. \( \square \)

One says, $T$ is measurable.
Connes’ Character Formula

**Theorem.** Let \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) be an even regular spectral triple, for which \(\mathcal{D}\) is invertible and \(\Delta^{-k} \in \mathcal{L}^{1,\infty}(\mathcal{H})\). The formula

\[
\phi_\omega(a^0, \ldots, a^{2k}) = \frac{1}{2^k} \text{Tr}_\omega(\varepsilon a^0[D, a^1][D, a^2] \cdots [D, a^{2k}]\Delta^{-k})
\]

defines a Hochschild cocycle on \(\mathcal{A}\). Its value on any Hochschild cycle is the same as that of the cocycle

\[
\phi(a^0, \ldots, a^{2k}) = \frac{1}{2} \text{Trace}(\varepsilon F[F, a^0][F, a^1][F, a^2] \cdots [F, a^{2k}]),
\]

where \(F = D\Delta^{-\frac{1}{2}}\). In particular, \(\phi_\omega\) is independent of \(\omega\).

**Corollary.** Suppose that the Hochschild cocycle \(\phi\) pairs nontrivially with some Hochschild cycle. Then \(\text{Tr}_\omega(\Delta^{-k}) \neq 0\).

The conclusion that \(\text{Tr}_\omega(\Delta^{-k}) \neq 0\) is in effect a lower bound on the eigenvalue growth for the operator \(\Delta\) (albeit an indirect one).
K-Theory
and
Noncommutative Geometry

Lecture 3
Cyclic Cohomology

Nigel Higson
Penn State University
July, 2002
Proposition. If $\tau: A \to \mathbb{C}$ is a trace on an algebra $A$ then the formula

$$\tau_*[P] = \sum_i \tau(P_{ii}),$$

where $P$ is an idempotent matrix over $A$, determines a homomorphism

$$\tau_*: K_0(A) \to \mathbb{C}.$$

Proof. Suppose $UV = P$ and $VU = Q$. Then

$$\tau_*[P] = \sum_i \tau(P_{ii})$$

$$= \sum_{i_1, i_2} \tau(U_{i_1 i_2} V_{i_2 i_1})$$

$$= \sum_{i_1, i_2} \tau(V_{i_2 i_1} U_{i_1 i_2})$$

$$= \sum_i \tau(Q_{ii}) = \tau_*[Q].$$
**Proposition (Connes).** If $\phi$ is a 3-linear functional on $A$ and if

(a) $\phi(a^0, a^1, a^2) = \phi(a^2, a^0, a^1)$, and

(b) $\phi(a^0a^1, a^2, a^3) - \phi(a^0, a^1a^2, a^3) + \phi(a^0, a^1, a^2a^3) - \phi(a^3a^0, a^1, a^2) = 0$,

*then the formula*

$$\phi_*[P] = \sum_{i_1, i_2, i_3} \phi(P_{i_1i_2}, P_{i_2i_3}, P_{i_3i_1})$$

*determines a homomorphism*

$$\phi_* : K_0(A) \to \mathbb{C}.$$ 

**Proof.** Exercise! \qed

**Example.** $A = C^\infty(\Sigma)$ and

$$\phi(f^0, f^1, f^2) = \int_{\Sigma} f^0 df^1 df^2.$$
Characteristic Numbers

Let $\mathcal{M}$ be a smooth manifold, $V \subseteq \mathcal{M}$ an oriented, closed submanifold. Let $P: \mathcal{M} \to \mathcal{M}_{K}(\mathbb{C})$ be a smooth, projection-valued function. Define

$$c_{V}(P) = \int_{V} \text{Trace}(PdPdP \cdots dPdP).$$

**Proposition.** Fixing $V$, the scalar $c_{V}(P)$ only depends on the class $[P] \in K^{0}(\mathcal{M})$.

**Proof.** First, $\text{Trace}(PdPdP \cdots dPdP)$ is a closed form. Second, given a projection-valued function $P: I \times \mathcal{M} \to \mathcal{M}_{n}(\mathbb{C})$, one has (by Stokes’ Theorem)

$$\int_{\partial I \times \mathcal{M}} \text{Trace}(PdPdP \cdots dPdP) = \int_{I \times \mathcal{M}} \text{Trace}(dPdPdP \cdots dPdP) = 0.$$

**Remark.** If $\dim(V)$ is odd then $c_{V}(P) = 0$ (in fact the differential form $\text{Trace}(PdPdP \cdots dP) \equiv 0$).
Noncommutative Generalization

\[ A = \text{any algebra, and } P \in M_K(A), \ P^2 = P. \]

Question. If \( c_V(P) = \int_V \text{Trace}(PdPdP \cdots dPdP) \) then \ldots What is \( V \)? What is \( \int \)? What is \( d \)?

Definition (Connes). An \( n \)-cycle over an algebra \( A \) is a pair \((\Omega, \int)\), where

(a) \( \Omega \) is a differential graded algebra, equipped with an algebra map from \( A \) into \( \Omega^0 \), and

(b) \( \int : \Omega^n \to \mathbb{C} \) is a closed, graded trace on \( \Omega^* \):

\[ \begin{align*}
    \text{(i) } \int \omega_1 \omega_2 &= (-1)^{\deg(\omega_1) \deg(\omega_2)} \int \omega_2 \omega_1, \\
    \text{(ii) } \int d\omega &= 0.
\end{align*} \]

Remark. It is not necessarily true that \( d1 = 0 \), nor that \( 1 \cdot \omega = \omega \), nor that \( \omega_1 \omega_2 = \pm \omega_2 \omega_1 \).

Proposition. If \((\Omega, \int)\) is an \( n \)-cycle then the characteristic number

\[ c(P) = \int \text{Trace}(PdPdP \cdots dPdP) \]

depends only on \([P] \in K_0(A)\). \( \square \)
Cyclic Cocycles

**Proposition.** Let \((\Omega, \int)\) be an \(n\)-cycle for \(\Lambda\). The formula

\[
\varphi(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \cdots da^n
\]

defines a multilinear functional on \(\Lambda\) with the following properties:

- \(\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^1, \ldots, a^n, a^0)\)
- \(b \varphi(a^0, \ldots, a^{n+1}) = 0\), where

\[
b \varphi(a^0, \ldots, a^{n+1}) = \varphi(a^0 a^1, \ldots, a^{n+1}) - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) + \cdots + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n).
\]

**Definition.** Let \(\Lambda\) be an algebra. A *cyclic \(n\)-cocycle* on \(\Lambda\) is an \((n+1)\)-multilinear functional on \(\Lambda\) with the above two properties (*cyclicity, coboundary zero*).
Proof of the Proposition. Cyclicity is proved as follows. First,

\[ \phi(a^0, \ldots, a^n) = \int a^0 da^1 \ldots da^n \]
\[ = \int da^1 \ldots da^n \cdot a^0. \]

Next

\[ da^n \cdot a^0 = -a^n da^0 + d(a^n a^0), \]

and so (elaborating on this observation)

\[ da^1 \ldots da^n \cdot a^0 = (-1)^n a^1 da^2 \ldots da^n + \text{exact form} \]

Finally, to prove \( b\phi = 0 \) use Leibniz's rule:

\[ (a^0 a^1) da^2 \ldots da^{n+1} \]
\[ -a^0 d(a^1 a^2) da^3 \ldots da^{n+1} \]
\[ + \cdots \]
\[ +(-1)^{n+1}(a^{n+1} a^0) da^1 \ldots da^n \]
\[ = (-1)^n (a^{n+1} a^0 da^1 \ldots da^n - a^0 da^1 \ldots da^n \cdot a^{n+1}) \]

\( \square \)
Examples of Cyclic Cocycles

Example. Let $G$ be a group and let $c : G^{n+1} \to \mathbb{C}$ be a group cocycle. Thus:

$$\begin{cases} c(gg_0, \ldots gg_n) = c(g_0, \ldots, g_n) \\ \sum (-1)^jc(g_0, \ldots, g_j, \ldots, g_{n+1}) = 0 \end{cases}$$

The following is a cyclic $n$-cocycle for $\mathbb{C}[G]$:

$$\varphi_c(g_0, \ldots, g_n) = \begin{cases} c(1, g_1, g_1g_2, \ldots) & \text{if } g_0 \cdots g_n = 1 \\ 0 & \text{if } g_0 \cdots g_n \neq 1 \end{cases}$$

Example. Suppose a Lie algebra $g$ acts on $\Lambda$ by derivations and that $\tau$ is an invariant trace on $\Lambda$: $\tau(X(a)) = 0$, $\forall X \in g$. Let $c \in \wedge^n g \subseteq \otimes^n g$ be a (Chevalley-Eilenberg) Lie algebra cycle. If we define

$$\phi_{\chi^1 \wedge \ldots \wedge \chi^n}(a^0, \ldots, a^n) = \tau(a^0\chi^1(a^1)\ldots\chi^n(a^n))$$

then $\phi_c$ is a cyclic $n$-cocycle on $\Lambda$. 
Remark. By ‘cycle’ we mean a cycle in the Chevalley-Eilenberg complex which computes the homology of \( \mathfrak{g} \) with \emph{trivial} coefficients \( \mathbb{C} \). The boundary operator in the complex is

\[
b(X^1 \wedge \cdots \wedge X^n) \\
= \sum_{i<j} (-1)^{i+j} [X^i, X^j] \wedge X^1 \wedge \cdots \wedge \hat{X}^i \wedge \cdots \hat{X}^j \wedge \cdots \wedge X^n.
\]

We embed the exterior powers \( \wedge^n \mathfrak{g} \) into \( \otimes^n \mathfrak{g} \) by total antisymmetrization.

Example. If \( \delta^1 \) and \( \delta^2 \) are commuting derivations on an algebra \( \mathcal{A} \), and if \( \tau \) is an invariant trace, then the formula

\[
\phi(a^0, a^1, a^2) = \tau (a^0 (\delta^1(a^1)\delta^2(a^2) - \delta^2(a^1)\delta^1(a^1)))
\]

is a cyclic 2-cocycle.
Cyclic Cohomology

**Proposition.** *Cyclic* $n$-*cocyles are precisely the functionals associated to $n$-*cycles* $(\Omega_{\text{univ}}, \int)$ on the universal differential graded algebra over $\mathbb{A}$.

**Lemma.** Let $\varphi$ be a cyclic $n$-*linear functional*. Then

- $b\varphi$ is a cyclic $(n + 1)$-*linear functional*, and
- $b^2 \varphi = 0$. □

**Definition.** Let $\mathbb{A}$ be an algebra. The $n$-*th cyclic cohomology group of* $\mathbb{A}$ is

$$\text{HC}^n(\mathbb{A}) = \left\{ \begin{array}{l} \text{cyclic } n\text{-cocycles} \\ \text{modulo cyclic coboundaries.} \end{array} \right\}$$

**Proposition.** The formula

$$\langle \varphi, P \rangle = \sum \varphi(P_{i_0i_1}, P_{i_1i_2}, \ldots, P_{i_ni_0})$$

defines a pairing $\text{HC}^{2n}(\mathbb{A}) \otimes K_0(\mathbb{A}) \to \mathbb{C}$. □
Cyclic Cohomology and Manifolds

For $V^n \subseteq M^d$, oriented, we define

$$\varphi_V(a^0, a^1, \ldots, a^n) = \int_V a^0 da^1 \cdots da^n,$$

where $d$ is the de Rham differential. We obtain maps

- geometric $n$-cycles $\rightarrow$ closed de Rham currents $\rightarrow$ cyclic $n$-cocycles

In fact Connes identified $\mathcal{H}^*(C^\infty(M))$ with de Rham homology (details later). Note however that:

- $b(n$-current$) = (n - 1)$-current
- $b(\text{cyclic } n$-cochain$) = \text{cyclic } (n + 1)$-cocycle
- de Rham currents (not closed) do not determine cyclic cocycles.

So the situation is not altogether straightforward.
Godbillon-Vey Class

Let \( \mathcal{A} = C^\infty(S^1) \ltimes \Gamma \), where \( \Gamma \subseteq \text{Diffeo}^+(S^1) \). Define, for \( a^j = \sum_{g \in \Gamma} a^j_g[g] \in \mathcal{A} \),

\[
\phi(a^0, a^1, a^2) = \sum_{g_0 g_1 g_2 = 1} \int_{S^1} a^0_{g_0} a^1_{g_1} a^2_{g_2} c(g_1, g_2),
\]

where

\[
c(g_1, g_2) = \log(g_2') \, d \log(g_1') - \log(g_1') \, d \log(g_2').
\]

This is Connes’ **Godbillon-Vey cocycle**, a cyclic 2-cocycle on \( \mathcal{A} \).

Suppose now that \( \Gamma = \pi_1(\mathcal{W}) \). Form the manifold \( \mathcal{M} = S^1 \times_{\Gamma} \widetilde{\mathcal{W}} \) and denote by \( T_{\mathcal{W}}\mathcal{M} \) the codimension 1 bundle of tangent vectors to \( \mathcal{M} \) which are tangent to \( \mathcal{W} \). According to Connes, the cocycle \( \phi \) corresponds to the Godbillon-Vey 3-form

\[
\omega = \alpha \wedge d\alpha, \quad \text{kernel}(\alpha) = T_{\mathcal{W}}\mathcal{M}
\]

on \( \mathcal{M} \).
Let $J_2(S^1) = S^1 \times \mathbb{R}^+ \times \mathbb{R}$. This is the bundle of 2-jets of orientation-preserving diffeomorphisms. The group $\text{Diffeo}^+(S^1)$ acts on $J_2(S^1)$ by

$$g: (t, a, b) \mapsto \left( g(t), g'(t)a, g'(t)b + g''(t)\frac{a^2}{2} \right).$$

(The formula comes from the computation

$$g(t + sa + s^2b)$$

$$= g(t) + sg'(t)a + s^2 \left( g'(t)b + g''(t)\frac{a^2}{2} \right) + o(s^2)$$

which proves that we get an action.)

**Lemma.** The differential 3-form $\sigma = -\frac{1}{a^3}dt\,d\alpha\,db$ on $J_2(S^1)$ is $\text{Diffeo}^+(S^1)$-invariant. \qed

**Lemma.** Suppose that $\Gamma \subseteq \text{Diffeo}^+(S^1)$ and that $\pi_1(\mathcal{W}) = \Gamma$. If $\omega = \alpha \wedge d\alpha$ is the Godbillon-Vey class on $S^1 \times_\Gamma \widetilde{\mathcal{W}}$ then the pullback of $\omega$ along the map

$$J_2(S^1) \times_\Gamma \widetilde{\mathcal{W}} \longrightarrow S^1 \times_\Gamma \widetilde{\mathcal{W}}$$

is cohomologous to $\sigma$. \qed
**Hochschild Cohomology**

**Definition.** The *Hochschild cohomology* of $\mathcal{A}$ is the cohomology $\text{HH}^n(\mathcal{A})$ of the complex

\[
\text{Hom}(\mathcal{A}, \mathbb{C}) \xrightarrow{b} \text{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathbb{C}) \xrightarrow{b} \cdots ,
\]

where, as before,

\[
b\varphi(a^0, \ldots, a^{n+1}) = \varphi(a^0 a^1, \ldots, a^{n+1}) - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) + \cdots + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n),
\]

**Example.** If $\mathcal{A} = C^\infty(M^d)$ then the Hochschild cohomology $\text{HH}^n(\mathcal{A})$ is isomorphic to $\Omega_n(M)$, the de Rham currents.\(^1\) Note that if $\nu \in \Omega_n(M)$ is an $n$-current then

\[
\phi_\nu(f^0, f^1, \ldots, f^n) = \int_{\nu} f^0 \, df^1 \cdots df^n
\]

is a Hochschild $n$-cocycle.

\(^1\)To be accurate, here $\text{HH}^*$ should be defined using the continuous multilinear forms on $\mathcal{A}$. 
**Proposition.** The de Rham differential on $\Omega_*(M)$ corresponds to the following operator:

$$
B\phi(a^0, \ldots, a^n) = \sum_{j=0}^{n} (-1)^{nj}\phi(1, a^j, a^{j+1}, \ldots, a^{j-1})
$$

$$
+ \sum_{j=0}^{n} (-1)^{n(j-1)}\phi(a^j, a^{j+1}, \ldots, a^{j-1}, 1).
$$

To be somewhat more accurate, $B\phi_n = n \cdot \phi_{\partial\phi_n}$.

It is therefore reasonable to expect that $B$ will play some role in the description of cyclic cohomology. This expectation is reinforced by the following formula: $B = NB_0$, where

$$
B_0\phi(a^0, \ldots, a^n) = \phi(1, a^0, \ldots, a^n)
$$

$$
- (-1)^{n+1} \phi(a^0, \ldots, a^n, 1)
$$

and

$$
N\psi(a^0, \ldots, a^n) = \sum_{j=0}^{n} (-1)^{nj}\psi(a^j, a^{j+1}, \ldots, a^{j-1}).
$$
Further important properties of $B$

- The image of $B$ is comprised of cyclic cochains.
- $B$ vanishes on cyclic cochains.
- $B^2 = 0$.
- $Bb + bB = 0$.
- $B$ defines a morphism
  
  $$HH^n(A) \overset{B}{\rightarrow} HC^{n-1}(A)$$

  and the composition

  $$HC^n(A) \overset{i}{\rightarrow} HH^n(A) \overset{B}{\rightarrow} HC^{n-1}(A)$$

  is zero. In fact this sequence is exact.
Cycles, Again

**Definition.** An $n$-cycle $X = (\Omega_X, \int_X)$ **bounds** if there exists a pair $W = (\Omega_W, \int_W)$, and a surjection $r: \Omega_W \to \Omega_X$, such that

$$\int_X r[\omega] = \int_W d\omega$$

**Lemma.** If $X = (\Omega_X, \int_X)$ bounds then $c_X(P) = 0$, for all projections $P$. \qed

**Remark.** This gives a natural context for showing that $[P] \mapsto \int_X \text{Trace}(PdPdP \cdots dPdP)$ is well defined (depends only on $[P] \in K_0(A)$).

**Theorem.** A cycle $(\Omega, \int)$ bounds iff its cyclic cohomology class is in the image of the map $B: \text{HH}^{n+1}(A) \to \text{HC}^n(A)$. \qed
**The S-Operator**

**Proposition.** The natural product operation on cycles

\[
(\Omega_{A_1}, \int_{A_1}) \times (\Omega_{A_2}, \int_{A_2}) = (\Omega_{A_1} \hat{\otimes} \Omega_{A_2}, \int_{A_1} \hat{\otimes} \int_{A_2})
\]

induces

\[
HC^{n_1}(A_1) \hat{\otimes} HC^{n_2}(A_2) \to HC^{n_1+n_2}(A_1 \hat{\otimes} A_2)
\]

**Example.** $HC^*(\mathbb{C})$ is a polynomial algebra with degree two generator $\varphi(1, 1, 1) = 1$.

**Definition.** Denote by

\[
S: HC^*(A) \to HC^{*+2}(A)
\]

the map obtained from the product operation

\[
HC^*(\mathbb{C}) \otimes HC^*(A) \to HC^*(A)
\]

and the generator of $HC^2(\mathbb{C})$.

**Proposition.** $\langle \varphi, \chi \rangle = \langle S\varphi, \chi \rangle$, $\forall \chi \in K_0(A)$. 
More Remarks on Cycles

**Theorem.** A cycle bounds iff its cyclic class is in the kernel of $S : \text{HC}^n(A) \to \text{HC}^{n+2}(A)$.

But if a cycle bounds it is in the image of $B : \text{HH}^{n+1}(A) \to \text{HC}^n(A)$. In fact there is an exact sequence

\[ \cdots \to \text{HH}^{n-1}(A) \xrightarrow{B} \text{HC}^n(A) \xrightarrow{S} \text{HC}^{n+2}(A) \xrightarrow{I} \cdots \]

**Example.** For $A = C^\infty(M)$ Connes showed\(^2\) that

\[
\begin{align*}
\text{HH}^n(A) &= \Omega_n(M) \\
\text{HC}^n(A) &= \mathbb{Z}\Omega_n(M) \oplus H_{n-2}(M) \oplus H_{n-4}(M) \oplus \cdots
\end{align*}
\]

with the obvious maps in the above sequence.

**Problem.** Obtain a description of $\text{HC}^n(A)$ in which $B$, $S$, the exact sequence, are as transparent as possible.

\(^2\)As before, one works with *continuous* multilinear maps.
The \((b,B)\)-Bicomplex

\[
\begin{align*}
\vdots & \\
\downarrow b & \downarrow b & \downarrow b & \downarrow b & \\
\text{Hom}(A \otimes A \otimes A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A \otimes A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A, C) \\
\downarrow b & \downarrow b & \\
\text{Hom}(A \otimes A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A, C) \\
\downarrow b & \\
\text{Hom}(A \otimes A, C) & \xrightarrow{B} & \text{Hom}(A, C) \\
\downarrow b & \\
\text{Hom}(A, C) & 
\end{align*}
\]
The (b,B)-Bicomplex, Continued

- The first column, a quotient of the totalized \((b, B)\) bicomplex, is the Hochschild complex.

- The second and higher columns give a subcomplex and a copy of the \((b, B)\)-bicomplex.

- We get the Hochschild-Cyclic long exact sequence from this short exact sequence of complexes.

- The cyclic complex is a subcomplex of the totalized \((b, B)\)-complex, concentrated in the first column.

- **Theorem.** *The inclusion is a quasi-isomorphism.*
The (b,B)-Bicomplex, Continued

A $2n$-cocycle for the $(b, B)$-bicomplex is a family

$$\Phi = (\Phi_0, \Phi_2, \Phi_4, \ldots, \Phi_{2n})$$

such that $b\Phi_{2j-2} + B\Phi_{2j} = 0$ for all $j$.

It is conventional to include cyclic cocycles into the $(b, B)$-bicomplex as follows:

$$\begin{cases} b\Phi_{2k} = 0 \\ \lambda\Phi_{2k} = \Phi_{2k} \end{cases} \sim \begin{cases} (0, \ldots, 0, \Phi_{2k}, 0, \ldots) \\ \Phi_{2k} = (-1)^k \frac{k!}{(2k)!} \Phi_{2k} \end{cases}$$

**Theorem.** The formula

$$\langle \Phi, e \rangle = \sum_{k=0}^{2n} (-1)^k \frac{k!}{(2k)!} \Phi_{2k}(e - \frac{1}{2}, e, e, \ldots, e)$$

defines a pairing $\text{HC}^{2n}(A) \otimes K_0(A) \to \mathbb{C}$ compatible with the previously defined pairing between $K_0(A)$ and cyclic cocycles.
Periodic Cyclic Theory

It is often convenient to periodize $HC$, as follows.

**Definition.** The *periodic cyclic cohomology groups* of $A$ are

$$HP^j(A) = \lim_{S} HC^{j+2k}(A).$$

**Theorem.** Let $A$ be an algebra over $\mathbb{C}$ with a multiplicative unit. The periodic cyclic cohomology of $A$, denoted $HP^*(A)$ is the cohomology of the (direct sum) totalization of the bicomplex

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\ldots & \xrightarrow{b} & \text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{b} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{b} & \text{Hom}(A, \mathbb{C}) \\
\ldots & \xrightarrow{b} & \text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{b} & \text{Hom}(A, \mathbb{C}) \\
\ldots & \xrightarrow{b} & \text{Hom}(A, \mathbb{C}) \\
\end{array}
\]

**Note.** Even cocycles are families $(\Phi_0, \Phi_2, \ldots)$ with $b\Phi_{2j-2} + B\Phi_{2j} = 0$ and with $\Phi_{2j} \equiv 0$ for $j \gg 0$. 

Construction of Cyclic Cocycles, I

**Definition.** Fix an algebra \( L \) over \( \mathbb{C} \). For \( n \geq 0 \) denote by \( \text{Hom}^n(A, L) \) the vector space of \( n \)-linear maps from \( A \) to \( L \). Let \( \text{Hom}^{**}(A, L) \) be the direct product

\[
\text{Hom}^{**}(A, L) = \prod_{n=0}^{\infty} \text{Hom}^n(A, L).
\]

**Definition.** If \( \phi \in \text{Hom}^{**}(A, L) \), \( \psi \in \text{Hom}^{**}(A, L) \), define

\[
\phi \vee \psi(a^1, \ldots, a^n)
= \sum_{p+q=n} \phi(a^1, \ldots, a^p)\psi(a^{p+1}, \ldots, a^n)
\]

\[
b'\phi(a^1, \ldots, a^{n+1})
= \sum_{i=1}^{n} (-1)^{i+1} \phi(a^1, \ldots, \alpha^i a^{i+1}, \ldots, a^{n+1}).
\]

**Lemma (Quillen).** The space \( \text{Hom}^{**}(A, L) \), so equipped, is a \( \mathbb{Z}/2 \)-graded differential algebra. \( \square \)
**Definition.** Suppose that on $L$ there is a trace $\tau: L \to \mathbb{C}$. For $\phi \in \text{Hom}^{**}(A, L)$ define

$$\tau^h(\phi)(a^0, \ldots, a^n) = \sum_{i=0}^{n} (-1)^i \tau(\phi(a^i, a^{i+1}, \ldots, a^{i-1})).$$

**Proposition.** The homogeneous parts of $\tau^h(\phi)$ (as above) are cyclic:

$$\tau^h(\phi)(a^0, \ldots, a^n) = (-1)^n \tau^h(\phi)(a^1, \ldots, a^n, a^0).$$

Moreover

$$b(\tau^h(\phi)) = \tau^h(b'(\phi)) \quad \text{and} \quad \tau^h([\phi, \psi]_-) = 0. \quad \square$$

**Corollary.** If $\phi \in \text{Hom}^{n+1}(A, L)$ and if

$$b'\phi = 0 \quad \text{modulo commutators in} \ \text{Hom}^{**}(A, L),$$

then $\tau^h(\phi)$ is a cyclic $n$-cocycle. \quad \square

**Example.** If $\theta \in \text{Hom}^1(A, L)$ is any element and if

$$K = b'\theta + \theta^2$$

then $\tau^h(K^n)$ is a cyclic $(2n - 1)$-cocycle.
To see this, note that
\[ b'K = b'(b'\theta + \theta^2) = b'(\theta^2) = b'\theta \vee \theta - \theta \vee b'\theta = [K, \theta], \]
and since both \( b' \) and \( ad_\theta \) are derivations,
\[ b'(K^n) = [K^n, \theta], \]
so that \( b'(K^n) \) is a commutator, as required.

**Example.** If \( \theta : A \to B \) is a linear map between algebras, if \( \sigma \) is multiplicative modulo an ideal \( J \) of \( B \), and if \( J^n \) maps to \( L \), we can form \( \tau^b(K^n) \) (slightly stretching the above analysis). We get
\[ K(a^1, a^2) = \theta(a^1 a^2) - \theta(a^1)\theta(a^2) \]
and a cyclic cocycle
\[
\frac{1}{n} \phi(a^0, a^1, \ldots, a^{2n}) \\
= \tau(K(a^1, a^2)K(a^3, a^4) \ldots K(a^{2n-1} a^{2n})) \\
- \tau(K(\alpha^{2n}, a^0)K(a^1, a^2) \ldots K(\alpha^{2n-2} a^{2n-1})).
\]

This is the cyclic cocycle Connes associates to an extension of algebras.
Preview of Next Lecture

Among other things, we shall discuss a method (also due to Quillen) for constructing cocycles in the \((b, B)\)-bicomplex. This is similar to the method just reviewed, but more complicated.

We shall look at the \textit{JLO cocycle}

\[
\Phi_{2n}(a^0, \ldots, a^{2n}) = \int_{\Sigma^{2n}} \text{Trace} \left( \varepsilon a^0 e^{-t_0 \Delta} [D, a^1] e^{-t_0 \Delta} \ldots e^{-t_1 \Delta} [D, a^{2n}] e^{-t_2 \Delta} \right) \, dt
\]

of a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) from this perspective. After that we shall turn to the \textit{residue cocycle}

\[
\Phi_{2n}(a^0, \ldots, a^{2n}) = \sum_{k \geq 0} c_{2n,k} \tau(\varepsilon a_0 [D, a^1]^{(k_1)} \ldots [D, a^{2n}]^{(k_{2n})} \Delta^{-(n+k)})
\]

of Connes and Moscovici.
Today’s References

Background reading:


Today’s main theorem is (essentially) from our basic reference


but the proof is rather different.
Index Pairings

Let $H$ be a $\mathbb{Z}/2$-graded Hilbert space: $H = H_0 \oplus H_1$.

Let $D$ be a self-adjoint operator on $H$, of odd grading-degree:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}. $$

Let $A$ be an algebra of even grading-degree operators on $H$, and suppose that

- each $a \in A$ maps $\text{Domain}(D)$ to $\text{Domain}(D)$, and

- the operators $[D, a] : \text{Domain}(D) \to H$ is compact, as are the operators $a(D \pm iI)^{-1}$.

Example. Let $(A, H, D)$ be a spectral triple.

Example. Let $A = C^\infty_c(M)$ and let $D$ be a self-adjoint extension of an order 2 (or higher) elliptic differential operator on $M$. 
If \( P \in A \) is an idempotent then \( PDP : PH \to PH \) is an (unbounded) Fredholm operator. Write

\[
\text{Index}_{\varepsilon}(PDP) = \begin{pmatrix} \text{Fredholm index of component} \\ \text{of } PDP \text{ mapping } PH_0 \text{ to } PH_1 \end{pmatrix}
\]

If \( A \) is \textit{unital} we obtain an index map

\[
\text{Index}_{\varepsilon,D} : K_0(A) \to \mathbb{Z}.
\]

If \( A \) is \textit{nonunital} we obtain a similar map by considering the Fredholm operators

\[
\begin{pmatrix} PDP & \varepsilon PQ \\ \varepsilon QP & QDQ \end{pmatrix} : PH \oplus QH^{\text{opp}} \to PH \oplus QH^{\text{opp}},
\]

where \( P \) and \( Q \) are idempotent operators in \( \tilde{A} \) whose difference is an operator in \( A \).

**Problem.** Compute these index maps in various examples arising in noncommutative geometry.
Index Cocycles

Definition. Given $A$ and $D$, as above, a cyclic cocycle, or $(b, B)$-cocycle $\Phi$ for the algebra $A$ is an index cocycle for the pair $(A, D)$ if

\[
\langle \Phi, x \rangle = \text{Index}_{\varepsilon, D}(x),
\]

“algebraic index” “analytic index”

for all $x \in K_0(A)$.

Remarks.

– On the left hand side is the pairing between cyclic theory and K-theory.

– Index cocycles have an obvious integrality property.

– Our objective is to construct index cocycles. This falls short of proving index theorems, for which we typically need to identify an index cocycle with something computable and concrete.
Connes’ Cyclic Chern Character

**Definition.** Let $p \geq 1$. The *Schatten $p$-ideal* is

$$\mathcal{L}^p(H) = \left\{ T: H \to H \mid \sum \mu_n(T)^p < \infty \right\}.$$  

One has $\mathcal{L}^1(H) \subseteq \mathcal{L}^p(H) \subseteq \mathcal{K}(H)$.

There is a version of *Holder’s Inequality*:

$$\mathcal{L}^p(H) \cdot \mathcal{L}^q(H) \subseteq \mathcal{L}^r(H), \quad \text{where} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$  

**Definition.** A spectral triple $(\mathbb{A}, H, \mathbb{D})$ is *$p$-summable* if $a(I + \Delta)^{-\frac{1}{2}} \in \mathcal{L}^p(H)$, for all $a \in \mathbb{A}$.

**Lemma.** Assume that $(\mathbb{A}, H, \mathbb{D})$ is $p$-summable, and, for simplicity, that $\mathbb{D}$ is invertible. Let

$$F = D\Delta^{-\frac{1}{2}}.$$  

*Then* $[a, F] \in \mathcal{L}^p(H)$, *for every* $a \in \mathbb{A}$.
Remarks. One has
\[ F^2 = D\Delta^{-\frac{1}{2}}D\Delta^{-\frac{1}{2}} = D^2\Delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} = D^2\Delta^{-1} = I. \]

If \( D\nu = \lambda \nu \) then \( F\nu = \text{sign}(\lambda)\nu \).

**Theorem (Connes).** If \((A, H, D)\) is \(p\)-summable and if \( n \geq p \) then the formula
\[
\phi_n(a^0, \ldots, a^n) = (-1)^{\frac{n}{2}} \text{Trace}(\varepsilon a^0[F, a^1] \cdots [F, a^n])
\]
defines a cyclic \(n\)-cocycle on \( A \).

**Remark.** If \( n \) is odd then \( \phi_n \) is identically zero.

**Remark.** An adaptation deals with the case where \( D \) is *not* invertible.

**Theorem (Connes).** If \( n \geq p \) is even then \( \phi_n \) is an index cocycle for \((A, D)\). Thus
\[
\text{Index} (\text{PDP: } PH_0 \to PH_1) = (-1)^{\frac{n}{2}} \text{Trace} (\varepsilon P[F, P]^n).
\]

**Remark.** Note the similarity with the formula \( \int \text{Trace}(PdP^n) \) for characteristic numbers.
A Small Refinement

**Definition.** Denote by $\text{Trace}'(X)$ the quantity

$$\text{Trace}'(X) = \frac{1}{2} \text{Trace}(\varepsilon F[F, X])$$

**Note.** If $X$ is trace-class then $\text{Trace}'(X) = \text{Trace}(X)$.

**Theorem.** If $(A, H, D)$ is $p$-summable and $n \geq p - 1$ then the formula

$$\phi_n(a^0, \ldots, a^n) = (-1)^{\frac{n}{2}} \text{Trace}'(a^0[F, a^1] \ldots [F, a^n])$$

$$= (-1)^{\frac{n}{2}} \text{Trace}(\varepsilon F[F, a^0][F, a^1] \ldots [F, a^n])$$

defines a cyclic $n$-cocycle on $A$ which is an index cocycle for $(A, D)$.

**Proof when $n = 0$.** The cocycle condition is that $a \mapsto \text{Trace}'(a)$ is a trace on $A$, which is easily checked. The index formula amounts to

$$\text{Trace}(P - Q) = \text{Index}(QP : PH \to QH)$$

when $P, Q$ are idempotents with $P - Q \in L^1(H)$.  □
The Chern Character as a \((b, B)\)-Cocycle

According to our conventions the 2k-th cyclic Chern character determines the \((b, B)\)-cocycle

\[
\Phi_n = \begin{cases} 
(-1)^{n/2} \frac{n!}{n!} \Gamma\left(\frac{n}{2} + 1\right) \phi_n & \text{if } n = 2k \\
0 & \text{if } n \neq 2k
\end{cases}
\]

The following computation helps confirm that our conventions are reasonable.

**Lemma.** The cyclic cohomology class of the \((b, B)\)-cocycle \(\Phi\) is independent of the choice of \(k \geq p/2\).

**Proof.** Let \(\Psi_{n+1}\) be the cochain

\[
(-1)^{n+1} \frac{\Gamma\left(\frac{n}{2} + 2\right)}{(n+2)!} \text{Trace}\left(\varepsilon \alpha^0 F[F, \alpha^1][F, \alpha^2] \cdots [F, \alpha^{n+1}]\right).
\]

Then one checks that \(b\Psi_{n+1} = (-1)^{n+2} \frac{n+2}{2} \Gamma\left(\frac{n+2}{2} + 1\right) \phi_{n+2}\)

while \(B\Psi_{n+1} = -(-1)^{n} \frac{n!}{n!} \Gamma\left(\frac{n}{2} + 1\right) \phi_n\). \(\square\)
The JLO cocycle
(J = Jaffe, L = Lesniewski and O = Osterwalder.)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and assume that

\[ \text{Trace}(e^{-t\Delta}) < \infty, \quad \text{for all } t > 0. \]

This is \textit{theta summability}, a very weak condition.

**Theorem (JLO).** \textit{The formula}

\[ \Phi_n(a^0, \ldots, a^n) = \int_{\Sigma^n} \text{Trace} \left( e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} \ldots e^{-t_{n-1} \Delta} [D, a^n] e^{-t_n \Delta} \right) \, dt \]

\textit{for } n = 0, 2, 4, \ldots \textit{ defines an (improper) index cocycle in the } (b, B)\text{-bicomplex.} \textit{(Here } \Sigma^n \text{ is the standard } n\text{-simplex.)}

\textbf{Note.} \textit{There are convergence problems to be addressed in pairing the cocycle with } K\text{-theory. This is the domain of } entire cyclic cohomology.
Quillen’s Approach to JLO

This is formal (it ignores analysis), but at the same time very suggestive. First, some background:

**Lemma.** *In a Banach algebra,*

\[ e^{a+b} = \sum_{n=0}^{\infty} \int_{\sum} e^{-t_0 a} e^{-t_1 a} b \ldots e^{-t_n a} b \, dt. \]

This follows from Duhamel’s equation

\[ e^{s(a+b)} = e^{sa} + \int_0^s e^{t(a+b)} b e^{(s-t)a} \, dt \]

and an iteration argument.

Is the JLO cocycle (a trace of) an exponential?
In the last lecture, we gave \( \text{Hom}^{**}(\Lambda, \mathbb{L}) \) the structure of a (differential) \( \mathbb{Z}/2 \)-graded algebra:

\[
\phi \vee \psi(a^1, \ldots, a^n) = \sum_{p+q=n} \phi(a^1, \ldots, a^p)\psi(a^{p+1}, \ldots, a^n)
\]

Denote by \( \partial_H(\phi) \) the \( \mathbb{Z}/2 \)-grading-degree of \( \phi \) (e.g. \( \partial_H(\phi) \) is even if the odd-multilinear components of \( \phi \) are zero).

If \( \mathbb{L} \) is \( \mathbb{Z}/2 \)-graded, modify the algebra structure to

\[
\phi \diamond \psi = (-1)^{\partial_H(\phi)\partial_L(\psi)} \phi \vee \psi
\]

and define \( \partial(\phi) = \partial_H(\phi) + \partial_L(\phi) \).

**Proposition.** \( \text{Hom}^{**}(\Lambda, \mathbb{L}) \) is once again a \( \mathbb{Z}/2 \)-graded algebra. \( \square \)

**Remark.** The operator \( d\phi = (-1)^{\partial_L(\phi)} b' \phi \) is a graded derivation.
Back to spectral triples and the JLO cocycle . . .

Let $L$ be the $\mathbb{Z}/2$-graded algebra of ‘operators’ on $H = H_0 \oplus H_1$. Let $\rho: A \to L$ be the given representation of $A$ on $H$.

Define the ‘superconnection form’

$$\theta = D - \rho \in \text{Hom}^{**}(A, L)$$

(of odd-grading-degree) and let $K$ be its ‘curvature’:

$$K = d\theta + \theta^2.$$  

**Lemma.** $K = \Delta - E$, where $E: A \to L$ is defined by

$$E(a) = [D, \rho(a)].$$

**Definition.** Denote by $e^{-K} \in \text{Hom}^{**}(A, L)$ the element

$$e^{-K} = \sum_{n=0}^{\infty} \left( \int_{\Sigma_n} e^{-t_0 \Delta} E e^{-t_1 \Delta} \ldots E e^{-t_n \Delta} \, dt. \right.$$ 

\[ \begin{array}{c}
\text{\textbf{n-linear map}} \\
\text{\textbf{\(a_1, \ldots, a_n \mapsto \int_{\Sigma_n} e^{-t_0 \Delta} [D, a_1] e^{-t_1 \Delta} \ldots [D, a_n] e^{-t_n \Delta} \, dt\)}}
\end{array} \]
Lemma (Bianchi Identity).

\[ d(e^{-K}) + [e^{-K}, \theta] = 0. \]

Lemma (Differential Equation). Suppose that \( \partial \) is a derivation of \( \text{Hom}^{**}(A, L) \) into a bimodule. Then

\[ \partial(e^{-K}) = -\partial(K)e^{-K}, \]

modulo (limits of) commutators.

These follow from the ‘Duhamel formula’

\[ D(e^{-K}) = \int_0^1 e^{-tK}D(K)e^{-(1-t)K} \, dt \]

where \( D \) is any derivation.

Note. We are disregarding analytic details, for now.
Construction of the JLO cocycle

From a (theta summable) spectral triple \((A, H, D)\) we have constructed a family of multilinear maps

\[(a^1, \ldots, a^n) \mapsto \int_{\Sigma^n} e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} [D, a^2] \ldots [D, a^n] e^{-t_n \Delta} \, dt\]

with values in \(L\) (actually the trace-class operators).

Suppose now we compose with the ‘supertrace’

\[\text{Trace}_\varepsilon(X) = \text{Trace}(\varepsilon X),\]

**Theorem (Quillen).** As a result of the Bianchi identity and the differential equation satisfied by \(e^{-K}\), the formula

\[\Phi_{2n}(a^0, \ldots, a^{2n}) = \int_{\Sigma^n} \text{Trace} \left( \varepsilon a^0 e^{-t_0 \Delta} [D, a^1] e^{-t_1 \Delta} [D, a^2] \ldots [D, a^n] e^{-t_n \Delta} \right) \, dt\]

defines a \((b, B)\)-cocycle. \(\square\)
The \((b,B)\)-Bicomplex
The Connes-Tsygan Complex

\[
\begin{array}{ccc}
\text{Hom}(A \otimes A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A \otimes A \otimes A, \mathbb{C}) \\
\downarrow{b} & & \downarrow{b'} \\
\text{Hom}(A \otimes A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A \otimes A, \mathbb{C}) \\
\downarrow{b} & & \downarrow{b'} \\
\text{Hom}(A, \mathbb{C}) & \xrightarrow{N'} & \text{Hom}(A, \mathbb{C})
\end{array}
\]

\[b' \phi(a^0, \ldots, a^{n+1}) = \sum_{i=0}^{n} \phi(a^0, \ldots, a^i a^{i+1}, \ldots, a^{n+1})\]

\[N = 1 + \lambda + \cdots + \lambda^n \quad \text{and} \quad N' = 1 - \lambda.\]

\[bN = Nb' \quad \text{and} \quad b'N' = N'b.\]
Remark. The two complexes are essentially the same. If we define $t$ by

$$t\phi(a^0, \ldots, a^n) = (-1)^n\phi(a^0, \ldots, a^n, 1)$$

then $B = NtN'$.

Quillen uses the Connes-Tsygan complex.

Definition. $\Phi \in \text{Hom}^{**}(A, \mathbb{C})$ and $\phi \in \text{Hom}^{**}(A, \mathbb{C})$ comprise a cyclic pair if

\[\clubsuit \quad b\Phi + N\phi = 0\]

\[\heartsuit \quad b'\phi + N'\Phi = 0\]

Definition. Suppose that the algebra $A$ is unital. A pair of elements $\Phi \in \text{Hom}^{**}(A, \mathbb{C})$ and $\phi \in \text{Hom}^{**}(A, \mathbb{C})$ is normalized if

$$\phi_n(a^1, \ldots, a^n) = \Phi(1, a^1, \ldots, a^n)$$

$$\Phi_n(a^0, \ldots, a^n) = 0 \quad \text{whenever} \quad a^i = 1 \quad \text{for some} \quad i \geq 1.$$
Lemma. If \((\Phi, \phi)\) is a normalized cyclic pair then the families

\[
(\Phi_0, \Phi_2, \Phi_4, \ldots) \quad \text{and} \quad (\Phi_1, \Phi_3, \Phi_5, \ldots)
\]

are respectively even and odd improper cocycles in the periodic \((b, B)\)-complex.

By improper we mean that the families are not necessarily eventually zero (as periodic cocycles should be).

Quillen shows that from \(e^{-k}\) is obtained a cyclic pair. (The proof uses a remarkable identification of the odd rows in the Connes-Tsygan complex as commutator quotients.)

\(\heartsuit\) corresponds to the Bianchi identity.

\(\clubsuit\) corresponds to the differential equation.

The corresponding \((b, B)\)-cocycle is the JLO cocycle.
Residue Cocycle — Preliminaries

We shall work with an *admissible* pseudodifferential operator algebra $\Psi^a(\mathcal{D}, \Delta) \triangleleft \Psi(\mathcal{D}, \Delta)$, as follows:

- $\mathcal{H}$ is $\mathbb{Z}/2$-graded, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, and the grading operator $\varepsilon = \left( \begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right)$ belongs to $\mathcal{D}_0$.

- $\Delta = \mathcal{D}^2$, where $\mathcal{D} = \left( \begin{array}{cc} 0 & \mathcal{D}_+^* \\ \mathcal{D}_- & 0 \end{array} \right)$, and $\mathcal{D} \in \mathcal{D}_r$, $r = \frac{p}{2}$.

- $\mathcal{A} \subseteq \mathcal{D}_0$ is a subalgebra of grading-degree zero operators, with $[\mathcal{A}, \mathcal{D}] \subseteq \mathcal{D}_{r-1}$.

- If $T \in \Psi^a_0(\mathcal{D}, \Delta)$ then $T(I + \Delta)^{-s}$ is trace-class for all $s > \frac{d}{p}$.

- The zeta functions $\zeta(s) = \text{Trace}(T(I + \Delta)^{-s})$ extend to meromorphic functions on $\mathbb{C}$, with only simple poles. Denote by $\tau: \Psi^a(\mathcal{D}, \Delta) \rightarrow \mathbb{C}$ the residue trace.

*Other variations are possible.*
Connes-Moscovici Index Theorem

**Theorem.** Let $\Psi^a(\mathcal{D}, \Delta)$ be an admissible pseudodifferential operator algebra. The formula

$$\Phi_n(a^0, \ldots, a^n) = \sum_{k \geq 0} c_{nk} \tau(\epsilon a_0[D, a^1]^{(k_1)}[D, a^2]^{(k_2)} \ldots [D, a^n]^{(k_n)} \Delta^{-\frac{n+2k}{2}})$$

where $n = 0, 2, \ldots$ and

$$c_{nk} = \frac{(-1)^{|k|}}{k!} \frac{\Gamma(|k| + \frac{n}{2})}{(k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \cdots + k_n + n)}.$$ 

defines an index cocycle for the pair $(\mathcal{D}, \mathcal{A})$ in the $(\mathcal{B}, \mathcal{B})$-bicomplex.
Remarks

- $k = (k_1, \ldots, k_n)$ is a multi-index with nonnegative integer components.

- $\tau(T\Delta^{-\frac{n+2k}{2}})$ is defined since $\Delta^\frac{1}{2}$ is invertible in $\Psi(\mathcal{D}, \Delta)$, modulo very low order operators.

- Note that if $n + k > d$ then the $(n, k)$-contribution to the index formula is identically zero. Thus the sum in the formula is finite, for each $n$, and is identically zero, for $n > d$.

- $c_{00}$ is not well defined by the above formula since the $\Gamma$-function has a pole at $z = 0$. If $\Delta$ is invertible we let

  $$c_{00}\tau(\varepsilon \alpha^0) = \frac{1}{2} \text{Res}_{s=0} \left( \Gamma(s) \text{Trace}(\varepsilon \alpha^0 \Delta^{-s}) \right).$$

- If $\Delta$ is not invertible this definition must be altered.
Classical Case

Let $M^d$ be a (complete) even-dimensional Spin manifold, let $D$ be the Dirac operator, and form the pseudodifferential operator algebra $\Psi(M, \Delta)$. Denote by $\Psi^a_n(M, \Delta)$ the operators $T$ which for every $k$ may be written

$$T = D\Delta^{-\frac{n+m}{2}} + R,$$

where $D$ is a compactly supported, order $m$ differential operator and $R$ is trace-class as an operator from $H^s$ to $H^{s+k}$, for all $s$ (in particular, order$_\Delta(R) < -k$).

**Theorem.** In the classical case, the $k \neq 0$ terms in the Connes-Moscovici formula vanish. Moreover

$$\tau(\varepsilon a_0[D, a^1][D, a^2] \ldots [D, a^n]\Delta^{-\frac{n}{2}})$$

$$= \text{constant} \cdot \int_M a^0 da^1 \ldots da^n \wedge \hat{A}(M).$$

This follows from Getzler’s approach to the Atiyah Singer Theorem.
Typical Case (Complexity Estimate)

In the \textit{simplest} case of interest to Connes and Moscovici one has

- $\Gamma \subseteq \text{Diffeo}^+(\mathbb{R})$
- $A = C_c^\infty(\mathbb{R}^2) \rtimes \Gamma$ (the crossed product algebra),
  $$g(x, t) = (g(x), t + \log(g'(x)))$$
- $D = \begin{pmatrix} e^t \frac{\partial}{\partial x} & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial x^2} & -e^t \frac{\partial}{\partial x} \end{pmatrix}$ (roughly speaking).

A typical generator of $A$ looks like $f \cdot g$, and

$$[D, f \cdot g] = f[D, g] + [D, f]g$$

The terms are of the form $f \cdot g$, or worse, and (by my rough count)

$$[\Delta, f \cdot g] = f[\Delta, g] + [\Delta, f]g$$

Thus $[D, f \cdot g]^{(1)}$ has say 65 terms. The full CM formula has $\gg 500$ terms!
Residue Cocycle: Conceptual Approach

We shall emulate Quillen’s approach to JLO: we shall construct other functions of $K$ — the complex powers.

**Lemma.** If $\lambda \notin \text{Spec}(\Delta)$ then $\lambda - K$ is invertible in $\text{Hom}^{**}(A, L)$.

**Proof.** Since $(\lambda - K) = (\lambda - \Delta + \varepsilon)$ we can write

$$
(\lambda - K)^{-1} = (\lambda - \Delta)^{-1}
$$

0-linear

$$
- (\lambda - \Delta)^{-1}\varepsilon(\lambda - \Delta)^{-1}
$$

1-linear

$$
+ (\lambda - \Delta)^{-1}\varepsilon(\lambda - \Delta)^{-1}\varepsilon(\lambda - \Delta)^{-1}
$$

2-linear

$$
- \cdots
$$

The infinite series has an obvious meaning in $\text{Hom}^{**}(A, L)$.

□
For simplicity let us now assume $\Delta$ is invertible.

**Definition.** For any complex $s$ with positive real part define $K^{-s} \in \text{Hom}^{**}(A, L)$ by

$$K^{-s} = \frac{1}{2\pi i} \int_{\ell} \lambda^{-s}(\lambda - K)^{-1} \, d\lambda,$$

where $\ell$ is a vertical line between 0 and $\text{Spec}(\Delta)$. Thus the degree $n$ component of $K^{-s}$ is

$$(a^1, \ldots, a^n) \mapsto \frac{(-1)^n}{2\pi i} \int_{\ell} \lambda^{-s}(\lambda - \Delta)^{-1} [D, a^1] \ldots [D, a^n](\lambda - \Delta)^{-1} \, d\lambda$$

The assumption that $\text{Re}(s) > 0$ guarantees convergence of the integral.

If $D$ is a derivation then

$$D(K^{-s}) = \frac{1}{2\pi i} \int_{\ell} \lambda^{-s}(\lambda - K)^{-1} D(K)(\lambda - K)^{-1} \, d\lambda.$$

Therefore …
Lemma (Bianchi Identity). \( d(K^{-s}) + [K^{-s}, \theta] = 0. \)

Lemma (Differential Equation). Suppose that \( \partial \) is a derivation of \( \text{Hom}^**(A, L) \) into a bimodule. Then

\[
\partial(K^{-s}) = -s\partial(K)K^{-s-1},
\]

modulo (limits of) commutators.

It is convenient to work with \( \Gamma(s)K^{-s} \), so that the differential equation becomes

\[
\partial(\Gamma(s)K^{-s}) = -\partial(K)\Gamma(s + 1)K^{-(s+1)}
\]

modulo (limits of commutators). Except for the appearance of \( s + 1 \) in place of \( s \) this is the same as the differential equation for \( e^{-K} \). Meanwhile, the Bianchi identity

\[
d(\Gamma(s)K^{-s}) + [\Gamma(s)K^{-s}, \theta] = 0
\]

still holds.
Following Quillen’s approach to JLO one obtains:

**Theorem.** For $\Re(s) > \frac{n}{2}$ define

$$\phi^s_n(a^0, \ldots, a^n) = \frac{(-1)^n \Gamma(s - \frac{n}{2})}{2\pi i} \text{Trace} \left( \int \lambda^{\frac{n}{2}-s} \varepsilon a^0 (\lambda - \Delta)^{-1} [D, a^1] \ldots [D, a^n] (\lambda - \Delta)^{-1} d\lambda \right)$$

Then $b\phi^s_n + B\phi^s_{n+2} = 0$.

We have *discovered* this theorem using Quillen’s formalism.\(^1\) But having discovered it it is not hard at all to *prove* the theorem *directly*. Getzler and Szenes follow this approach for JLO, and their arguments may be copied here.

---

\(^1\)Actually it is easier here to make Quillen’s approach rigorous than it is for JLO.
Residues

We do not yet have an cohomologically interesting cocycle since $\phi^s$ is not finitely supported (or even more than ‘locally’ defined, thanks to the restriction $\text{Re}(s) > \frac{n}{2}$). However:

**Proposition.** The function $s \mapsto \phi^s_n(a^0, \ldots, a^n)$ is **meromorphic**. Moreover if $n > d$ then this function is **holomorphic near the origin**.

We shall see why this is so in a moment.

**Proposition.** The formula

$$\Phi_n(a^0, \ldots, a^n) = \text{Res}_{s=0} \left( \phi^s_n(a^1, \ldots, a^n) \right)$$

for $n = 0, 2, 4, \ldots$ defines a properly supported $(b, B)$-cocycle.

This is clear: $\text{Res}_{s=0}$ is a linear functional on meromorphic functions, so the cocycle condition is preserved. Moreover there are no poles when $n > d$ so $\Phi_n = 0$ here.
Proposition. The cocycle $\Phi$ may be evaluated as follows:

$$\Phi(a^0, \ldots, a^{2n}) = \sum_k c_{nk} \tau(e[D, a^1]^{(k_1)} \cdots [D, a^n]^{(k_n)} \Delta^{n-k}),$$

where the constants are as in the CM index theorem.

Note. It is not at all easy to prove directly that the formula for $\Phi$ above is a $(b, B)$-cocycle.

Proof. The idea is to move all the terms $(\lambda - \Delta)^{-1}$ in the integral defining $\phi^s$ to the right, using the binomial formula, then take residues. Thus we start from

$$(\lambda - \Delta)^{-h}E = E(\lambda - \Delta)^{-h} + h \cdot E^{(1)}(\lambda - \Delta)^{-(h+1)}$$

$$+ \frac{h(h+1)}{2!} E^{(2)}(\lambda - \Delta)^{-(h+2)} + \cdots$$

$$= \sum_{k \geq 0} \frac{h(h+1) \cdots (h+k-1)}{k!} E^{(k)}(\lambda - \Delta)^{-(h+k)}.$$
Setting $\lambda = 1$ we get

\[(\lambda - \Delta)^{-1}E_1 = \sum_{k_1 \geq 0} C(k_1) E_1^{(k_1)} (\lambda - \Delta)^{-(k_1+1)}\]

where $C(k_1) = \frac{k_1!}{k_1!} = 1$. Next, we get

\[(\lambda - \Delta)^{-1}E_1(\lambda - \Delta)^{-1}E_2 = \sum_{k_1, k_2 \geq 0} C(k_1, k_2) E_1^{(k_1)} E_2^{(k_2)} (\lambda - \Delta)^{-(k_1+k_2+2)}\]

where

\[C(k) = \frac{k_1!(k_1 + 2)(k_1 + 3) \ldots (k_1 + k_2 + 1)}{k_1! \frac{k_2!}{k_2!}} = \frac{(k_1 + k_2 + 2)!}{k_1!k_2!(k_1 + 1)(k_1 + k_2 + 2)}.

Finally

\[(\lambda - \Delta)^{-1}E_1 \ldots (\lambda - \Delta)^{-1}E_n \]

\[= \sum_{k \geq 0} C(k_1, \ldots, k_n) E_1^{(k_1)} \ldots E_n^{(k_n)} (\lambda - \Delta)^{-(k_1+n)}\]

where

\[C(k_1, \ldots, k_n) = \frac{(k_1 + \cdots + k_n + n)!}{k_1! \ldots k_n!(k_1 + 1) \ldots (k_1 + \cdots + k_n + n)}.

30
It follows that
\[
\int \lambda^{\frac{n}{2} - s} a^0 (\lambda - \Delta)^{-1} E_1 \cdots (\lambda - \Delta)^{-1} E_n (\lambda - \Delta)^{-1} \, d\lambda
\]
\[= \sum_{k \geq 0} C(k) E_1^{(k_1)} \cdots E_n^{(k_n)} \int \lambda^{\frac{n}{2} - s} (\lambda - \Delta)^{-(|k| + n + 1)} \, d\lambda\]

Now we can use Cauchy’s integral formula to compute
\[
\frac{1}{2\pi i} \int \lambda^{\frac{n}{2} - s} (\lambda - \Delta)^{-(|k| + n + 1)} \, d\lambda
\]
\[= \frac{(\frac{n}{2} - s)(\frac{n}{2} - s - 1) \cdots (\frac{n}{2} - s - n - |k| + 1)}{(|k| + n)!} \Delta^{-\frac{n}{2} - |k| - s}\]

It therefore follows from the functional equation for \( \Gamma(s) \) that
\[
\frac{(-1)^n \Gamma(s - \frac{n}{2})}{2\pi i} \int \lambda^{\frac{n}{2} - s} (\lambda - \Delta)^{-(|k| + n + 1)} \, d\lambda
\]
\[= (-1)^n \Gamma(s - \frac{n}{2}) \frac{(\frac{n}{2} - s)(\frac{n}{2} - s - 1) \cdots (\frac{n}{2} - s - n - |k| + 1)}{(|k| + n)!} \Delta^{-\frac{n}{2} - |k| - s}\]
\[= (-1)^{|k|} \Gamma(s + \frac{n}{2} + |k|) \frac{1}{(|k| + n)!} \Delta^{-\frac{n}{2} - |k| - s}\]
Putting everything together we get

\[
\frac{(-1)^n}{2\pi i} \Gamma\left(\frac{n}{2} - s\right) \text{Trace} \left( \int \lambda^{\frac{n}{2} - s} a^0(\lambda - \Delta)^{-1} E_1 \cdots E_n(\lambda - \Delta)^{-1} d\lambda \right) \\
= \sum_{k \geq 0} (-1)^{|k|} \Gamma\left(s + \frac{n}{2} + |k|\right) \frac{1}{(|k| + n)!} c(k) \\
\times \text{Trace} \left( \varepsilon a^0 E_1^{(k_1)} \cdots E_n^{(k_n)} \Delta^{-\frac{n}{2} - |k| - s} \right)
\]

and taking residues at \( s = 0 \) we get the result. \( \square \)

**Remark.** The formula for \( \phi^s \) obtained toward the end of the proof shows that \( \phi^s(a^0, \ldots, a^n) \) is meromorphic since in any half plane it is a finite sum of zeta functions (each assumed to be meromorphic), plus a holomorphic `error’ term.
Homotopy Invariance

**Proposition.** If $D_t$ is a smooth homotopy of order one operators, all equal modulo order zero operators, then the residue cocycles associated to the operators $D_t$ are all cohomologous.

*Idea of the Proof.* We first find $\psi^{s,t}$ so that

$$b\psi^{s,t} + B\psi^{s,t} = \frac{d}{dt}\phi^{s,t}.$$ 

We obtain it from the integral

$$\frac{\Gamma(s)}{2\pi i} \int_{\ell} \lambda^{-s}(\lambda - K_t)^{-1} \frac{dD_t}{dt}(\lambda - K_t)^{-1} d\lambda$$

in $\text{Hom}^{**}(A, L)$. Then we integrate $\psi^{s,t}$ from $t = 0$ to $t = 1$ to solve

$$b\psi^{s} + B\psi^{s} = \phi^{s,1} - \phi^{s,0}.$$ 

Then we take residues at $s = 0$. \qed
Theorem. The residue cocycle $\Phi$ of Connes and Moscovici is an index cocycle.

Proof. Given $P \in A$, we want to evaluate the pairing $\langle \Phi, P \rangle$ and obtain $\text{Index}_{\varepsilon,D}(P)$.

Replace $D$ by $PDP + P\perp DP\perp$.

The straight line from $D$ to its replacement is a homotopy to which the previous proposition applies. So we can assume that $P$ and $D$ commute. Now the result is easy.

Remark. The argument requires that we deal with operators which are not invertible (we have not discussed this nuance here).

Remark. The same argument proves that the JLO cocycle is an index cocycle. This is the approach of Getzler and Szenes.
K-Theory
and
Noncommutative Geometry

Lecture 5
Cyclic Cohomology
for
Hopf Algebras

Nigel Higson
Penn State University

July, 2002
References

The main paper is one which was already cited in Lecture 1:


Various papers (available on the ArXiv server) provide improvements and surveys:

A. Connes and Moscovici, math.QA/9904154, math.QA/05013, math.OA/0002125, …

The following elegant paper develops cyclic theory for Hopf algebras from the point of view of Cuntz-Quillen theory:

Overview

Theme of the Lecture: There is a general construction

\[ \xymatrix{ \text{HC}^* (\mathcal{H}) \ar[r]^{\tau^i} & \text{HC}^* (A) } \]

which accounts for many geometric constructions of cyclic cocycles.

The index cocycles of the last lecture (analytic constructions) tend to be exceptions.

Next Lecture: We shall show that the residue cocycle is a Hopf cocycle.

Roughly speaking, to prove an index theorem is to identify an index cocycle with an explicit Hopf cocycle (say at the level of cohomology).
Recall that \( \varphi : A \otimes^{n+1} \rightarrow \mathbb{C} \) is a \textit{cyclic n-cocycle} if

1. \( \varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^n, a^0, \ldots, a^{n-1}) \)

2. \( b \varphi(a^0, \ldots, a^{n+1}) = 0 \), where

\[
\begin{align*}
b \varphi(a^0, \ldots, a^{n+1}) &= \varphi(a^0 a^1, \ldots, a^{n+1}) \\
&\quad - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\
&\quad + \ldots \\
&\quad + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n).
\end{align*}
\]

The formula

\[
\langle \varphi, p \rangle = \varphi(p, p, \ldots, p)
\]

determines a pairing

\[
HC^{2n}(A) \otimes K_0(A) \rightarrow \mathbb{C},
\]

between cyclic cohomology (cyclic cocycles modulo coboundaries) and K-theory.
Examples of Cyclic Cocycles

From Lecture 3 . . .

\[ g \rightarrow \text{Lie algebra} \]

\[ g \otimes A \rightarrow A \quad \text{Action of } g \text{ by derivations} \]

\[ \tau: A \rightarrow \mathbb{C} \quad \text{Invariant trace : } \tau(X(a)) = 0 \]

The homology of \( g \) with coefficients \( \mathbb{C} \) is computed from the ‘Chevalley-Eilenberg’ complex

\[ \begin{align*}
    g & \xleftarrow{\delta} g \wedge g \xleftarrow{\delta} g \wedge g \wedge g \xleftarrow{\delta} \cdots \\
\end{align*} \]

where

\[ \delta(X_1 \wedge \cdots \wedge X_n) = \]

\[ \sum_{i < j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n. \]

Now, embed \( \wedge^n g \) into \( \otimes^n g \) by total antisymmetrization.

\[ X_1 \wedge \cdots \wedge X_n \mapsto \sum_{\sigma} (-1)^{\sigma} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}. \]
**Proposition.** The map from $\otimes^n g$ into $\text{Hom}(A \otimes^n, \mathbb{C})$ defined by the formula

$$\phi_{x_1 \otimes \cdots \otimes x_n}(a^0, \ldots, a^n) = \tau(a^0X_1(a^1) \cdots X_n(a^n)).$$

takes Lie algebra cycles to cyclic cocycles.

**Proof of Cyclicity.** Tricky. From

$$0 = \tau\left( X_n\left( a^0X_1(a^1) \cdots X_{n-1}(a^{n-1})a^n \right) \right)$$

$$= \tau\left( X_n(a^0)X_1(a^1) \cdots X_{n-1}(a^{n-1})a^n \right)$$

$$+ \sum \tau\left( a^0X_1(a^1) \cdots X_n(X_i(a^i)) \cdots X_{n-1}(a^{n-1})a^n \right)$$

$$+ \tau\left( a^0X_1(a^1) \cdots X_{n-1}(a^{n-1})X_n(a^n) \right)$$

we get, for $c \in \wedge^n g$, $\phi_c - \lambda \phi_c = (-1)^{n-1}\psi_{\delta_c}$, where

$$\psi_{\gamma_1 \otimes \cdots \gamma_{n-1}}(a^0, \ldots, a^n) = \psi(a^0\gamma_1(a^1) \cdots \gamma^{n-1}(a^{n-1})a^n).$$

$\square$
Proof that Coboundary is Zero. Easy. One has $b\varphi_c = 0$ for any Lie algebra chain $c$ (not necessarily a cycle).

Example. Let $X_1$ and $X_2$ be derivations on $A$, let $\tau$ be an invariant trace, and let

$$\phi(a^0, a^1, a^2) = \tau(a^0X_1(a^1)X_2(a^2)) - \tau(a^0X_2(a^1)X_1(a^2)).$$

One has

$$\phi(a^0, a^1, a^2) - \phi(a^2, a^1, a^0) = \tau(a^0 Y(a^1) a^2),$$

where

$$Y = [X_1, X_2] = -\delta(X_1 \wedge X_2).$$

So if $X_1$ and $X_2$ are commuting derivations then $\phi$ is a cyclic 2-cocycle.

(The irrational rotation algebra carries such a cyclic 2-cocycle.)
Index Theory

\[ \Psi(\mathcal{D}, \Delta) = \text{Abstract pseudodifferential operators.} \]

Assume the zeta-type functions

\[ \zeta(s) = \text{Trace}(T(I + \Delta)^{-s}) \]

have meromorphic extensions (as in the classical case) and form

\[ \tau(T) = \frac{1}{\text{order}(\Delta)} \text{Res}_{s=0} \left( \text{Trace}(T(I + \Delta)^{-s}) \right). \]

Suppose given \( D \in \mathcal{D}, D^2 = \Delta \) and \( D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \).

If \( A \subseteq \Psi_0(\mathcal{D}, \Delta) \) is comprised of even-order operators commuting with \( D \) modulo lower order terms then we get

\[ \text{Index}_{\varepsilon, \mathcal{D}} : K_0(A) \to \mathbb{Z}. \]
Theorem (Connes and Moscovici). The formula

\[ \Phi_n(a^0, \ldots, a^n) = \sum_{k \geq 0} c_{nk} \tau(\varepsilon a_0[D, a^1]^{(k_1)}[D, a^2]^{(k_2)} \ldots [D, a^n]^{(k_n)} \Delta^{\frac{n+2k}{2}}) \]

is an index cocycle in the \((b, B)\)-bicomplex. \(\square\)

The \((b, B)\)-bicomplex.
Classical Case

\( M^{2m} \) Spin Manifold

D Dirac Operator

\( \tau: \Psi(M, \Delta) \to \mathbb{C} \) Wodzicki Residue

A priori there are many terms in the C-M formula (e.g. 8 for \( \dim(M) = 4 \)). However:

**Theorem.** In the classical case, the \( k \neq 0 \) terms in the Connes-Moscovici formula vanish. \( \square \)

Moreover:

**Theorem.** In the classical case

\[
\tau(\varepsilon a_0[D, a^1][D, a^2] \ldots [D, a^n]\Delta^{-\frac{n}{2}}) = \text{constant} \cdot \int_M a^0 da^1 \ldots da^n \wedge \hat{\Lambda}(M).
\]

\( \square \)

This follows from Getzler’s approach to the Atiyah Singer Theorem.
**Typical Case (Complexity Estimate)**

In the *simplest* case of interest to Connes and Moscovici one has

- \( \Gamma \subseteq \text{Diffeo}^+(\mathbb{R}) \)
- \( A = C_c^\infty(\mathbb{R}^2) \rtimes \Gamma \) (the crossed product algebra),
  \[ g(x, t) = (g(x), t + \log(g'(x))) \]
- \( D = \begin{pmatrix} e^t \frac{\partial}{\partial x} & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial x^2} & -e^t \frac{\partial}{\partial x} \end{pmatrix} \) (roughly speaking).

A typical generator of \( A \) looks like \( f \cdot g \), and

\[
[D, f \cdot g] = f[D, g] + [D, f]g
\]

The terms are of the form \( f \cdot g \), or worse, and (by my rough count)

\[
[D, f \cdot g] = f[D, g] + [D, f]g
\]

Thus \([D, f \cdot g]^{(1)}\) has say 65 terms. The full CM formula has \( \gg 500 \) terms!
We are going to generalize the construction of cyclic cocycles from Lie algebra cycles . . .

**Definition.** A *bi-algebra* is an associative algebra $\mathcal{H}$ with unit, equipped with algebra homomorphisms

\[
\Delta: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \quad \text{(comultiplication)}
\]

and

\[
\varepsilon: \mathcal{H} \rightarrow \mathbb{C} \quad \text{(co-unit)}
\]

such that the following diagrams commute:

- **Co-associativity:**
  \[
  \begin{array}{ccc}
  \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H} \otimes \mathcal{H} \\
  \Delta \downarrow & & \downarrow 1 \otimes \Delta \\
  \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta \otimes 1} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}
  \end{array}
  \]

- **Co-unit Property:**
  \[
  \begin{array}{ccc}
  \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\varepsilon \otimes 1} & \mathcal{H} \\
  \Delta \uparrow & & \uparrow 1 \otimes \varepsilon \\
  \mathcal{H} & \xrightarrow{\Delta} & \mathcal{H} \otimes \mathcal{H}
  \end{array}
  \]
Example. Let $g$ be a Lie algebra and let $\mathcal{H}$ be its enveloping algebra. For $X \in g$ define

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad \text{and} \quad \varepsilon(X) = 0.$$ 

Since for example $\Delta: g \to g \otimes g$ is a Lie algebra map we obtain $\Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$.

Example. Let $G$ be a discrete group and $\mathcal{H} = \mathbb{C}[G]$. Define $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

Example. Let $G$ be a group and let $\mathcal{H} = \mathcal{F}(G)$ be a suitable algebra of functions on $G$. Define

$$\Delta: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \quad \text{and} \quad \varepsilon: \mathcal{H} \to \mathbb{C}$$

by $\Delta(f)(g_1, g_2) = f(g_1g_2)$ and $\varepsilon(f) = f(e)$.

Remark. On a finite group we can take $\mathcal{F} =$ all functions. On an algebraic group we can take $\mathcal{F} =$ regular coordinate functions.

These will combine to form our main examples.
**Actions of Bi-Algebras**

**Sweedler Notation.** Write $\Delta h = \sum h_1 \otimes h_2$.

**Example.** With this notation,

$$(\Delta \otimes 1)(\Delta(h)) = \sum h_{11} \otimes h_{12} \otimes h_2,$$

and by co-associativity,

$$\sum h_{11} \otimes h_{12} \otimes h_2 = \sum h_1 \otimes h_{21} \otimes h_{22}.$$

**Definition.** An *action* of $\mathcal{H}$ on an associative algebra $\mathcal{A}$ is a unital homomorphism $\mathcal{H} \to \text{End}_\mathbb{C}(\mathcal{A})$ for which

- $h(1_\mathcal{A}) = \varepsilon(h)1_\mathcal{A}$, and

- $h(a_1a_2) = \sum h_1(a_1)h_2(a_2)$.

**Example.** If $h \in \mathcal{H}$ is *group-like*, meaning $\Delta(h) = h \otimes h$, then $h$ acts as an automorphism. If $h$ is *primitive*, meaning $\Delta(h) = h \otimes 1 + 1 \otimes h$, then $h$ acts as a derivation.
Example. If $\mathcal{H}$ is an enveloping algebra then from
$\Delta(X) = X \otimes 1 + 1 \otimes X$, for $X \in g$, we get

$$X(ab) = X(a)b + aX(b)$$

Thus actions of $\mathcal{H}$ correspond to actions of $g$ by derivations.

Example. If $\mathcal{H} = \mathbb{C}[G]$ then actions of $\mathcal{H}$ on $A$ correspond to actions of $G$ by algebra automorphisms.

Example. Actions of $\mathcal{F}(G)$ correspond to ‘coactions’. An important instance is $A = B \rtimes G$ and

$$h(a)(g) = h(g)a(g)$$

(think of $A$ as functions $a: G \to B$ with twisted convolution multiplication).

Remark. If $G$ is abelian then

$$\mathbb{C}[\hat{G}] \cong \mathcal{F}(G) \quad \text{(Fourier duality)}.$$ 

Actions of $\mathcal{H} = \mathcal{F}(G)$ correspond to actions of $\hat{G}$.
Construction of Bi-Algebras

\[ G = G_1 \cdot G_2 \]

G finite

Identify \( G/G_2 \) with \( G_1 \) and \( G_1 \backslash G \) with \( G_2 \) to form

\[ \mathcal{H} = \mathcal{F}(G_1) \Join \mathcal{F}(G_2) \quad \mathcal{A} = G_1 \Join \mathcal{F}(G_2). \]

There is a natural algebra homomorphism

\[ \mathcal{H} \to \text{End}_\mathbb{C} (\mathcal{A}). \]

If \( G_2 \) is normal then there is a natural coproduct on \( \mathcal{H} \), assembled from the coproducts on \( \mathcal{F}(G/G_2) \) and \( \mathbb{C}[G_2] \):

\[ \Delta(f \cdot g_2) = \Delta(f) \cdot \Delta(g_2). \]

The action of \( \mathcal{H} \) on \( \mathcal{A} \) is a bi-algebra action.

**Amazing Fact.** There is *always* a coproduct:

\[ \Delta(g_2) = \sum_{g_1} q(g_2 g_1) \Join p(g_1) \cdot g_2. \]

More on this *matched pair construction* next lecture.
Antipodes

Definition. A Hopf algebra is a bi-algebra for which there is a linear map $S : \mathcal{H} \to \mathcal{H}$ such that

$$\sum S(h_1)h_2 = \epsilon(h) = \sum h_1S(h_2),$$

for every $h \in \mathcal{H}$. Terminology: $S = \text{Antipode}$.

Example. For enveloping algebras, $S(X) = -X$.

Example. For group algebras, $S(g) = g^{-1}$.

Example. For $\mathcal{F}(G)$, $S(f)(g) = f(g^{-1})$.

Lemma. The antipode $S$ is unique, supposing it exists at all.

Lemma. The antipode is anti-multiplicative and anti-co-multiplicative:\footnote{1}{If that is a word.}

$$S(hk) = S(k)S(h) \quad \text{and} \quad \Delta(S(h)) = \sum S(h_2) \otimes S(h_1).$$

Warning. It is not true that $S^2 = 1$.\footnote{1}{If that is a word.}
Invariant Traces

To construct cyclic cocycles from Lie algebra cycles we also required a trace . . .

**Definition.** A functional $\tau: A \to \mathbb{C}$ is **invariant** if

$$\tau(h(a)) = \varepsilon(h)\tau(a) \quad \forall a \in A, \forall h \in H.$$

**Example.**

- $h \in H$ group-like $\Rightarrow \tau(h(a)) = \tau(a)$
- $h \in H$ primitive $\Rightarrow \tau(h(a)) = 0$

**Example.** Let $A = \mathbb{C}[G]$. The canonical trace $\tau: \mathbb{C}[G] \to \mathbb{C}$,

$$\tau(f) = f(e),$$

is invariant for the action of $H = F(G)$,

$$h(f)(g) = h(g)f(g).$$
Cyclic Cocycles from Hopf Algebras

We want to construct cyclic cocycles from the correspondence

\[
\left( h^1 \otimes \cdots \otimes h^n \right) \quad \leftrightarrow \quad \tau(a^0 h^1(a^1) \cdots h^n(a^n)).
\]

Element of \( \mathcal{H} \otimes \cdots \otimes \mathcal{H} \)

\( \tau \) (Multi-linear functional on \( \Lambda \))

We have:

\[
1 \otimes h^1 \otimes \cdots \otimes h^n \leftrightarrow \tau(a^0 a^1 h^1(a^2) \cdots h^n(a^{n+1}))
\]

\[
h^1 \otimes \cdots \otimes \Delta h^i \otimes \cdots \otimes h^n \leftrightarrow \tau(a^0 h^1(a^1) \cdots h^i(a^i a^{i+1}) \cdots h^n(a^{n+1}))
\]

\[
h^1 \otimes \cdots \otimes h^n \otimes 1 \leftrightarrow \tau(a^{n+1} a^0 h^1(a^1) \cdots h^n(a^n))
\]

**Conclusion.** Using \( \Delta \) and the unit 1 we can construct a complex from \( \mathcal{H} \), mapping to the Hochschild complex of \( \Lambda \).
Cyclicity

**Problem.** When is $\tau(a^0h^1(a^1) \ldots h^n(a^n))$ cyclic?

Take for example $n = 1$. We want to fill in the blank:

\[
h^1 \leftrightarrow \tau(a^0h^1(a^1))
\]

?? \leftrightarrow \tau(a^1h^1(a^0))

And for $n = 2$,

\[
h^1 \otimes h^2 \leftrightarrow \tau(a^0h^1(a^1)h^2(a^2))
\]

?????? \leftrightarrow \tau(a^2h^1(a^0)h^2(a^1))

**Solution.**

For $n = 1$: $S(h^1)$

For $n = 2$: $\Delta(S(h^1)) \cdot (h^2 \otimes 1)$

General case: $\Delta^{n-1}(S(h^1)) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1)$. 

19
Lemma. If the Hopf algebra $\mathcal{H}$ acts on $A$ and if $\tau$ is invariant then

$$\tau(h(a)b) = \tau(aS(h)(b)),$$

for every $h \in \mathcal{H}$ and $a, b \in A$.

Proof. From $h = \sum h_1 \varepsilon(h_2)$ (co-unit property) we get

$$h(a)b = \sum h_1(a) \varepsilon(h_2)b$$

(co-unit)

$$= \sum h_1(a) h_{21}(S(h_{22})(b))$$

(antipode)

$$= \sum h_{11}(a) h_{12}(S(h_2)(b))$$

(co-associativity)

$$= \sum h_1(aS(h_2)(b))$$

(action)

Taking traces we get

$$\tau(h(a)b) = \sum \tau(h_1(aS(h_2)(b)))$$

$$= \sum \varepsilon(h_1) \tau(aS(h_2)(b))$$

(invariance)

$$= \sum \tau(aS(\varepsilon(h_1)h_2)(b))$$

$$= \tau(aS(h)(b))$$

(co-unit) \qed
Explanation: Cyclicality

We have

\[ H \xrightarrow{\Delta} H \otimes H \xrightarrow{\Delta} H \otimes H \otimes H \xrightarrow{\Delta} \ldots \xrightarrow{\Delta} H \otimes \ldots \otimes H \]

and so \( \Delta^{n-1}(h) \in H^\otimes n \).

From the definition of action,

\[ \Delta^{n-1}(h) \leftrightarrow \tau(a^0 h(a^1 \ldots a^n)) \]

Therefore

\[
\Delta^{n-1}(h^1) \cdot (h^2 \otimes \ldots \otimes h^n \otimes 1) \\
\leftrightarrow \tau(a^0 S(h^1)(h^2(a^1) \ldots h^n(a^{n-1})a^n))
\]

By the lemma and the trace property

\[
\Delta^{n-1}(h^1) \cdot (h^2 \otimes \ldots \otimes h^n \otimes 1) \\
\leftrightarrow \tau(h^1(a^0)h^2(a^1) \ldots h^n(a^{n-1})a^n)) \\
= \tau(a^n h^1(a^0)h^2(a^1) \ldots h^n(a^{n-1})).
\]
Cyclic Cohomology Again

\[ A = \text{Algebra over } \mathbb{C}, \quad C^n = \text{Hom}(A \otimes^{n+1}, \mathbb{C}) \]

Define maps

\[ \delta_i: C^n \rightarrow C^{n+1}, \quad \sigma_i: C^n \rightarrow C^{n-1} \quad \gamma: C^n \rightarrow C^n \]

by the formulas

\[ \delta^i \phi(a^0, \ldots, a^{n+1}) = \phi(a^0, \ldots, a^i a^{i+1}, \ldots, a^{n+1}) \]
\[ \sigma_i \phi(a^0, \ldots, a^{n-1}) = \phi(a^0, \ldots, a^{i-1}, 1, a^i, \ldots, a^{n-1}) \]
\[ \gamma \phi(a^0, \ldots, a^n) = \phi(a^n, a^0, \ldots, a^{n-1}) \]

The \( \delta_i \) and \( \sigma_i \) satisfy face-degeneracy relations, and in addition

\[ \gamma \delta_i = \delta_{i-1} \gamma \quad \tau \delta_0 = \delta_{n+1} \]
\[ \gamma \sigma_j = \sigma_{j-1} \gamma \quad \gamma \sigma_0 = \sigma_n \gamma^2 \]
\[ \gamma^{n+1} = 1 \]

These relations define the \textit{cyclic category}. 
Cyclic Objects

The complexes to compute cyclic cohomology are constructed from $\delta$, $\sigma$ and $\gamma$. For example:

$$\lambda = (-1)^n \gamma$$

$$b = \sum_{i=0}^{n+1} (-1)^i \delta_j, \quad B = \left( \sum_{i=0}^{n-1} \lambda^i \right) (\sigma_n \gamma)(1 - \lambda),$$
on the object $C^n$.

**Definition.** A cyclic object in the category of abelian groups is a functor from the cyclic category to abelian groups. Its cyclic cohomology is (for example) the cohomology of the $(b, B)$-bicomplex constructed as above.

**Definition.** The cyclic cohomology of a Hopf algebra $\mathcal{H}$ for which $S^2 = 1$ is the cyclic cohomology of the cyclic object obtained from the following operators . . .
\[
\begin{align*}
\delta_0(h^1 \otimes \cdots \otimes h^n) &= 1 \otimes h^1 \otimes \cdots \otimes h^n \\
\delta_i(h^1 \otimes \cdots \otimes h^n) &= h^1 \otimes \cdots \otimes \Delta h^i \otimes \cdots \otimes h^n \\
\delta_{n+1}(h^1 \otimes \cdots \otimes h^n) &= h^1 \otimes \cdots \otimes h^n \otimes 1 \\
\sigma_i(h^1 \otimes \cdots \otimes h^n) &= \varepsilon(h^i)h^1 \otimes \cdots \otimes \hat{h}^i \otimes \cdots \otimes h^n \\
\gamma(h^1 \otimes \cdots \otimes h^n) &= \Delta^{n-1}(S(h^1)) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1)
\end{align*}
\]

**Theorem (Connes and Moscovici).** As long as \( S^2 = 1 \) these formulas do indeed define a cyclic object. \( \square \)

**Definition.** Let \( \mathcal{H} \) be a Hopf algebra for which \( S^2 = 1 \). If \( \mathcal{H} \) acts on an algebra \( A \), and if \( \tau \) is an invariant trace on \( A \), then define

\[
\tau^\mathcal{H} : HC^*(\mathcal{H}) \longrightarrow HC^*(A)
\]

by the correspondence

\[
h^1 \otimes \cdots \otimes h^n \mapsto \tau(a^0 h^1(a^1) \cdots h^n(a^n)).
\]
Example

\[ g \quad \text{Lie algebra} \]
\[ g \otimes \mathcal{A} \to \mathcal{A} \quad \text{Action of } g \text{ by derivations} \]
\[ \tau: \mathcal{A} \to \mathbb{C} \quad \text{Invariant trace} \]
\[ \mathcal{H} \quad \text{Enveloping algebra} \]

**Theorem.** The Hopf algebra periodic cyclic cohomology of the enveloping algebra \( \mathcal{H} \) is isomorphic to the Lie algebra homology of \( g \) (with trivial coefficients),

\[ HP^{\text{even/odd}}(\mathcal{H}) = H_{\text{even/odd}}(g, \mathbb{C}), \]

in such a way that the characteristic map

\[ \tau^\sharp: HP^*(\mathcal{H}) \to HC^*(\mathcal{A}), \]

associates to the class of a Lie algebra cycle the cyclic cocycle constructed at the beginning of the lecture.
Sketch of the Proof. We shall use the \((b, \mathcal{B})\)-bicomplex.

Step 1. The inclusion \(\wedge^n g \subseteq \otimes^n g \subseteq \otimes^n \mathcal{H}\) gives

\[
\text{Kernel}(b) = \wedge^n g \oplus \text{Image}(b).
\]

(The definition of \(b\) does not invoke the Lie bracket \([, ,]\). In effect, we can assume \(g\) is abelian.)

Step 2. The operator \(B : \otimes^n \mathcal{H} \to \otimes^{n-1} \mathcal{H}\) maps \(\wedge^n g\) to \(\wedge^{n-1} g\) and coincides with the Chevalley-Eilenberg boundary map. (A direct computation.)

Step 3. The result follows from the first two steps, plus some bookkeeping. \(\square\)

Remark. As Connes and Moscovici observe, the same argument is used to compute \(HC^\ast(C^\infty(M))\).
A First Generalization

Unfortunately, in important examples $S^2 \neq 1$.

**Definition.** A *character* of $\mathcal{H}$ is an algebra homomorphism $\delta: \mathcal{H} \to \mathbb{C}$.

**Definition.** A trace $\tau: A \to \mathbb{C}$ is $\delta$-*invariant* if

$$\tau(h(a)) = \delta(h)\tau(a)$$

for all $h \in \mathcal{H}$ and $a \in A$.

**Lemma.** *If $\tau$ is $\delta$-invariant then*

$$\tau(h(a)b) = \tau(aS_\delta(h)(b)),$$

*where*

$$S_\delta(h) = \sum \delta(h_1)S(h_2).$$

□
**Theorem.** Assume that $S^2_\delta = 1$. The twisted cyclic operator

$$\gamma(h^1 \otimes \cdots \otimes h^n) = \Delta^{n-1}(S_\delta(h^1))(h^2 \otimes \cdots \otimes h^n \otimes \mathbb{1})$$

and the previous face and degeneracy operators constitute a cyclic object. 

**Definition.** Denote by $HC^*_\delta(\mathcal{H})$ the associated cyclic cohomology groups, and by

$$\tau^\mathcal{H} : HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A)$$

the characteristic map associated to a $\delta$-invariant trace.

**Theorem.** Let $\mathcal{H}$ be the enveloping algebra of $\mathfrak{g}$ and let $\delta$ be a character of $\mathcal{H}$. Then $S^2_\delta = 1$ and

$$HP^\text{even/odd}_\delta(\mathcal{H}) = H^\text{even/odd}(\mathfrak{g}, \mathbb{C}_\delta).$$
Ultimate Generalization

It is to replace the trace property by a *modular condition*:

**Definition.** A *modular pair* for a Hopf algebra $\mathcal{H}$ consists of a character $\delta : \mathcal{H} \to \mathbb{C}$ and a group-like element $u \in \mathcal{H}$ such that $\delta(u) = 1$. The pair $(\delta, u)$ is *involutive* if

$$S_\delta^2 = \text{Ad}(u) : \mathcal{H} \to \mathcal{H}.$$ 

The definition is suggested by the conditions

$$\tau(ab) = \tau(bu(a))$$
$$\tau(h(a)) = \delta(h)\tau(a)$$

on a linear functional $\tau : \Lambda \to \mathbb{C}$, which imply

$$\tau(h(a)b) = \tau(aS_\delta(h)(b))$$

as before.
Theorem. The amended formulas

\[ \delta_{n+1}(h^1 \otimes \cdots \otimes h^n) = (h^1 \otimes \cdots \otimes h^n \otimes u) \]

and

\[ \gamma(h^1 \otimes \cdots \otimes h^n) = \Delta^{n-1}(S_\delta(h^1))(h^2 \otimes \cdots \otimes h^n \otimes u) \]

determine a cyclic object.

We obtain a characteristic map

\[ \tau^\delta: HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A) \]

as before.

The present generalization treats the algebra and co-algebra structures of \( \mathcal{H} \) more symmetrically than the previous generalization.

We shall consider examples in the next lecture (time permitting), but in our main examples we shall have \( u = 1 \).


For the connection between Hopf algebras and rooted trees . . . which we have not time to discuss:


Putting Everything Together

The program for today . . .

- **Construct a class of ‘noncommutative spaces’** modelled on, but much broader than, the class of smooth manifolds (included in the new class are orbit spaces and leaf spaces of foliations).

- **Develop pseudodifferential operator theory, the residue trace, index theory, etc.,** for these new spaces.

- **Partially compute the class of the residue cocycle** in cyclic cohomology.

- For manifolds the computation will show that the residue cocycle attached to the signature operator of $M^d$ is a **universal** polynomial in the Pontrjagin classes.

- Compare this with the Patodi, Gilkey, *et al* approach to the Atiyah-Singer index theorem.
\{U_1, \ldots, U_N\} cover of \(M\) by open sets.

The algebra \(C^\infty_c(M)\) may be assembled from the algebras \(C^\infty_c(U)\), as follows. Define

\[
A = \left\{ [f_{ij}] \in \mathcal{M}_N(C^\infty_c(M)) \mid \text{Supp}(f_{ij}) \subseteq U_i \cap U_j \right\}.
\]

**Lemma.** The algebras \(C = C^\infty_c(M)\) and \(A\) are Morita equivalent.

**Proof.** Let \(B \subseteq \mathcal{M}_{N+1}(C)\) be associated, as above, to the cover \(\{M, U_1, \ldots, U_n\}\). There are inclusions \(A \subseteq B\) and \(C \subseteq B\), and commuting diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & B \\
\downarrow & & \downarrow \cong \\
A & \xrightarrow{\iota} & \mathcal{M}_\infty(A)
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
C & \xrightarrow{\iota} & B \\
\downarrow & & \downarrow \cong \\
C & \xrightarrow{\iota} & \mathcal{M}_\infty(C)
\end{array}
\]

where the maps \(\iota\) are the standard inclusions. \(\square\)
It is convenient now to assume the $U_i$ are coordinate charts, and identify them with \emph{disjoint} open sets in $\mathbb{R}^d$. Retain the transition functions between them:

\begin{equation*}
\varphi_j \circ \varphi_i^{-1}
\end{equation*}

We obtain a smooth (étale) groupoid:

- Object space $=$ Open set in $\mathbb{R}^d$.

- Source, range maps are local diffeomorphisms.

\textbf{Generalization.} If $\Gamma$ acts on $M$ by diffeomorphisms then we obtain a groupoid with the same object space but with more morphisms:

\[ U_i \ni p \overset{\varphi_j \circ \gamma \circ \varphi_i^{-1}}{\longrightarrow} q \in U_j. \]
Summary, In Pictures

Manifold

Charts

Groupoid
Groupoid Algebras

Let $\mathcal{G}$ be a smooth, étale groupoid.

The groupoid algebra of $\mathcal{G}$ is the linear space

$$A = C^\infty_c(\mathcal{G})$$

(smooth, compactly supported functions on the space of all morphisms, which is a manifold), with the convolution multiplication

$$f_1 \ast f_2(\gamma) = \sum_{\gamma_1 \circ \gamma_2 = \gamma} f_1(\gamma_1)f_2(\gamma_2).$$

**Example.** If $\mathcal{G}$ is the transformation groupoid $M \times \Gamma$ for a smooth action of a group $\Gamma$ on $M$ then $C^\infty_c(\mathcal{G})$ is the crossed product algebra:

$$C^\infty_c(\mathcal{G}) \cong C^\infty_c(M) \rtimes \Gamma.$$

**Remark.** The groupoid algebra constructed from an action an a system of charts (as on previous page) is Morita equivalent to the crossed product algebra $C^\infty_c(M) \rtimes \Gamma$. 
**Second Generalization**

**Definition.** A *codimension d foliation* of a smooth manifold $V$ is a rank $d$ sub-bundle $F$ of the tangent bundle $TV$ which is locally the kernel of submersions to $\mathbb{R}^d$.

**Lemma.** If $F$ is a codimension $d$ foliation of $V$ and if $p \in V$ then there is a unique, maximal connected $d$-dimensional submanifold $L \hookrightarrow V$ which is tangent to $F$ and which contains $p$. \(\square\)

The submanifolds $L$ are the *leaves* of $F$. They partition $V$, onion-like. They need not be closed.

The leaves of the *Kronecker Foliation*. 

---

7
Holonomy

Let $γ$ be a curve between points $p$ and $q$ in $V$, contained within a leaf $L$.

Let $φ_p$ and $φ_q$ be submersions from $V$ to $\mathbb{R}^d$, defined near $p$ and $q$.

**Definition.** The *holonomy* of $γ$, relative to $φ_p$ and $φ_q$ is the germ of the diffeomorphism $\text{Holonomy}(γ)_{φ_q,φ_p} = φ_{q,p} : \mathbb{R}^d → \mathbb{R}^d$ defined by the following picture.

\[
φ_{q,p} = \text{germ of } (φ_{n,n−1} ∘ ⋯ ∘ φ_{3,2} ∘ φ_{2,1})
\]

This depends only on $φ_p$, $φ_q$, and the leafwise homotopy class of $γ$. 
If $T_1, \ldots, T_N$ is a complete set of transversals for $(V, F)$, each identified with an open set in $\mathbb{R}^d$ by some locally defined submersion from $V$ to $\mathbb{R}^d$, then we obtain a groupoid:

- **Object space** = Disjoint union of the transversals in $\mathbb{R}^d$.

- **Morphisms are holonomies**:

$$T_i \ni p \xrightarrow{\text{Holonomy}(\gamma)_{\varphi_j, \varphi_i}} q \in T_j.$$

This is a smooth, étale groupoid, as before.\(^1\) The groupoid algebra is independent of the choice of $\{T_j\}$, up to Morita equivalence.

\(^1\)Warning: the groupoid may fail to be Hausdorff.
Differential Operator Theory

For simplicity we’ll consider groupoids

\[ G = \mathbb{R}^d \times \Gamma. \]

We want to develop differential operator theory with ‘scalar functions’ \( \mathcal{A} = C_c(G) \).

We require a ‘Laplace operator’ \( \Delta \) (on something like \( H = L^2(\mathbb{R}^d) \)) so that

\[ \text{‘Order’}(\Delta a - a\Delta) < \text{‘Order’}(\Delta), \]

among other things (see Lecture 2 for details).

Unfortunately no such \( \Delta \) presents itself, unless for example \( \Gamma \) preserves a Riemannian structure on \( \mathbb{R}^d \).

To solve this problem we are going to modify the algebra \( \mathcal{A} \) a bit . . .
**Definition.** Denote by

\[ P^d = P(\mathbb{R}^d) \]

the bundle over \( \mathbb{R}^d \) whose fiber at \( x \in \mathbb{R}^d \) is the space of all inner products on \( T_x \mathbb{R}^d \).

**Remark.** Thus

\[ P(\mathbb{R}^d) \cong \mathbb{R}^d \times \text{GL}(n, \mathbb{R})/\text{O}(n). \]

A section of \( P \) is a Riemannian metric on \( \mathbb{R}^d \).

**Remark.** The action of \( \Gamma \) on \( \mathbb{R}^d \) lifts canonically to an action of \( \Gamma \) on \( P(\mathbb{R}^d) \).

**Definition.** An *upper triangular structure* on a manifold \( M \) is a sub-bundle \( E \subseteq TM \), together with metrics on \( E \) and \( TM/E \).

**Proposition.** The space \( P \) has an upper triangular structure (with \( E \) the vertical tangent bundle) which is preserved by the action of \( \text{Diffeo}(\mathbb{R}^d) \). \( \square \)
From now on we shall work with

$$A_{\text{new}} = C_c^\infty(P(\mathbb{R}^d)) \rtimes \Gamma.$$  

(One can similarly form $A_{\text{new}} = C_c^\infty(G)$.) We shall assume that the action of $\Gamma$ is free on $P(\mathbb{R}^d)$, i.e. that the action on $\mathbb{R}^d$ has no nondegenerate fixed points.

Define $B = C_c^\infty(P(\mathbb{R}^d))$ and

$$\mathcal{D}(B) = \text{Compactly supported diff. ops on } P$$

$$\mathcal{D}(A) = \mathcal{D}(B) \rtimes \Gamma$$

These act on, for example the vector space of smooth, compactly supported functions (or differential forms) on $P$.

Filter $\mathcal{D}(A)$ by

$$\text{order}_\mathcal{D}(a) = 0 \quad \text{if } a \in A$$

$$\text{order}_\mathcal{D}(Y) \leq 1 \quad \text{if } Y \text{ is a vertical vector field}$$

$$\text{order}_\mathcal{D}(X) \leq 2 \quad \text{if } X \text{ is a non-vertical vector field}$$
Define an operator (on forms) by splitting $TP(\mathbb{R}^d)$ into vertical and horizontal spaces, and forming

$$D = d_{\text{horiz}} + d^*_{\text{horiz}} + \gamma(d^*_{\text{vert}}d_{\text{vert}} - d_{\text{vert}}d^*_{\text{vert}})$$

($\gamma$ is a grading-type operator). One has, roughly speaking,

$$D^2 = \Delta_{\text{horiz}} + \Delta_{\text{vert}},$$

where:

- $\Delta_{\text{horiz}}$ is elliptic of (usual) order 2 in the horizontal direction, and

- $\Delta_{\text{vert}}$ is elliptic of (usual) order 4 in the vertical direction.

$D$ is an (odd-graded) square root of a positive, hypoelliptic operator. It is a signature-type operator.
**Theorem.** The pair \((\mathcal{D}(A), \Delta)\) satisfies the axioms required to define a pseudodifferential operator algebra.\(^2\)

**Proof.** The key point is that if \(g \in \Gamma\) then

\[
g \cdot D \cdot g^{-1} = D + \text{lower } \mathcal{D}\text{-order operator,}
\]

Everything else is done by copying the proofs of the usual elliptic estimates. \(\square\)

**Theorem.** The zeta functions

\[
\zeta(s) = \text{Trace}(T(I + \Delta)^{-s})
\]

are defined for \(\text{Re}(s) \gg 0\), and extend to meromorphic functions, with only simple poles.

**Proof.** Guillemin’s lemma is applicable. \(\square\)

**Remark.** The horizontal directions count double, so the ‘analytic’ dimension of \(P(\mathbb{R}^d)\) is \(2d + \frac{d(d+1)}{2}\).

\(^2\)In the terminology of Lecture 2, \(\Delta\) is of Laplace type for \(\mathcal{D}(A)\).
Computation of Residues

We want to (partially) compute the terms which appear in the residue index formula.

We shall consider the dimension $d = 1$.\(^3\)

**First Case.** $a^0, \ldots, a^n \in C_c^\infty(P)$ (no $\Gamma$).

- $P$ identifies with the ‘$ax + b$ group’ $G_1$ of orientation preserving affine diffeomorphisms of $\mathbb{R}^1$.

- $D$ belongs to the enveloping algebra $\mathcal{H}(g_1)$ (acting as left-invariant differential operators).

Computing the commutators we get

$$\tau\left(\varepsilon a^0[D, a^1]^{(k_1)} \cdots [D, a^n]^{(k_n)} \Delta^{-\frac{n+2k}{2}}\right)$$

$$= \text{sum of terms } \tau\left(a^0 h^1(a^1) \cdots h^n(a^n) T \Delta^{-\frac{n+2k}{2}}\right),$$

\(^3\)In higher dimensions, one must distinguish between the bundle $P$ and the bundle of oriented frames $J$ (one divides $J$ by $SO(d)$ to get $P$).
where $h^1, \ldots, h^n \in \mathcal{H}(g_1)$ (this Hopf algebra acts on $A$ by differentiation) and $T \in \mathcal{H}(g_1)$.

**Lemma.** The functional

$$a \mapsto \tau(a T \Delta^{-\frac{n+2k}{2}})$$

on $C_c^\infty(P)$ is a multiple of $a \mapsto \int_P a(g) \, dg$ (left-invariant Haar measure).

**Proof.** The functional is continuous and left translation invariant. $\square$

Now, denote by $\text{Tr}: C_c^\infty(P) \to \mathbb{C}$ the Haar integral. It is a trace, of course, but not invariant for $\mathcal{H}(g_1)$. However

$$\text{Tr}(h(a)) = \delta(h) \, \text{Tr}(a),$$

where $\delta: g_1 \to \mathbb{R}$ is the modular character.

**Proposition.** On $B = C_c^\infty(P)$ the residue cocycle is in the image of the character map

$$\text{Tr}^\delta: HP_\delta^*(\mathcal{H}(g_1)) \to HP^*(B).$$

$\square$
Second Case. Assume that there is a Hopf algebra $\mathcal{H}$ which contains $\mathcal{H}(g_1)$ (as a subalgebra\(^4\)) and assume that the action of $\mathcal{H}(g_1)$ on $B = \mathcal{C}_c(P)$ extends to an action of $\mathcal{H}$ on $A = \mathcal{C}_c(P) \rtimes \Gamma$.

Denote by $\text{Tr}: A \to \mathbb{C}$ the trace

$$\text{Tr} \left( \sum f_g[g] \right) = \int \! f_e(p) \, dp.$$ 

Assume that this is $\delta$-invariant, for some extension of the modular character to $\mathcal{H}$.

Arguing as above, we get:

**Proposition.** The residue cocycle on the algebra $A = \mathcal{C}_c(P) \rtimes \Gamma$ is in the image of the character map

$$\text{Tr}_\natural: \mathcal{H}P_\delta^*(\mathcal{H}(g_1)) \to \mathcal{H}P^*(A).$$

\(^4\)The comultiplication $\Delta$ can differ.
Construction of a Hopf Algebra

We shall continue to consider only \( d = 1 \).\(^5\) Write

\[
G = \text{Diffeo}^+ (\mathbb{R}) = G_1 \cdot G_2,
\]

where \( G_1 \) is the \( ax + b \) group of affine diffeomorphisms as before, and

\[
G_2 = \left\{ \text{Diffeomorphisms } \varphi \text{ s.t. } \varphi(0) = 0 \text{ and } \varphi'(0) = 1 \right\}.
\]

One has

\[
P \cong G_1 \cong G/G_2,
\]

so \( G \) acts on \( G_1 \) (this gives the action of \( \text{Diffeo}^+ (\mathbb{R}) \) on \( P \)). Similarly

\[
G_2 \cong G_1 \setminus G,
\]

so \( G \), and in particular \( G_1 \), acts on \( G_2 \) (on the right).

**Summary.** Each of \( G_1, G_2 \) acts on the other.

\(^5\)For the same reason as before.
Define

\[ \mathcal{H}(G_2) = \left\{ \text{Polynomial functions of } \phi''(0), \phi'''(0), \text{etc.} \right\}. \]

This is a Hopf algebra of functions on \( G_2 \).

The Lie algebra \( g_1 \) acts by derivations on \( \mathcal{H}(G_2) \), and so we can form the twisted product,

\[ \mathcal{H} = \mathcal{H}(g_1) \Join \mathcal{H}(G_2). \]

This is an \textit{algebra} (not yet a Hopf algebra).

There is an algebra representation of \( \mathcal{H} \) as \( \mathbb{C} \)-linear operators on \( E = C_c^\infty(G_1) \Join G_2 \),

\[ \mathcal{H} \longrightarrow \text{End}_\mathbb{C}(E) \]

(think of \( G_2 \) as discrete here).

**Theorem.** \textit{There is a Hopf algebra structure on } \( \mathcal{H} \) \textit{for which this is an action, and } \( \text{Tr} \) \textit{is a } \( \delta \)-\textit{invariant trace.} \hfill \Box
Consider the case

\[ G = G_1 G_2, \quad G \text{ finite.} \]

Construct the algebras

\[ \mathcal{H} = G_1 \Join \mathcal{F}(G_2), \]
\[ E = \mathcal{F}(G_2) \Join G_2. \]

One has an algebra representation

\[ \mathcal{H} \longrightarrow \text{End}_\mathbb{C}(E). \]

Here again, there is a Hopf algebra structure for which this is an action.

The dual of \( \mathcal{H} \) identifies with \( E \), and the coproduct on \( \mathcal{H} \) identifies with the product on \( E \).

This is the matched pair construction (discovered by G.I. Kac).
The algebra $\mathcal{A} = \mathcal{A}(G_1) \triangleright \mathcal{A}(G_2)$ acts on the linear space $E = \mathcal{C}_c^\infty(G_1) \triangleright G_2$.

There is a canonical Hopf algebra structure on $\mathcal{H}$ (dual to the algebra structure of $E$) for which this action is a Hopf algebra action.

The action of $\mathcal{H}$ on $A$ extends to the larger algebra $E' = \mathcal{C}_c^\infty(G_1) \triangleright G$ and restricts to any $E'' = \mathcal{C}_c^\infty(G_1) \triangleright \Gamma$.

There is a character map

$$\text{Tr}^\mathcal{H} : \mathcal{H}^* \to \mathcal{H}^*(E'') .$$

But this is not quite what we want, since we’re interested in $A = \mathcal{C}_c(P) \triangleright \Gamma$, not $E'' = \mathcal{C}_c^\infty(G_1)$.

Note that $P = G_1/\text{SO}(d)$ and $A = [E'']^{\text{SO}(d)}$. There is a character map

$$\text{Tr}^\mathcal{H} : \mathcal{H}^*(\mathcal{H}, \text{SO}(d)) \to \mathcal{H}^*(A) .$$

On the left side is the $\text{SO}(d)$ invariant part of $\mathcal{H}^*(\mathcal{H})$ (the cohomology of the $\text{SO}(d)$-invariant subcomplex).
Explicit Structure of the Hopf Algebra

\[ [Y, X] = X. \]

Define \( \delta_n \in \mathcal{F}(G_2) \) by

\[ \delta_n(\varphi) = \left. \frac{d^n}{dx^n} \log \left( \frac{d\varphi^{-1}(x)}{dx} \right) \right|_{x=0}. \]

Then \( \mathcal{H} \) is generated (as an algebra) by \( X, Y \) and \( \delta_n \), subject to the relations

\[ [Y, X] = X \quad [\delta_n, \delta_m] = 0 \]
\[ [Y, \delta_n] = n\delta_n \quad [X, \delta_n] = \delta_{n+1} \]

Moreover one has

\[ \Delta Y = Y \otimes 1 + 1 \otimes Y \]
\[ \Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \]
\[ \Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1 \]
\[ J_k M = \{ k\text{-jets of diffeomorphisms} \} \]

A point in the fiber over \( p \in M \) is an equivalence class of diffeomorphisms from a neighborhood of \( p \) to a neighborhood of \( 0 \in \mathbb{R}^d \) (taking \( p \) to 0).

Two diffeomorphisms are equivalent if their derivatives, up to order \( k \), agree at \( p \).

**Example.** \( J_1 M \) is the bundle of frames of \( TM \).

The group of (germs of) diffeomorphisms of \( \mathbb{R}^d \) fixing 0 acts on \( J_k(M) \) on the left, \( \text{Diffeo}(M) \) acts on the right.

There are principal fibrations

\[
M \leftarrow J_1(M) \leftarrow J_2(M) \leftarrow J_3(M) \leftarrow \cdots
\]

All but the last is a homotopy equivalence.

If we replace \( J_k(M) \) by \( J_k(M)/SO(d) \), then all the maps are homotopy equivalences.
Gelfand-Fuks Theory

**Definition.** Denote by \( \Omega^*_\text{nat}(JM) \) the complex of Diffeo(M)-invariant forms on \( JM \) (take a direct limit over \( k \)).

**Problem.** Compute the cohomology of \( O(d) \)-invariant forms in \( \Omega^*_\text{nat}(JM) \). This cohomology is independent of \( M \) and maps to \( H^*(M) \).

**Solution, Part One.** Denote by \( A^*_0(\mathbb{R}^d) \) the complex which computes the continuous cohomology of the Lie algebra of formal vector fields on \( \mathbb{R}^d \). There is an isomorphism

\[
A^*_0(\mathbb{R}^d) \xrightarrow{\cong} \Omega^*_\text{nat}(JM).
\]

**Solution, Part Two** The DGA

\[
WO(d) = \mathbb{R}[c_1, c_2, \ldots] \otimes \wedge^*[h_1, h_5, h_9, \ldots]
\]

\[
\text{deg}(c_i) = 2i, \quad \text{deg}(h_j) = j, \quad dc_i = 0, \quad dh_{4i+1} = c_{2i+1}
\]

maps to \( A^*_0(\mathbb{R}^d) \). After truncation, the map is a quasi-isomorphism to the \( O(d) \)-invariant complex.
Example

In dimension 1 we get

\[ WO(1) = \langle c_1, h_1 \mid c_1^2 = 0, h_1^2 = 0 \rangle, \]

with \( dh_1 = c_1 \) and \( dc_1 = 0 \). We get

\[
H^p(WO(1)) = \begin{cases} 
\mathbb{R} & \text{if } p = 0, 3 \\
0 & \text{else}
\end{cases}
\]

Now \( J_2(S^1) = S^1 \times \mathbb{R}^\times \times \mathbb{R} \). The group \( \text{Diffeo}(S^1) \) acts on \( J_2(S^1) \) by

\[
g: (t, a, b) \mapsto \left( g(t), g'(t)a, g'(t)b + g''(t)\frac{a^2}{2} \right). \]

**Lemma.** The differential 3-form \( \sigma = \pm \frac{1}{a^3} dt da db \) on \( J_2(S^1) \) is \( O(1) \) and \( \text{Diffeo}(S^1) \)-invariant. \( \square \)

This is the *Godbillon-Vey* form.
Theorem (Connes and Moscovici). There are isomorphisms

\[ \text{HP}_\delta^*(\mathcal{H}(d)) \cong H^{\text{ev/odd}}(A_0^*(\mathbb{R}^d)) \]
\[ \text{HP}_\delta^*(\mathcal{H}(d), \text{SO}(d)) \cong H^{\text{ev/odd}}(A_0^*(\mathbb{R}^d), \text{SO}(d)). \]

Roughly speaking, the theorem can be understood as follows. (We’ll ignore \( \text{SO}(n) \)-invariance.)

- If \( \mathcal{H}(g) \) is the enveloping algebra of a Lie algebra \( g \) then \( \text{HP}_*^*(\mathcal{H}) \cong H_*(g, \mathbb{C}) \) (direct sum of even/odd homology groups).

- Twisted version: If \( \delta: g \to \mathbb{C} \) is a character then \( \text{HP}_\delta^*(\mathcal{H}(g)) \cong H_*(g, \mathbb{C}_\delta) \).

- Poincaré duality. For \( \delta: g \to \wedge^{\text{top}} g \) we get \( H_*(g, \mathbb{C}_\delta) \cong H^*(g, \mathbb{C}) \).

- Hence for this \( \delta \), \( \text{HP}_\delta^*(\mathcal{H}(g)) \cong H^*(g, \mathbb{C}) \).

At this point, we have accounted for ‘half’ of \( \mathcal{H}(d) \) and its cyclic theory.
Let \( \mathcal{U} \) be a unipotent group and let \( \mathfrak{u} \) be its Lie algebra. Let \( \mathcal{H}(\mathcal{U}) \) be the Hopf algebra of polynomial functions on \( \mathcal{U} \). One has

\[ \mathcal{H}(\mathcal{U}) \rightarrow \mathcal{H}(\mathfrak{u})^* \]

by the pairing \( \langle f, D \rangle = (Df)(e) \) (thinking of \( \mathcal{H}(\mathfrak{u}) \) as differential operators on \( \mathcal{U} \)).

The above is a Hopf algebra isomorphism onto the continuous dual of \( \mathcal{H}(\mathfrak{u}) \), denoted \( \mathcal{H}(\mathfrak{u})_{\text{cont}}^* \).

\( \text{HP}^*(\mathcal{H}(\mathfrak{u})_{\text{cont}}^*) \cong H^*(\mathfrak{u}, \mathbb{C}) \) (Lie algebra cohomology).

By a spectral sequence argument, \( \text{HP}_{\delta}^*(\mathcal{H}(\mathfrak{d})) \) is assembled from \( \text{HP}_{\delta}^*(\mathcal{H}(\mathfrak{g_1})) \) and \( \text{HP}^*(\mathcal{H}(\mathfrak{G_2})) \).

Thus \( \text{HP}^*(\mathcal{H}(\mathfrak{d})) \) is assembled from \( H^*(\mathfrak{g_1}, \mathbb{C}) \) and \( H_{\text{cont}}^*(\mathfrak{g_2}, \mathbb{C}) \).

In fact \( \text{HP}_{\delta}^*(\mathcal{H}(\mathfrak{d})) \cong H_{\text{cont}}^*(\mathfrak{g_1} + \mathfrak{g_2}, \mathbb{C}) \).

Thus \( \text{HP}_{\delta}^*(\mathcal{H}(\mathfrak{d})) \) is identified with the continuous cohomology of the Lie algebra of formal vector fields, as required.
Final Comments

We have constructed (quite indirectly) a characteristic homomorphism

\[ \text{Tr}_d^\triangledown: H^*(\text{WSO}(d)) \longrightarrow \text{HP}^*(A) \]

for which the residue cocycle is the image of a distinguished Hopf cocycle (depending only on the dimension \(d\)).

**Problem.** What is this class?

**Partial Solution.** In low degrees it is the \(L\)-polynomial (up to powers of 2). This is because \(H^*(\text{WSO}(d))\) is a polynomial algebra in the Pontrjagin classes in low degrees, so we can appeal to the Atiyah-Singer theorem.

**Full Solution?** This appears to require a new approach, probably organized around a new view of the characteristic map.

\(^6H^*(\text{WSO}(d))\) is a small variation on \(H^*(\text{WO}(d))\).