A coarse Mayer–Vietoris principle

BY NIGEL HIGSON

Department of Mathematics, Pennsylvania State University, University Park,
PA 16802

JOHN ROE

Jesus College, Oxford, OX1 3DW

AND GUOLIANG YU

Department of Mathematics, University of Colorado, Boulder, CO 80309

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Introduction

In [1], [4], and [6] the authors have studied index problems associated with the
‘coarse geometry’ of a metric space, which typically might be a complete noncompact
Riemannian manifold or a group equipped with a word metric. The second author
has introduced a cohomology theory, coarse cohomology, which is functorial on the
category of metric spaces and coarse maps, and which can be computed in many
examples. Associated to such a metric space there is also a $C^*$-algebra generated by
locally compact operators with finite propagation. In this note we will show that for
suitable decompositions of a metric space there are Mayer–Vietoris sequences both
in coarse cohomology and in the $K$-theory of the $C^*$-algebra. As an application we
shall calculate the $K$-theory of the $C^*$-algebra associated to a metric cone. The result
is consistent with the calculation of the coarse cohomology of the cone, and with a
‘coarse’ version of the Baum–Connes conjecture.

1. Mayer–Vietoris sequence in coarse cohomology

In [4], the second author remarked that there is not in general a Mayer–Vietoris
sequence for coarse cohomology. In other words, if $M$ is a proper metric space
(‘proper’ means that closed and bounded sets are compact), and if $A$ and $B$ are closed
subspaces with $M = A \cup B$, then it is not in general true that there is a long exact
sequence

$$\ldots \to HX^q(M) \to HX^q(A) \oplus HX^q(B) \to HX^q(A \cap B) \to HX^{q+1}(M) \to \ldots$$

One can see this simply by taking $M$ to be a two point space, and $A$ and $B$ disjoint
one point subspaces.

Even in ordinary cohomology, though, one does not expect to have a
Mayer–Vietoris sequence for every decomposition of a space; some kind of
excissiveness property is needed, for instance that $A^\circ \cup B^\circ = M$ (compare section 4.6
of [5]). Since in coarse theory definitions involving small open sets get replaced by
definitions involving large bounded neighbourhoods, the following is perhaps not
entirely unexpected.
Definition 1. Let \( M \) be a proper metric space, and let \( A \) and \( B \) be closed subspaces with \( M = A \cup B \). We say that \((A, B)\) is an \( \omega \)-excisive couple, or that \( X = A \cup B \) is an \( \omega \)-excisive decomposition, if for each \( R > 0 \) there is some \( S > 0 \) such that

\[
\text{Pen}(A; R) \cap \text{Pen}(B; R) \subseteq \text{Pen}(A \cap B; S).
\]

(As in [4], \( \text{Pen}(A; R) \) denotes the set of points in \( M \) of distance at most \( R \) from \( A \).)

Example 1. Let \( M = \mathbb{R} \), with \( A = \{ x \in \mathbb{R} : x \geq 0 \} \) and \( B = \{ x \in \mathbb{R} : x \leq 0 \} \). Then \((A, B)\) is an \( \omega \)-excisive couple. More generally let \( N \) be a compact path metric space and let \( \Phi : [0, \infty) \rightarrow [0, \infty) \) be a weight function, tending to infinity, describing a metric on the cone \( CN \) (see paragraph 3·46 in [4]). If \( N = N_1 \cup N_2 \) is a decomposition into closed subspaces, the corresponding decomposition \( C_\Phi N = C_\Phi N_1 \cup C_\Phi N_2 \) is \( \omega \)-excisive.

Example 2. Let \( M \) be the space of Remark 2·70 in [4], that is,

\[ M = \{ (x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y \in (0, 1), \text{ or } x = 0 \text{ and } 0 \leq y \leq 1 \}, \]
equipped with the metric induced from \( \mathbb{R}^2 \). Let

\[ A = \{ (x, y) \in M : y \leq \frac{1}{2} \} \quad \text{and} \quad B = \{ (x, y) \in M : y \geq \frac{1}{2} \}. \]

Then \( A \cap B \) contains just one point, but \( \text{Pen}(A; 1) \cap \text{Pen}(B; 1) = M \), so that this decomposition is not \( \omega \)-excisive.

Lemma 1. The decomposition \( M = A \cup B \) is \( \omega \)-excisive if and only if for each \( R > 0 \), the natural map

\[ A \cap B \rightarrow \text{Pen}(A; R) \cap \text{Pen}(B; R) \]
is a bornotopy-equivalence.

We remind the reader that two coarse maps \( F_1, F_2 : M \rightarrow M' \) are bornotopic if there is a constant \( R > 0 \) such that \( d(F_1(m), F_2(m)) \leq R \), for all \( m \in M \) (the definition of coarse map is given in Section 4). This notion of bornotopy leads to a notion of bornotopy equivalence, just as from homotopy we derive the notion of homotopy equivalence.

Proof. If \((A, B)\) is \( \omega \)-excisive there is an \( S > 0 \) such that

\[ \text{Pen}(A; R) \cap \text{Pen}(B; R) \subseteq \text{Pen}(A \cap B; S). \]

Therefore, \( A \cap B \) is \( \omega \)-dense in \( \text{Pen}(A; R) \cap \text{Pen}(B; R) \), and by Proposition 2·6 of [4] the inclusion is a bornotopy equivalence. Conversely, if the natural map is a bornotopy equivalence then the existence of a bornotopy inverse implies the existence of a suitable \( S > 0 \) as in the definition above.

The main result of this section is as follows.

Theorem 1. Suppose that \( M = A \cup B \) is an \( \omega \)-excisive decomposition. Then there is an exact Mayer–Vietoris sequence in coarse cohomology, of the form

\[ \ldots \rightarrow \text{H}^q(M) \rightarrow \text{H}^q(A) \oplus \text{H}^q(B) \rightarrow \text{H}^q(A \cap B) \rightarrow \text{H}^{q+1}(M) \rightarrow \ldots. \]

The proof of this theorem requires a couple of lemmas. We begin by considering certain inverse limit complexes. For \( n = 0, 1, 2, \ldots \) let

\[ C^*_n = \text{CX}^*(\text{Pen}(A; n)) \oplus \text{CX}^*(\text{Pen}(B; n)). \]
The complexes $C^*_n$ form an inverse sequence under the obvious surjective restriction maps, and we define

$$C^* = \lim C^*_n.$$  

We may define $C^*$ concretely as follows: an element of $C^q$ is a pair $(\phi_A, \phi_B)$ of $\omega$-bounded Borel functions on $M^{q+1}$, such that the restriction of $\phi_A$ to any penumbra $\text{Pen}(A; n)$ is a coarse co-chain, and similarly for $\phi_B$. We also let

$$D^*_n = CX^*(\text{Pen}(A; n) \cap \text{Pen}(B; n)),$$

and let

$$D^* = \lim D^*_n.$$  

It has a similar explicit description.

**Lemma 2.** If $C^*$ and $D^*$ are the complexes defined above for an $\omega$-excisive decomposition $X = A \cup B$, then the natural restriction maps induce isomorphisms

$$H^q(C) \cong HX^q(A) \oplus HX^q(B)$$

and

$$H^q(D) \cong HX^q(A \cap B).$$

**Proof.** By standard results on cohomology and inverse limits [2], there is a short exact sequence

$$0 \to \lim \uparrow H^{q-1}(C_n) \to H^q(C) \to \lim H^q(C) \to 0.$$  

But since the inclusions $A \to \text{Pen}(A; n)$ and $B \to \text{Pen}(B; n)$ are bornotopy equivalences, it follows from Proposition 2.6 of [4] that the cohomology groups $H^q(C_n)$ are all isomorphic by restriction to $HX^q(A) \oplus HX^q(B)$. The result for the complex $C^*$ follows. The proof for $D^*$ is similar, making use of Lemma 1.

Consider the sequences of complexes

$$0 \to CX^*(M) \to C^*_n \to D^*_n \to 0,$$

where the maps are the usual ones of the Mayer–Vietoris sequence, that is, $i_n$ is a sum of two restriction maps and $j_n$ is a difference of two restriction maps. These sequences are not exact in general. However, by proceeding to the inverse limit we obtain a sequence

$$(\ast) \quad 0 \to CX^*(M) \to C^* \to D^* \to 0,$$

and we have:

**Lemma 3.** The sequence $(\ast)$ is exact (whether or not $(A, B)$ is $\omega$-excisive).

**Proof.** We will make use of the explicit descriptions of the inverse limit complexes $C^*$ and $D^*$ given above. It is clear that $i$ is injective, so that the sequence is exact at $CX^*$. An element of $\text{Ker}(j)$ can be described as a function $\phi: M^{q+1} \to \mathbb{R}$ such that the restriction of $\phi$ to each of the sets $\text{Pen}(A; n)$ and $\text{Pen}(B; n)$ is a coarse co-chain there. Let $\phi$ be such a function. Suppose that

$$(x_0, \ldots, x_q) \in \text{Supp}(\phi) \cap \text{Pen}(\Delta; R).$$
Then $d(x_0, x_k) \leq 2R$ for $k = 0, \ldots, q$, and so if $n$ is the least integer greater than $2R$, then either all the $x_k$ belong to $\text{Pen}(A; n)$ or else all the $x_k$ belong to $\text{Pen}(B; n)$. Since $\phi$ restricts a coarse cocycle on each of these two sets, we find that $\text{Supp}(\phi) \cap \text{Pen}(\Delta; R)$ is compact. In other words, $\phi \in \text{Image}(i)$. This shows that the sequence is exact at $C^*.$

Finally we must prove the exactness at $D^*$. An element of $D^q$ is a function $\psi: M^{q+1} \to \mathbb{R}$ whose restriction to each $\text{Pen}(A; n) \cap \text{Pen}(B; n)$ is a coarse co-chain. Choose a bounded, continuous bump function $\beta$ on $M$ with $\text{Supp}(\beta) \subseteq \text{Pen}(A; 1)$ and $\text{Supp}(1 - \beta) \subseteq \text{Pen}(B; 1)$, and define functions $\phi_A$ and $\phi_B$ on $M^{q+1}$ by

$$\phi_A(x_0, \ldots, x_q) = (1 - \beta(x_0)) \psi(x_0, \ldots, x_q),$$

$$\phi_B(x_0, \ldots, x_q) = \beta(x_0) \psi(x_0, \ldots, x_q).$$

Then $\psi = \phi_A + \phi_B$, and we claim that $(\phi_A, -\phi_B) \in C^*$; this will then show that $j$ is surjective. It is enough to show that $\phi_B$ restricts to a coarse co-chain on each $\text{Pen}(B; n)$, the proof for $\phi_A$ being analogous. Suppose then that

$$(x_0, \ldots, x_q) \in \text{Supp}(\phi_B) \cap \text{Pen}(\Delta; R),$$

with each $x_k \in \text{Pen}(B; n)$. Necessarily, $x_0 \in \text{Pen}(A; 1)$, and so each $x_k \in \text{Pen}(A; m)$, where $m$ is the least integer greater than $2R + 1$. Thus $(x_0, \ldots, x_q)$ belongs to the support of the restriction of $\psi$ to $\text{Pen}(A; m) \cap \text{Pen}(B; n)$, which is, by hypothesis, a compact set.

We can now prove Theorem 1. By Lemma 3, the sequence $(*)$ is a short exact sequence of complexes. By standard homological algebra, there is associated to it a long exact sequence of cohomology groups. Lemma 2 identifies the cohomology groups of the complexes $C^*$ and $D^*$, and thereby shows that this long exact sequence is the Mayer–Vietoris sequence we require.

2. Decompositions of the coarse compactification

The following ideas were introduced in [1] and [4].

**Definition 1.** Let $M$ be a proper metric space. A bounded continuous function $f$ on $M$ has vanishing variation at infinity if for every $R > 0$ the function

$$V_R f(x) = \max \{|f(x) - f(y)| : d(x, y) \leq R\}$$

converges to zero at infinity. Denote by $C_b(M)$ the $C^*$-algebra of all bounded continuous functions on $M$ with vanishing variation at infinity.

**Definition 2.** A coarse compactification of $M$ is a compactification $\overline{M}$ (that is, a compact Hausdorff space which contains $M$ as a dense open subset) with the property that every continuous function on $M$ restricts to a bounded continuous function on $\overline{M}$ with vanishing variation at infinity.

There is a universal coarse compactification, characterized by the property that every bounded continuous function on $M$ with vanishing variation at infinity extends to a continuous function on $\overline{M}$. See [1,4]. Thus $C_b(M) \cong C(\overline{M})$ if $\overline{M}$ is universal.

In this section we shall prove the following result.

**Proposition 1.** Let $M$ be a proper metric space, and let $A$ and $B$ be closed subspaces whose union is $M$. The decomposition $(A, B)$ is $\omega$-excisive if and only if

$$\overline{A} \cap \overline{B} = \overline{A \cap B},$$

where the bar denotes the closure inside the universal compactification.
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For $F$ a closed subset of $M$ denote by $\mathcal{I}(F)$ the ideal in $C_b(M)$ consisting of functions which vanish on $F$. In view of the Gelfand–Neumark correspondence between compact spaces and commutative $C^*$-algebras, Proposition 1 is easily seen to be equivalent to the following assertion about $C_b(M)$.

**Proposition 2.** The decomposition $M = A \cup B$ is $\omega$-excisive if and only if

$$\mathcal{I}(A) + \mathcal{I}(B) = \mathcal{I}(A \cap B).$$

**Proof.** Let $f \in \mathcal{I}(A \cap B)$, and choose a continuous partition of unity $\{i_A, i_B\}$ with $i_A$ and $i_B$ supported within distance 1 of $A$ and $B$ respectively. Then

$$f = i_A f + i_B f,$$

and the functions $i_A f$ and $i_B f$ are continuous and vanish on $B \setminus \mathrm{Pen}(A; 1)$ and $A \setminus \mathrm{Pen}(B; 1)$ respectively. Suppose now that $(A, B)$ is $\omega$-excisive. Given $R > 1$, choose $S > R$ such that

$$\mathrm{Pen}(A; 2R) \cap \mathrm{Pen}(B; 2R) \subseteq \mathrm{Pen}(A \cap B; S).$$

The set $M \setminus \mathrm{Pen}(A \cap B; S)$ falls into two pieces, one contained in $A$ and one in $B$, with a distance of more than $R$ separating the two. On the first we have $i_A f = f$; on the second we have $i_A f = 0$; and on $\mathrm{Pen}(A \cap B; S)$ we have $f \to 0$ at infinity, since $f \in \mathcal{I}(A \cap B)$. Considering $\mathrm{Pen}(A \cap B; S)$ and these two pieces separately it follows easily that the variation $V_{\mathcal{I}}(i_A f)$ vanishes at infinity on $M$, so that $i_A f, i_B f \in C_b(M)$. This shows that if $(A, B)$ is $\omega$-excisive then $\mathcal{I}(A) + \mathcal{I}(B) = \mathcal{I}(A \cap B)$. Suppose, on the other hand, that $(A, B)$ is not $\omega$-excisive. Then for some $R > 0$ there is a sequence of points $x_n \in M$ such that

$$d(x_n, A) \leq R \quad \text{and} \quad d(x_n, B) \leq R, \quad \text{but} \quad d(x_n, A \cap B) \geq 2^n.$$  

We may also arrange that $d(x_n, x_k) \geq 2^n$, for $k < n$, and then it is a simple matter to build a bounded continuous function $f$ on $M$, as a sum of smoother and smoother bump functions centred at the points $x_n$, for which $V_{\mathcal{I}}(f) \to 0$, as $x \to \infty$, and $f = 0$ on $A \cap B$, but $f(x_n) = 1$ for all $n$. Note that if $g \in \mathcal{I}(A) + \mathcal{I}(B)$ then $g(x_n) \to 0$. So our function $f \in \mathcal{I}(A \cap B)$ does not lie in $\mathcal{I}(A) + \mathcal{I}(B)$.

3. Some $K$-theory preliminaries

We gather together a few facts from $K$-theory (none of them are new) which we shall need in the remaining sections of the paper.

**Lemma 1.** Let $\mathcal{A}$ and $\mathcal{B}$ be closed, two-sided ideals in a $C^*$-algebra $\mathcal{M}$. Assume that $\mathcal{A} + \mathcal{B}$ is dense in $\mathcal{M}$. Then $\mathcal{A} + \mathcal{B} = \mathcal{M}$, and the map $a \oplus b \mapsto a + b$ produces an isomorphism of $C^*$-algebras

$$\mathcal{A} / (\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{B} / (\mathcal{A} \cap \mathcal{B}) \cong \mathcal{M} / (\mathcal{A} \cap \mathcal{B}).$$

**Proof.** Since $\mathcal{A} \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$ the map $a \oplus b \mapsto a + b$ passes to an injective $*$-homomorphism

$$\mathcal{A} / (\mathcal{A} \cap \mathcal{B}) \oplus \mathcal{B} / (\mathcal{A} \cap \mathcal{B}) \to \mathcal{M} / (\mathcal{A} \cap \mathcal{B}).$$

By basic $C^*$-algebra theory the range is closed, while by hypothesis the range is dense. Consequently our map is an isomorphism. The fact that $\mathcal{A} + \mathcal{B} = \mathcal{M}$ follows immediately from this. 

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ be $C^*$-algebras, as in Lemma 1. There is a Mayer–Vietoris sequence in $K$-theory:
\[ \ldots \to K_j(\mathcal{A} \cap \mathcal{B}) \to K_j(\mathcal{A}) \oplus K_j(\mathcal{B}) \to K_j(\mathcal{M}) \to K_{j-1}(\mathcal{A} \cap \mathcal{B}) \to \ldots \]

One way to define this is to form the $C^*$-algebra
\[ \mathcal{C} = \{ f \in C([0,1], \mathcal{M}) : f(0) \in \mathcal{A}, f(1) \in \mathcal{B} \}, \]
and analyse the exact sequence in $K$-theory arising from the ideal
\[ \mathcal{I} = \{ f \in C([0,1], \mathcal{M}) : f(0) = f(1) = 0 \}. \]

Since $\mathcal{I}$ is just the suspension of $\mathcal{M}$, we have that $K_\ast(\mathcal{I}) \cong K_{\ast+1}(\mathcal{M})$. The quotient $\mathcal{C}/\mathcal{I}$ is isomorphic to $\mathcal{A} \oplus \mathcal{B}$. The inclusion into $\mathcal{C}$ of the algebra of continuous $\mathcal{A} \cap \mathcal{B}$-valued functions on $[0,1]$ is easily seen to induce an isomorphism on $K$-theory. So the exact $K$-theory sequence associated to $\mathcal{C}$ and $\mathcal{I}$ gives a Mayer–Vietoris sequence as claimed. It is functorial, in the sense that if $\mathcal{M}', \mathcal{A}', \mathcal{B}'$ is another system of $C^*$-algebras, as in Lemma 1, and if $\Phi: \mathcal{M}' \to \mathcal{M}$ maps $\mathcal{A}'$ into $\mathcal{A}$, and $\mathcal{B}'$ into $\mathcal{B}$, then the obvious diagram relating Mayer–Vietoris sequences commutes.

At several points we shall need the following observation.

**Lemma 2.** Let $\Phi: \mathcal{A} \to \mathcal{B}$ be a homomorphism of $C^*$-algebras and let $W$ be a partial isometry in the multiplier algebra of $\mathcal{B}$ such that $\Phi(a)W*W = \Phi(a)$, for all $a \in \mathcal{A}$. Then $\Ad(W) \circ \Phi(a) = W\Phi(a)W^*$ is a $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ and passing to the induced maps on $K$-theory we have
\[ (\Ad(W) \circ \Phi)_\ast = \Phi_\ast: K_\ast(\mathcal{A}) \to K_\ast(\mathcal{B}). \]

**Proof.** Embedding $\mathcal{B}$ into $\text{Mat}_2(\mathcal{B})$ in the ‘top left corner’ (which gives an isomorphism on $K$-theory), and replacing $W$ by
\[ \begin{pmatrix} W & 1 - WW^* \\ 1 - W^*W & W^* \end{pmatrix}, \]
we reduce to the case where $W$ is a unitary, which is well known.

Finally, we shall need

**Lemma 3.** Let $\Phi, \Psi: \mathcal{A} \to \mathcal{B}$ be orthogonal homomorphisms of $C^*$-algebras, meaning that $\Phi[\mathcal{A}] \Psi[\mathcal{A}] = 0$. Suppose that there is an isometry $V$ in the multiplier algebra of $\mathcal{B}$ such that $V(\Phi(a) + \Psi(a))V^* = \Psi(a)$, for all $a \in \mathcal{A}$.

Then the induced map
\[ \Phi_\ast: K_\ast(\mathcal{A}) \to K_\ast(\mathcal{B}) \]
is the zero map.

**Proof.** We note that under the hypothesis of orthogonality the map $\Psi + \Phi$ is a $*$-homomorphism. By hypothesis, $\Ad(V) \circ (\Phi + \Psi) = \Psi$. Passing to the induced maps on $K$-theory and using Lemma 2, we get
\[ \Psi_\ast = (\Phi + \Psi)_\ast. \]

But it is easily shown that
\[ (\Phi + \Psi)_\ast = \Phi_\ast + \Psi_\ast, \]
and so subtracting $\Psi_\ast$ from everything we get $\Phi_\ast = 0$. 

4. The algebra $C^*(M)$

Let $M$ be a proper metric space. Recall from [4] that a standard $M$-module is a separable Hilbert space equipped with a faithful and non-degenerate representation of $C_0(M)$ whose range contains no non-zero compact operator.

**Definition 1.** Let $H_M$ and $H_{M'}$ be standard $M$ and $M'$-modules, respectively. The support of a bounded linear operator $T: H_M \to H_{M'}$ is the complement of the set of points $(m, m') \in M \times M'$ for which there exist functions $f \in C_0(M)$ and $f' \in C_0(M')$ such that $f'Tf = 0$, $f(m) \neq 0$, and $f'(m') \neq 0$.

We shall say that $T$ is properly supported if the projection from $\text{Supp}(T)$ to $M$ and $M'$ are proper maps.

**Definition 2.** A bounded linear operator $T: H_M \to H_{M'}$ is locally compact if the operators $f'T$ and $Tf$ are compact, for every $f \in C_0(M)$ and $f' \in C_0(M')$.

**Lemma 1.** (a) If $T: H_M \to H_{M'}$ and $T': H_{M'} \to H_M$ are bounded operators then

\[
\text{Supp}(T'T) \subseteq \left\{ (m, m') \in M \times M' : \exists m' \in M' : (m, m') \in \text{Supp}(T) \text{ and } (m', m'') \in \text{Supp}(T') \right\}.
\]

(b) If $T$ is properly supported and $S$ is locally compact then (assuming the compositions make sense) the operators $ST$ and $TS$ are locally compact.

**Proof.** Straightforward.

**Definition 3.** An operator $T: H_M \to H_M$ has finite propagation if

\[
\sup \{d(m_1, m_2) : (m_1, m_2) \in \text{Supp}(T)\} < \infty.
\]

It follows from part (a) of Lemma 1 that the set of finite propagation operators on $H_M$ is a $*$-subalgebra of the algebra of all bounded operators on $H_M$.

**Definition 4.** Denote by $C^*(M, H_M)$ the norm-closure of the $*$-algebra of all locally compact, finite propagation operators on $H_M$.

It is easy to prove that $C^*(M, H_M)$ is the same as the $C^*$-algebra $\mathcal{B}_{H_M}$ of [4]. It follows from Lemma 1 that any finite propagation operator is a multiplier of $C^*(M, H_M)$; this fact will be useful later.

We are interested in investigating the functoriality of $C^*(M, H_M)$ within the context of coarse geometry.

**Definition 5.** A coarse map from $M$ to $M'$ is a proper† Borel map $F: M \to M'$ such that for every $R > 0$ there exists $S > 0$ with

\[
d(m_1, m_2) \leq R \Rightarrow d(F(m_1), F(m_2)) \leq S.
\]

The composition of coarse maps is a coarse map, and we obtain the coarse category of proper metric spaces, denoted UBB in [4].

† We say that a Borel map between proper metric spaces is a proper map if the inverse image of any bounded set is bounded.
Lemma 2. Let $H_M$ and $H_{M'}$ be standard $M$ and $M'$-modules and let $F: M \to M'$ be a coarse map. There exists an isometry $V: H_M \to H_{M'}$ such that for some $R > 0$

$$\text{Supp}(V) \subseteq \{(m, m') \in M \times M' : d(F(m), m') \leq R\}.$$  

Proof. By spectral theory we can extend the representations of $C_0(M)$ and $C_0(M')$ on $H_M$ and $H_{M'}$ to representations of the algebras of bounded Borel functions. Partition $M'$ into Borel components $M'_j$, each with non-empty interior and uniformly bounded diameter. Denote by $\mu_j$ and $\mu'_j$ the characteristic functions of $F^{-1}[M'_j]$ and $M'_j$. Define an isometry $V$ by taking an arbitrary direct sum of isometries $V_i: \mu_j H_M \to \mu'_j H_{M'}$. If we choose $S > 0$ so that

$$d(m_1, m_2) \leq 1 \Rightarrow d(F(m_1), F(m_2)) \leq S$$

then our isometry $V$ satisfies the required support condition with

$$R = S + \sup \text{diam}(M'_j) + 1.$$  

With $V$ as in the Lemma, it follows from Lemma 1 that the homomorphism $\text{Ad}(V)$ maps $C^*(M, H_M)$ into $C^*(M', H_{M'})$.

Lemma 3. Let $F: M \to M'$ be a morphism and let $V_1, V_2: H_M \to H_{M'}$ be isometries satisfying the support condition in Lemma 2. The induced maps on K-theory are equal:

$$\text{Ad}(V_1)_* = \text{Ad}(V_2)_*: K_\bullet(C^*(M, H_M)) \to K_\bullet(C^*(M', H_{M'})).$$

Proof. It follows from Lemma 1 that the partial isometry $V_2 V_1^*$ is a multiplier of $C^*(M', H_{M'})$. So the result follows from Lemma 2 of the previous section.

The correspondence $M \mapsto K_\bullet(C^*(M, H_M))$ becomes a functor on the category whose objects are pairs $(M, H_M)$ and whose morphisms are coarse maps $F: M \to M'$. But it follows from functoriality that if $H_M$ and $H_{M'}$ are two standard $M$-modules then the map $\text{Id}_*: K_\bullet(C^*(M, H_M)) \to K_\bullet(C^*(M, H_{M'}))$ is an isomorphism, so up to canonical isomorphism the group $K_\bullet(C^*(M, H_M))$ does not depend on the choice of module.† So we might as well view $K_\bullet(C^*(M, H_M))$ as a functor on the coarse category of proper metric spaces.

We note that our functor is ‘bornotopy invariant’, in the sense that bornotopic morphisms give rise to the same map in K-theory. This is because if $F_1$ and $F_2$ are bornotopic then the same isometry $V$ will satisfy the support condition in Lemma 2 for both $F_1$ and $F_2$.

5. Mayer–Vietoris sequence for $K_\bullet(C^*(M))$

In this section we shall drop the module $H_M$ from our notation and write $C^*(M)$ in place of $C^*(M, H_M)$.

Definition 1. Let $A$ be a closed subspace of a proper metric space $M$ and let $H_M$ be a standard $M$-module. Denote by $C^*(A, M)$ the operator-norm closure of the set of all locally compact, finite propagation operators $T$ on $H_M$ whose support is contained in $\text{Pen}(A; R) \times \text{Pen}(A; R)$, for some $R > 0$ (depending on $T$).

† It is easy to check that up to non-canonical isomorphism the $C^*$-algebra $C^*(M, H_M)$ itself does not depend on $H_M$. 
We note that $C^*(A,M)$ is a closed two sided ideal in $C^*(M)$. If $V: H_A \to H_M$ is an isometry associated to the inclusion morphism $A \to M$ (as in Lemma 2 of the previous section) then the range of the map $\text{Ad}(V): C^*(A) \to C^*(M)$ lies within $C^*(A,M)$.

**Lemma 1.** The induced map

$$\text{Ad}(V): K_\ast(C^*(A)) \to K_\ast(C^*(A,M))$$

is an isomorphism.

**Proof.** The $C^*$-algebra $C^*(A,M)$ is an inductive limit

$$C^*(A,M) = \lim_{\to} C^*(\text{Pen}(A; n)) = \bigcup_{n=1}^{\infty} C^*(\text{Pen}(A; n)),$$

where $C^*(\text{Pen}(A, n))$ is viewed as acting on the standard module $C_0(\text{Pen}(A; n))H_M$. Consequently

$$K_\ast(C^*(A,M)) = \lim_{\to} K_\ast(C^*(\text{Pen}(A,n))).$$

Since the inclusions $A \subset \text{Pen}(A; n)$ and $\text{Pen}(A; n) \subset \text{Pen}(A; n+1)$ are bornotopy equivalences the induced maps on $K$-theory are all isomorphisms.

**Lemma 2.** Let $(A, B)$ be a decomposition of $M$. Then


If $(A, B)$ is $\omega$-excisive then

$$C^*(A,M) \cap C^*(B,M) = C^*(A \cap B,M).$$

**Proof.** Let $T$ be a locally compact, finite propagation operator on $H_M$. Extend the representation of $C_0(M)$ on $H_M$ to a representation of the bounded Borel functions, and let $P: H_M \to H_M$ be the projection operator corresponding to the characteristic function of $A$. Then $T = PT + (I - P)T$ is a decomposition of $T$ into a sum of an operator in $C^*(A,M)$ and an operator in $C^*(B,N)$. This shows that $C^*(A,M) + C^*(B,M)$ is dense in $C^*(M)$, and we can apply Lemma 1 of Section 3 to complete the first part of the proof.

For the second part, note that $C^*(A \cap B,M) \subseteq C^*(A,M) \cap C^*(B,M)$, whether or not the decomposition is $\omega$-excisive. For the converse, recall that by basic $C^*$-algebra theory the intersection of the ideals $C^*(A,M)$ and $C^*(B,M)$ is equal to their product. So it suffices to show that if $(A, B)$ is $\omega$-excisive, and if

$$\text{Supp}(T_A) \subseteq \text{Pen}(A; R') \times \text{Pen}(A; R')$$

and

$$\text{Supp}(T_B) \subseteq \text{Pen}(B; R') \times \text{Pen}(B; R''),$$

then

$$\text{Supp}(T_A T_B) \subseteq \text{Pen}(A \cap B; S) \times \text{Pen}(A \cap B; S),$$

for some $S > 0$. But this follows immediately from Lemma 1 of Section 4, together with the definition of $\omega$-excisiveness.

Combining Lemmas 1 and 2 with the discussion in Section 3 we obtain the following Mayer–Vietoris sequence for an $\omega$-excisive decomposition of $M$:

$$\ldots \to K_j(C^*(A \cap B)) \to K_j(C^*(A)) \oplus K_j(C^*(B)) \to K_j(C^*(M)) \to K_{j-1}(C^*(A \cap B)) \to \ldots$$
6. Relation with $K$-homology

**Definition 1.** Let $X$ be a compact metric space and let $Y$ be a closed subset of $X$. Let $H$ be a Hilbert space equipped with a faithful non-degenerate representation of $C(X)$ whose range contains no non-zero compact operator. Denote by $D^*(X, Y)$ the $C^*$-algebra of bounded operators $T$ on $H$ such that

1. if $f \in C(X)$ then $fT - Tf$ is a compact operator; and
2. if $f \in C(X)$ and $f = 0$ on $Y$ then $Tf$ and $fT$ are compact operators.

This definition is taken from [1], where the notation

$$D^*(X, Y) = \overline{D(C(X), C_0(X \setminus Y))}$$

is used. The following result is proved in [1].

**Theorem 1.** Suppose that $X$ and $Y$ are as above, with $Y$ non-empty. Denote by $\tilde{K}_*(Y)$ the reduced Steenrod $K$-homology of $Y$. There is a natural isomorphism

$$K_f(D^*(X, Y)) \cong \tilde{K}_{f-1}(Y).$$

Of course if $Y$ is empty (but $X$ is not) then $D^*(X, Y)$ is just the algebra of compact operators, so that $K_0(D^*(X, Y)) \cong \mathbb{Z}$ and $K_1(D^*(X, Y)) \cong 0$.

The term ‘natural’ in the statement of the theorem is explained by the following result.

**Proposition 1.** If $F: (X, Y) \rightarrow (X', Y')$ is a continuous map of compact metric space pairs then there is an isometry

$$V: H \rightarrow H'$$

with the property that $V(f \circ F) - fV$ is a compact operator, for every $f \in C(X')$. The homomorphism $Ad(V)$ maps $D^*(X, Y)$ into $D^*(X', Y')$, and the induced map on $K$-theory is independent of the choice of $V$.

**Proof.** See [1].

It follows that up to canonical isomorphism, $K_*(D^*(X, Y))$ does not depend on the choice of Hilbert space $H$, and we obtain a functor $(X, Y) \mapsto K_*(D^*(X, Y))$ on the category of compact metric space pairs. Of course, in view of Theorem 1 this functor factors through the functor $(X, Y) \mapsto Y$.

Suppose now that $X_M = \overline{M}$ is a metrizable coarse compactification of $M$. Let $H_M$ be a standard $M$-module. As we have pointed out earlier, the representation of $C_0(M)$ on $H_M$ extends to a representation of the bounded Borel functions; in particular it extends to a representation of $C(X_M) = C(\overline{M})$. Let

$$Y_M = \overline{M} \setminus M$$

be the ‘corona’ of $M$ in $X_M$ and form the algebra of operators $D^*(X_M, Y_M)$ on $H_M$.

**Lemma 1.**

(a) $C^*(M) \subseteq D^*(X_M, Y_M)$.

(b) Let $A$ be a closed subset of $M$, and let $Y_A = Y_M \cap \overline{A}$ (the bar denotes closure in $X_M$). Then $C^*(A; M) \subseteq D^*(X_M, Y_A)$.

† In fact different choices of $H$ lead to isomorphic $C^*$-algebras $D^*(X, Y)$, but the isomorphism is not canonical.
Proof. See Proposition 5.18 in [4].

Definition 2. Let $M$ be a proper metric space and let $X_M$ be a metrizable coarse compactification of $M$ with corona $Y_M$. We define a homomorphism

$$\beta(M, Y_M) : K_*(C^*(M)) \to \tilde{K}_{*-1}(Y_M)$$

by composing the $K$-theory map $K_*(C^*(M)) \to K_*(D(X_M, Y_M))$ induced by the inclusion in Lemma 1(a) with the isomorphism $K(D^*(X_M, Y_M)) \cong \tilde{K}_{*-1}(Y_M)$ given by Theorem 1.

The main result of this section is as follows. Let $M = A \cap B$ be an $\omega$-excisive decomposition of a proper metric space. Let $X_M$ and $Y_M$ be as above and let

$$Y_A = Y_M \cap \overline{A} \quad \text{and} \quad Y_B = Y_M \cap \overline{B},$$

so that $Y_A$, $Y_B$, and $Y_A \cap Y_B$ may be regarded as coronas of $A$, $B$ and $A \cap B$, respectively. Notice that Proposition 1 of Section 3 states that this assumption always holds for the universal compactification; however, since the universal compactification is not metrizable, it does not seem possible to use it directly in this context.

Theorem 2. If the maps $\beta(A, Y_A)$, $\beta(B, Y_B)$ and $\beta(A \cap B, Y_A \cap Y_B)$ are isomorphisms then so is $\beta(M, N_M)$.

The key to the proof is the following observation. Fix a Hilbert space $H$, equipped with a faithful non-degenerate representation of $C(X)$ whose range contains no non-zero compact operator, and view all the $C^*$-algebras below as subalgebras of $B(H)$.

Lemma 2. Let $Y = Y_1 \cup Y_2$ be any decomposition of $Y$ into closed subsets, and form the $C^*$-subalgebras $D^*(X, Y_1 \cap Y_2)$, $D^*(X, Y_1)$, $D^*(X, Y_2)$, and $D^*(X, Y)$ of $B(H)$. Then

(a) $D^*(X, Y_1)$ and $D^*(X, Y_2)$ are ideals in $D^*(X, Y)$.
(b) $D^*(X, Y_1) + D^*(X, Y_2) = D^*(X, Y)$.
(c) $D^*(X, Y_1) \cap D^*(X, Y_2) = D^*(X, Y_1 \cap Y_2)$.

Proof. A simple partition of unity argument.

Proof of Theorem 2. The inclusion maps provided by Lemma 2 give rise to a commutative diagram of Mayer–Vietoris sequences

$$
\begin{array}{ccccccccc}
K_j(C^*(A \cap B; M)) & \rightarrow & K_j(C^*(A; M)) \oplus K_j(C^*(B; M)) & \rightarrow \\
\downarrow i_A \cap B & & i_A \oplus i_B & \downarrow & \\
K_j(D^*(X_M, Y_A \cap Y_B)) & \rightarrow & K_j(D^*(X_M, Y_A)) \oplus K_j(D^*(X_M, Y_B)) & \rightarrow \\
\downarrow & & & & \downarrow & \\
K_j(C^*(M)) & \rightarrow & K_{j-1}(C^*(A \cap B; M)) & \rightarrow \\
\downarrow i_M & & \downarrow i_A \cap B & & \\
K_j(D^*(X_M, Y_M)) & \rightarrow & K_{j-1}(D^*(X_M, Y_A \cap Y_B)) & \rightarrow 
\end{array}
$$
It follows from the hypotheses, together with Lemma 1 in Section 5 and Theorem 1 above, that the maps \( i_A, i_B \) and \( i_{AB} \) are isomorphisms. So it follows from the Five Lemma that \( i_M \) is an isomorphism.

**Remark.** There is a similar result in coarse cohomology, based on the commutativity of the diagram

\[
\begin{array}{cccccc}
... & \rightarrow & H^{r-1}(Y_M) & \rightarrow & H^{r-1}(Y_A) \oplus H^{r-1}(Y_B) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
... & \rightarrow & HX^q(M) & \rightarrow & HX^q(A) \oplus HX^q(B) & \\
& & \downarrow & & \downarrow & \\
& & H^{r-1}(Y_A \cap Y_B) & \rightarrow & H^q(Y_M) & \rightarrow ... \\
& & \downarrow & & \downarrow & \\
& & HX^q(A \cap B) & \rightarrow & HX^{q+1}(M) & \rightarrow ...
\end{array}
\]

in which the top row is the Mayer–Vietoris sequence of ordinary cohomology, the bottom row is the Mayer–Vietoris sequence in coarse cohomology, and the vertical maps are the transgressions of [4] (which exist as long as the spaces \( Y \) are sufficiently well behaved, e.g. finite polyhedra).

7. **Cones**

In this section we will use the Mayer–Vietoris sequence to calculate the \( K \)-theory of \( C^* \)-algebra for a space \( M \) which is a Euclidean cone \( CN \), where \( N \) is a finite simplicial complex.

The metric space \( CN \) may be defined as follows. Embed \( N \) piecewise linearly (or piecewise smoothly) into a sphere centred at the origin in a Euclidean space. Then \( CN \) is the union of all half lines beginning at the origin and passing through a point in \( N \). We give \( CN \) the metric it inherits as a subspace of Euclidean space. Up to bornotopy equivalence the space \( CN \) is independent of the embedding of \( N \) used. We note that \( CN \) has an obvious coarse compactification, for which the corona is \( N \).

**Proposition 1.** Let \( C\Delta \) be the Euclidean cone on a single simplex \( \Delta \). Then

\[
K_{q}(C^*(C\Delta)) = 0.
\]

**Proof.** The cone on an \( n \)-simplex is (bornotopy equivalent to) the octant

\[
M = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0\}
\]

in Euclidean space. For our standard module we take \( L^2(M) \) (with respect to Lebesgue measure). Let

\[
L^2(M) = L^2(M) \oplus L^2(M) \oplus \ldots,
\]

and consider the inclusion

\[
\Phi : T \rightarrow T \oplus 0 \oplus 0 \oplus \ldots
\]
of $C^*(M, L^2(M))$ into $C^*(M, L^2(M)_\infty)$. By Lemma 3 of Section 4, and the remarks following it, the induced map

$$\Phi_* : K_*(C^*(M, L^2(M))) \to K_*(C^*(M, L^2(M)_\infty))$$

is an isomorphism, so it suffices to show that $\Phi_* = 0$. Define an isometry $W$ on $L^2(M)$ by

$$W\phi(x_0, x_1, \ldots, x_n) = \phi(x_0 + 1, x_1, \ldots, x_n),$$

and define an isometry $V$ on $L^2(M)_\infty$ by

$$V(\phi_1 \oplus \phi_2 \oplus \ldots) = (0 \oplus W\phi_1 \oplus W\phi_2 \oplus \ldots).$$

It has finite propagation, and is consequently a multiplier of $C^*(M, L^2(M)_\infty)$. Define a $*$-homomorphism

$$\Psi : C^*(M, L^2(M)) \to C^*(M, L^2(M)_\infty)$$

by the formula

$$\Psi(T) = 0 \oplus WTW^* \oplus W^2TW^* \oplus W^3TW^* \oplus \ldots.$$  

Note that despite the fact that the direct sum defining $\Psi(T)$ is infinite the resulting operator is still locally compact and finite propagation. To complete the proof we note that the homomorphisms $\Phi$ and $\Psi$ are orthogonal, and that $\text{Ad}(V) \circ (\Phi + \Psi) = \Psi$, so that by Lemma 3 of Section 4, $\Phi_* = 0$.

**Proposition 2.** Let $N$ be a finite simplicial complex. Then the map

$$\beta : K_*(C^*(C(N))) \to \tilde{K}^*_{n-1}(N)$$

is an isomorphism.

**Proof.** If $N$ is empty, let us define $\tilde{K}^*_{n-1}(N)$ to be $K_*(D^*(CN,N))$; $CN$ is a single point, and $D^*(CN,N)$ is the algebra of compact operators. Since $C^*(CN)$ is also the algebra of compact operators, the result is true for $N$ empty. If $N$ consists of a single simplex, the result is true by Proposition 1. The general result now follows by induction on the number of simplices, using Theorem 2 of Section 6.

This result is a $C^*$-analogue of a purely algebraic theorem of Pedersen and Weibel[3]. As suggested in [4], the result can also be considered to be a verification of the Baum–Connes conjecture in the context of coarse geometry.

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