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E-theory and KK-theory for groups which act properly and isometrically on Hilbert space

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1. Introduction

A good deal of research in C^*-algebra K-theory in recent years has been devoted to the Baum-Connes conjecture [3], which proposes a formula for the K-theory of group C^*-algebras that blends group homology with the representation theory of compact subgroups. The conjecture has brought C^*-algebra theory into close contact with manifold theory through its obvious similarity to the Borel conjecture of surgery theory [9, 31] and its links with the theory of positive scalar curvature [27]. In addition there are points of contact with harmonic analysis, particularly the tempered representation theory of semisimple groups, although the proper relation between the Baum-Connes conjecture and representation theory is not well understood.

The conjecture is most easily formulated for groups which are discrete and torsion-free. For such a group G there is a natural homomorphism

$$\mu_{\mathrm{red}}\colon K_*(BG) \to K_*\big(C^*_{\mathrm{red}}(G)\big),$$

mapping the K-homology of the classifying space of G to the K-theory of the reduced C^*-algebra of G (the reduced C^*-algebra is the completion of the complex group algebra of G in the regular representation as operators on $\ell^2(G)$). This *assembly map* is a counterpart of its namesake in surgery theory, and, in line with the Borel conjecture, the Baum-Connes conjecture asserts that the assembly map is an isomorphism.

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The statement of the conjecture for discrete groups with torsion, or for non-discrete groups, uses equivariant KK-theory [22]. Associated to any second countable, locally compact group G is a universal proper G-space $\mathcal{E}G$, which is unique up to equivariant homotopy [3], and using KK-theory we may form its equivariant K-homology $K_*^G(\mathcal{E}G)$. There is then an assembly map

$$\mu_{\mathrm{red}}\colon K_*^G(\mathcal{E}G) \to K_*\big(C^*_{\mathrm{red}}(G)\big)$$

which is conjectured by Baum and Connes to be an isomorphism. For more details see [3].

The decoration 'red' – short for 'reduced' – is used because there is also an assembly map

$$\mu_{\max}\colon K_*^G(\mathcal{E}G) \to K_*\big(C^*_{\max}(G)\big)$$

involving the *full* group C^*-algebra $C^*_{\max}(G)$ (which is the enveloping C^*-algebra of the complex group algebra in the discrete case, and the enveloping C^*-algebra of $L^1(G)$ in general). This 'max' assembly map is not generally an isomorphism. Both the assembly maps μ_{red} and $\mu_{\max}$ may be generalized by introducing a coefficient C^*-algebra A, on which G acts continuously by C^*-algebra automorphisms. If the associated reduced crossed product C^*-algebra is denoted $C^*_{\mathrm{red}}(G, A)$ then there is a reduced assembly map

$$\mu_{\mathrm{red}}\colon KK_*^G(\mathcal{E}G, A) \to K_*\big(C^*_{\mathrm{red}}(G, A)\big)$$

defined again using KK-theory [3], along with a similar map $\mu_{\max}$ for the full crossed product algebra $C^*_{\max}(G, A)$.

The main purpose of this article is to prove the Baum-Connes conjecture for an interesting and fairly broad class of groups, called by Gromov *a-T-menable* [10], and known to harmonic analysts as groups with the *Haagerup approximation property* [16]. These are those locally compact groups which admit continuous, affine, isometric and *metrically proper* actions on a Hilbert space, the latter term meaning that

$$\lim_{g\to\infty} \|g \cdot v\| = \infty,$$

for every vector v in the Hilbert space. Gromov's terminology is explained by the twin facts that all (second countable) amenable groups admit such action [4], whereas no non-compact group with Kazhdan's property T does [13]. The Haagerup approximation property first arose in an investigation of the Banach space structure of C^*-algebras [12].

Our main theorem is as follows.

1.1. Theorem. *If G is a second countable, locally compact group, and if G has the Haagerup approximation property, then for any separable G-C^*-algebra A the Baum-Connes assembly maps*

$$\mu_{\mathrm{red}} \colon KK_*^G(\mathcal{E}G, A) \to K_*\big(C^*_{\mathrm{red}}(G, A)\big)$$

and

$$\mu_{\max} \colon KK_*^G(\mathcal{E}G, A) \to K_*\big(C^*_{\max}(G, A)\big)$$

are isomorphisms of abelian groups.

Remark. A recent article of Chabert, Echterhoff and Meyer [5] asserts that the definition of proper action used in the standard formulation of the Baum-Connes conjecture [3] agrees with the usual notion of a proper action of a locally compact group on a locally compact space (this is obvious for discrete groups, but rather less clear otherwise). We shall use the latter notion throughout this paper.

Apart from amenable groups, important examples of groups covered by Theorem 1.1 are free groups, the real and complex hyperbolic groups $SO(n, 1)$ and $SU(n, 1)$, and Coxeter groups. A recent Seminaire Bourbaki of P. Julg [16] gives a good account of these and other examples. The Baum-Connes conjecture was previously established for real and complex hyperbolic groups [21, 17], but the argument we give here is quite different in character. In fact, in view of the arguments used in [17] to prove the Baum-Connes conjecture for $SU(n, 1)$, which blend contact geometry with unitary representation theory, it is quite remarkable that the arguments of this paper rely on essentially no geometry or representation theory at all.

Theorem 1.1 and a sketch of its proof, at least for discrete groups, were the content of our recent announcement [14]. We shall give a full account of that argument, as it applies to the assembly map $\mu_{\max}$, in the first six sections of this paper. This much of Theorem 1.1 is sufficient for applications to the Novikov conjecture. Of course if G is amenable then $\mu_{\max} = \mu_{\mathrm{red}}$, so we will at this stage have completed the proof of Theorem 1.1 for discrete, amenable groups.

The arguments in [14] use asymptotic morphisms and E-theory [6, 11], and unfortunately this theory is not well suited to dealing directly with μ_{red}. In our announcement we indicated an *ad hoc* way of circumventing this problem. Here we shall follow a somewhat different course. A *proper* G-C^*-algebra is a mildly non-commutative generalization of the notion of a proper G-space (the algebra of continuous functions, vanishing at infinity, on a locally compact proper G-space is the prototypical example of a proper G-C^*-algebra). It is proved in [11] that if G is discrete and A is proper then the assembly map

$$\mu_{\max} \colon E_*^G(\mathcal{E}G, A) \to K_*\big(C^*_{\max}(G, A)\big)$$

is an isomorphism. Roughly speaking, this is because proper actions are locally modelled by actions of finite groups, while the Baum-Connes conjecture is readily verified for finite groups. We used this result in our announcement [14], by noting that if the C^*-algebra $\mathbb{C}$ (with trivial G-action) is isomorphic in equivariant E-theory to a proper G-C^*-algebra then every separable G-C^*-algebra B is E-theoretically a direct summand of a proper G-C^*-algebra, so that the Baum-Connes assembly map for B, being a direct summand of the assembly map for a proper G-C^*-algebra, is an isomorphism. Here we shall use an improved result of Tu [29]: if the C^*-algebra $\mathbb{C}$ (with trivial G-action) is isomorphic in equivariant KK-theory to a proper G-C^*-algebra then both of the Baum-Connes assembly maps μ_{red} and μ_{max} are isomorphisms, for any coefficient C^*-algebra.

The problem of showing that $\mathbb{C}$ is isomorphic in equivariant KK-theory to a proper G-C^*-algebra reduces, as in [14], to Bott periodicity. If G acts linearly and isometrically on a *finite-dimensional* Hilbert space H then the tangent space TH is equivalent in equivariant K-theory to a point. If G acts properly and affine-isometrically on H then the G-C^*-algebra $C_0(TH)$ is proper. Using the fact that any affine-isometric action on the flat Euclidean space H is homotopic to a linear-isometric action, it follows that if G acts properly and affine-isometrically on a finite-dimensional Euclidean space H then the proper G-C^*-algebra $C_0(TH)$ is equivalent in equivariant K-theory, and indeed in equivariant KK-theory, to $\mathbb{C}$. We shall take the same approach in the infinite-dimensional case, following earlier work of ours with J. Trout on the periodicity phenomenon in infinite dimensions [15]. The main problems here are to find a suitable substitute for $C_0(TH)$, which does not make sense in infinite dimensions, and to replicate Atiyah's index-theoretic Bott periodicity argument [2] in the new context. The first problem was solved in [15] and is reviewed here in Sect. 4. The requisite index theory, which is the heart of the matter, is developed in Sects. 2 and 3, and is applied to our problem in the remaining sections of the paper.

To carry out the details of the above argument within KK-theory rather than E-theory (as was the case in [14] and [15]) we must recast our basic E-theory constructions in the language of KK-theory. The somewhat delicate arguments needed for this are presented in Sects. 7 and 8. Having done so, an attractive consequence presents itself:

1.2. Theorem. *Every second countable, locally compact topological group with the Haagerup approximation property is K-amenable.*

We shall not give the definition of K-amenability here (see [7,18]), but we recall for the reader that it implies that at the level of K-theory there is no difference between $C^*_{\mathrm{max}}(G, A)$ and $C^*_{\mathrm{red}}(G, A)$. It follows that if G is K-amenable then the Baum-Connes assembly maps μ_{max} and μ_{red} actually coincide. Thus if μ_{max} is an isomorphism then so of course is μ_{red}.

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2. The Bott-Dirac operator in infinite dimensions

Let H be a separable, infinite-dimensional, real Hilbert space. The purpose of this section is to construct a certain $\mathbb{Z}/2$-graded, complex Hilbert space $\mathcal{H}(H)$, comprised of differential forms on H. We shall also define a grading-degree one, index one, unbounded operator B on $\mathcal{H}(H)$ which is central to the Bott periodicity phenomenon in infinite dimensions [15].

The complex Hilbert space $\mathcal{H}(H)$ will depend only on the affine structure of H, and in fact we shall construct not one Hilbert space but an entire continuous field over the positive reals $0 < \alpha < \infty$. These features will be used in later sections.

For any finite-dimensional, affine subspace V of H, we denote by V_0 the underlying vector subspace of H, comprised of differences of elements in V. We will call V a *linear* subspace of H if it contains the point $0 \in H$. In this case V and V_0 are one and the same. We will consider the exterior algebra $\Lambda^*(V_0) \otimes \mathbb{C}$ as endowed with its natural Euclidean structure, induced by that of V_0.

Fix a positive real number α.

2.1. Definition. (i) Let $\mathcal{H}(V)$ be the $\mathbb{Z}/2$-graded Hilbert space of square-integrable functions from V into $\Lambda^*(V_0) \otimes \mathbb{C}$.

(ii) If W is a finite-dimensional *linear* subspace of H then denote by $\xi_W \in \mathcal{H}(W)$ the L^2-normalized scalar function

$$\xi_W(w) = (\pi\alpha)^{-m/4} \exp(-\|w\|^2/2\alpha),$$

where $m = dim(W)$.

(iii) If V' is an affine subspace of V, then the orthogonal complement W of V' in V is defined as the orthogonal complement of V'_0 in V_0. It is a linear subspace. We define an isometry $\mathcal{H}(V') \to \mathcal{H}(V)$ by mapping $f \in \mathcal{H}(V')$ to the L^2-form $f(v') \cdot \xi_W(w)$. We regard the latter as a function of $v = v' + w$, where $v' \in V'$ and $w \in W$.

Notice that the isometry $\mathcal{H}(V') \to \mathcal{H}(V)$ depends on $\alpha > 0$.

If $V'' \subset V' \subset V$ then the composition of isometries

$$\mathcal{H}(V'') \to \mathcal{H}(V') \to \mathcal{H}(V)$$

is equal to the isometry associated to the inclusion $V'' \subset V$. This allows us to make the following construction:

2.2. Definition. Denote by $\mathcal{H} = \mathcal{H}(H)$ the Hilbert space direct limit

$$\mathcal{H}(H) = \varinjlim_{V \subset H} \mathcal{H}(V),$$

taken over the directed system of finite-dimensional affine subspaces of H, using the isometries $\mathcal{H}(V') \to \mathcal{H}(V)$ in Definition 2.1. Denote by $\xi_H \in \mathcal{H}(H)$ the unit vector corresponding to any $\xi_W \in \mathcal{H}(W)$ in the directed system, where W is any finite-dimensional *linear* subspace (all W produce the same ξ_H).

The Hilbert space $\mathcal{H}(H)$ certainly depends on our choice of $\alpha > 0$. Later on, where necessary, we shall write $\mathcal{H}_\alpha(H)$ in place of $\mathcal{H}(H)$ to indicate this dependence.

Since the family of all finite-dimensional, affine subspaces of H is rather unwieldy, we shall often use the following simple approximation result, whose proof is omitted.

2.3. Lemma. *If H' is a dense subspace of H then the inclusion*

$$\varinjlim_{V \subset H'} \mathcal{H}(V) \hookrightarrow \varinjlim_{V \subset H} \mathcal{H}(V)$$

is an isomorphism of Hilbert spaces. In particular, if $V_1 \subset V_2 \subset \cdots$ is an increasing family of finite-dimensional affine subspaces of H whose union is dense in H then the canonical isometric inclusion of the Hilbert space direct limit $\varinjlim \mathcal{H}(V_n)$ into $\mathcal{H}(H)$ is a unitary isomorphism. □

2.4. Definition. If W is a finite-dimensional *linear* subspace of H, and if $w \in W$, then denote by $\operatorname{ext}(w)$ the operator of exterior multiplication by w on $\Lambda^*(W) \otimes \mathbb{C}$ and by $\operatorname{int}(w)$ its adjoint (i.e. interior multiplication by w). The *Clifford multiplication* operators $\bar{c}(w)$ and $c(w)$ are defined by

$$\begin{aligned}\bar{c}(w) &= \operatorname{ext}(w) - \operatorname{int}(w)\\ c(w) &= \operatorname{ext}(w) + \operatorname{int}(w).\end{aligned}$$

2.5. Definition. If V is a finite-dimensional affine subspace of H then denote by $\mathfrak{s}(V)$ the dense $\mathbb{Z}/2$-graded subspace of $\mathcal{H}(V)$ comprised of Schwartz functions from V into $\Lambda^*(V_0) \otimes \mathbb{C}$.

2.6. Definition. If W is a finite-dimensional linear subspace of H then we define a partial differential operator

$$B_W \colon \mathfrak{s}(W) \to \mathfrak{s}(W)$$

by the formula

$$B_W = \sum_{i=1}^{m} \alpha \bar{c}(w_i) \frac{\partial}{\partial x_i} + c(w_i) x_i,$$

where $w_1, \dots, w_m$ is an orthonormal basis for W and $x_1, \dots, x_m$ are coordinates in W dual to the orthonormal basis. This operator will be called the *Bott-Dirac operator* on W.

The operator B_W is independent of the choice of the basis $w_1, \dots, w_m$ and is a symmetric, grading-degree one operator on $\mathfrak{s}(W)$. The square of B_W is

$$B_W^2 = \sum_{i=1}^{m} \left(-\alpha^2 \frac{\partial^2}{\partial x_i^2} + x_i^2 \right) + 2\alpha N - \alpha m,$$

where N is the operator which assigns to a differential form its degree. It follows from the well-known spectral theory for the harmonic oscillator Hamiltonian[1] $-\alpha^2 d^2/dx^2 + x^2$ that the grading-degree zero and degree one subspaces of $\mathfrak{s}(W)$ each contain an orthonormal basis comprised of eigenfunctions for B_W^2. The eigenvalues are $0, 2\alpha, 4\alpha, \dots$; each is of finite multiplicity; and 0 occurs only in grading-degree zero, and with multiplicity one. The L^2-normalized 0-eigenfunction is the function

$$\xi_W(w) = (\pi\alpha)^{-m/4} \exp(-\|w\|^2/2\alpha)$$

of Definition 2.1.

It follows from the existence of an eigenbasis, or from standard facts about first order linear partial differential operators, that B_W is an essentially self-adjoint operator on the Hilbert space $\mathcal{H}(W)$, with domain $\mathfrak{s}(W)$. It has compact resolvent, and index one (in the graded sense of the word index).

Note that if $W_1, W_2, \dots, W_j \subset W$ then we can certainly regard the operators B_{W_j} as operators on $\mathfrak{s}(W)$. If $W = W_1 \oplus \cdots \oplus W_j$ then

$$B_W = B_{W_1} + \cdots + B_{W_j}$$

(the addition here refers to the ordinary sum of operators defined on the common domain $\mathfrak{s}(W)$; we are not concerned yet with the domains of the self-adjoint extensions of any of these operators). In addition it is easily verified that $B_i B_j + B_j B_i = 0$ if $i \neq j$, so that

$$B_W^2 = B_{W_1}^2 + \cdots + B_{W_j}^2.$$

To define a Bott-Dirac operator on the whole of H we shall use the following calculation.

[1] For each $n = 0, 1, 2, \dots$ the harmonic oscillator $-\alpha^2 d^2/dx^2 + x^2$ has an eigenfunction of the form $p_n(x)e^{-x^2/2\alpha}$ with eigenvalue $(2n+1)\alpha$, where p_n is a polynomial of degree n. The lowest eigenfunction is $e^{-x^2/2\alpha}$.

2.7. Lemma. *Let W be a finite-dimensional linear subspace of H. Let W_1 be a linear subspace of W and form the diagram*

$$\begin{array}{ccc}\mathfrak{s}(W_1) & \longrightarrow & \mathfrak{s}(W)\\ {\scriptstyle B_{W_1}}\downarrow & & \downarrow{\scriptstyle B_W}\\ \mathfrak{s}(W_1) & \longrightarrow & \mathfrak{s}(W)\end{array}$$

in which the horizontal maps $\mathfrak{s}(W_1) \to \mathfrak{s}(W)$ are the restrictions to Schwartz space of the isometries $\mathcal{H}(W_1) \to \mathcal{H}(W)$ in Definition 2.1. This diagram is commutative.

Proof. Write $W = W_1 \oplus W_2$. The inclusion $\mathfrak{s}(W_1) \to \mathfrak{s}(W)$ maps $f \in \mathfrak{s}(W_1)$ to the function $f \cdot \xi_{W_2}$, and the function ξ_{W_2} lies in the kernel of B_{W_2}. So it follows from the identity $B_W = B_{W_1} + B_{W_2}$ that $B_W(f \cdot \xi_{W_2}) = (B_{W_1} f) \cdot \xi_{W_2}$, which proves the lemma. □

2.8. Definition. Denote by $\mathfrak{s}(H)$ the algebraic direct limit of the spaces $\mathfrak{s}(W)$, as W ranges over the finite-dimensional linear subspaces of H. Define a symmetric operator $B\colon \mathfrak{s}(H) \to \mathfrak{s}(H)$ by requiring that $Bf = B_W f$ whenever $f \in \mathfrak{s}(W) \subset \mathfrak{s}(H)$.

Lemma 2.7 shows that this definition of B is unambiguous. Our previous analysis of B_W shows that $\mathfrak{s}(H)$ contains an orthonormal basis for $\mathcal{H}(H)$ comprised of eigenvectors for B. The kernel of B is spanned by the vector $\xi_H \in \mathcal{H}(H)$. The lowest non-zero eigenvalue of B^2 is 2α. In view of the existence of an eigenbasis, B is an essentially self-adjoint operator on $\mathcal{H}(H)$.

If $V_1 \subset V_2 \subset V_3 \subset \cdots$ is an increasing sequence of finite-dimensional linear subspaces of H, whose union is dense in H, and if we write $W_j = V_j \ominus V_{j-1}$, so that $V_j = W_1 \oplus \cdots \oplus W_j$ and

$$H = W_1 \oplus W_2 \oplus W_3 \oplus \cdots ,$$

then on the subspace $\varinjlim \mathfrak{s}(V_j) \subset \mathfrak{s}(H)$ the operator B is given by the alternate formula

$$B = B_{W_1} + B_{W_2} + B_{W_3} + \cdots .$$

This infinite sum of operators is well defined on $\varinjlim \mathfrak{s}(V_j)$ because if $f \in \mathfrak{s}(V_k) \subset \varinjlim \mathfrak{s}(V_j)$ then $B_{W_l} f = 0$ for all $l > k$, by an argument like the one used to prove Lemma 2.7. The subspace $\varinjlim \mathfrak{s}(V_j) \subset \mathcal{H}(H)$ is dense by Lemma 2.3, and it follows from diagonalizability that B is essentially self-adjoint on the domain $\varinjlim \mathfrak{s}(V_j)$ as well as on $\mathfrak{s}(H)$. Obviously $\mathfrak{s}(H)$ is a more canonical choice of domain (for instance if a group G acts linearly and isometrically on H then G acts isometrically on $\mathfrak{s}(H)$, and the operator B, defined on this domain, is equivariant). But $\varinjlim \mathfrak{s}(V_j)$ is sometimes better suited to computations.

Because the operator B is so central to what follows, we conclude this section with two other descriptions of it.

First, if $w_1, w_2, \dots$ is an orthonormal basis for H, and if V_j is the span of $w_1, \dots, w_j$, then on the subspace $\varinjlim \mathfrak{s}(V_j) \subset \mathfrak{s}(H)$ the operator B is given by the infinite series

$$B = \sum_{i=1}^{\infty} \left(\alpha \bar{c}(w_i) \frac{\partial}{\partial x_i} + c(w_i) x_i \right).$$

As before, the infinite sum is well-defined since if $\eta \in \mathcal{H}(V_j)$ then η is in the kernel of all the operators $\alpha \bar{c}(w_i) \frac{\partial}{\partial x_i} + c(w_i) x_i$ for $i > j$.

Second, if W is a finite-dimensional linear subspace of H then denote by $\Omega_{\mathrm{alg}}(W)$ the linear space of polynomial differential forms on W (in other words $\Omega_{\mathrm{alg}}(W)$ is the linear span, within all differential forms on W, of all polynomials on W, all de Rham differentials of polynomials, and all sums of products of these). If $W_1 \subset W$ then the operation of pull-back of differential forms along the orthogonal projection $W \to W_1$ defines an embedding $\Omega_{\mathrm{alg}}(W_1) \to \Omega_{\mathrm{alg}}(W)$, and we define

$$\Omega_{\mathrm{alg}}(H) = \varinjlim_{W \subset H} \Omega_{\mathrm{alg}}(W).$$

The de Rham differential gives a well-defined operator $d\colon \Omega_{\mathrm{alg}}(H) \to \Omega_{\mathrm{alg}}(H)$. Now define an inner product on $\Omega_{\mathrm{alg}}(H)$ by the formula

$$\langle \omega_1, \omega_2 \rangle = (\pi\alpha)^{-\dim(W)/2} \int_W (\omega_1, \omega_2)_w e^{-\|w\|^2/\alpha}\, dw \qquad (\omega_1, \omega_2 \in \Omega_{\mathrm{alg}}(W)),$$

where $(\omega_1, \omega_2)_w$ denotes the pointwise inner product of ω_1 and ω_2 at $w \in W$. The map $\omega \mapsto \omega \cdot \xi_W$ is an isometry from $\Omega_{\mathrm{alg}}(W)$ into $\mathfrak{s}(W)$, and from $\Omega_{\mathrm{alg}}(H)$ into $\mathfrak{s}(H)$.

2.9. Lemma. *Under the inclusion $\Omega_{\mathrm{alg}}(H) \subset \mathfrak{s}(H)$ the Bott-Dirac operator B on $\mathfrak{s}(H)$ corresponds to the de Rham-type operator $\alpha d + \alpha d^*$ on $\Omega_{\mathrm{alg}}(H)$ where d is the de Rham differential and d^* denotes its formal adjoint with respect to the given inner product on $\Omega_{\mathrm{alg}}(H)$.* □

Remark. Part of the lemma is the assertion that the formal adjoint exists, as an operator on $\Omega_{\mathrm{alg}}(V)$.

We shall not use the Lemma below. Its simple proof is left to the reader.

3. A non-commutative functional calculus for the operator B

If W is finite-dimensional then the resolvents $(B_W \pm i)^{-1}$ of the Bott-Dirac operator are compact operators. This is no longer true in infinite dimensions, and the main purpose of this section is to introduce perturbations of the Bott-Dirac operator B for which this property is restored. Unfortunately these perturbations destroy the symmetries of B, which is initially equivariant for any isometric linear action of a group on H. But we shall also investigate the extent to which a limited form of equivariance is retained.

To accomplish these goals we shall introduce an interesting 'functional calculus' for B which associates to any self-adjoint (but not necessarily bounded) operator h on H a self-adjoint operator $h(B)$ on $\mathcal{H}(H)$, in such a way that for instance if $h = I$ then $h(B) = B$, while if h has compact resolvent then so does $h(B)$. The basic construction appears in our previous work [14] and [15]. A recent article of Tu [30], extending our original work to groupoids, sketches a similar construction to the one below.

Our first definition of $h(B)$ will require that h be diagonalizable. Later we will extend the definition to arbitrary h (see Definition 3.5).

3.1. Definition. Let h be a self-adjoint operator on the real Hilbert space H. Assume that h is diagonalizable. Write H as an orthogonal sum of eigenspaces W_j for h, and denote by λ_j the corresponding eigenvalues. Let $V_j = W_1 \oplus \cdots \oplus W_j$ and define the operator $h(B)$, acting on the space $\varinjlim \mathfrak{s}(V_j) \subset \mathcal{H}(H)$, by

$$h(B) = \lambda_1 B_{W_1} + \lambda_2 B_{W_2} + \dots .$$

Remark. In the cases of most importance to us the eigenspaces W_j will be finite-dimensional. But if some W_j is infinite-dimensional then by B_{W_j} we will of course mean the operator constructed by the direct limit procedure of the previous section.

By a calculation like the ones done in the previous section, when the infinite sum $\sum_j \lambda_j B_{W_j}$ is applied to any vector in $\varinjlim \mathfrak{s}(V_j)$ the result has only finitely many non-zero terms. Hence there is no difficulty with convergence of the sum in Definition 3.1.

3.2. Lemma. *Let h be a self-adjoint diagonalizable operator on H. Then the operator $h(B)$ is essentially self-adjoint. If h has compact resolvent then $h(B)$ also has compact resolvent.*

Proof. We shall use the notation of Definition 3.1. In complete analogy with the case of B, the space $\varinjlim \mathfrak{s}(V_j)$ admits an orthonormal eigenbasis for $h(B)$, therefore $h(B)$ is essentially self-adjoint. If h has compact resolvent the each W_j is finite-dimensional and the values λ_j^2 tend to infinity as j tends to infinity. The square of $h(B)$ is

$$h(B)^2 = \lambda_1^2 B_{W_1}^2 + \lambda_2^2 B_{W_2}^2 + \cdots ,$$

from which it follows that $h(B)^2$ has an eigenbasis within $\varinjlim \mathfrak{s}(V_j)$ whose eigenvalues are all sums

$$2n_1\lambda_1^2\alpha + 2n_2\lambda_2^2\alpha + \cdots ,$$

where $n_1, n_2, \ldots$ are non-negative integers, almost all zero. Associated to each such sequence $n_1, n_2, \ldots$ there are finitely many eigenfunctions in the eigenbasis, so we see that

(1) the eigenvalues of the $h(B)^2$ form a sequence converging to ∞; and
(2) each eigenvalue is of finite multiplicity.

The lemma follows immediately from this. □

In the following lemma, by the *self-adjoint domain* of an essentially self-adjoint operator we mean the domain of its self-adjoint extension, or in other words the domain of its closure.

3.3. Lemma. *Let h be a self-adjoint operator on H with compact resolvent, and suppose that $h^2 \geq 1$. Then the self-adjoint domain of $h(B)$ is contained in the self-adjoint domain of B, and $\|h(B)\xi\| \geq \|B\xi\|$ for all vectors ξ in the self-adjoint domain of $h(B)$.*

Proof. Let $V_1, V_2, \ldots$ be as in Definition 3.1. If ξ lies in $\varinjlim \mathcal{S}(V_j)$ then the inequality $\|h(B)\xi\|^2 \geq \|B\xi\|^2$ is clear from the formula for $h(B)^2$ given in the proof of Lemma 3.2, together with the fact that $\lambda_j^2 \geq 1$, for all j. The inequality $\|h(B)\xi\|^2 \geq \|B\xi\|^2$ implies that the domain of the closure of B contains the domain of the closure of $h(B)$. Furthermore the inequality extends by continuity to the domain of the closure of $h(B)$. □

To prove more about the operator $h(B)$ we need an alternate description which gives an explicit formula for $h(B)$ on a larger domain than $\varinjlim{}_j \mathfrak{s}(V_j)$. Let h be any self-adjoint operator on H and let $H_h \subset H$ be the domain of h. Denote by $\mathfrak{s}(H_h) \subset \mathfrak{s}(H)$ the direct limit

$$\mathfrak{s}(H_h) = \varinjlim_{V \subset H_h} \mathfrak{s}(V)$$

over all finite-dimensional subspaces of H_h. We are going to define a symmetric operator

$$\tilde{h}(B)\colon \mathfrak{s}(H_h) \to \mathfrak{s}(H)$$

which, in the case where h is diagonalizable, will be an extension of the operator $h(B)$ of Definition 3.1 (note that $\mathfrak{s}(H_h)$ contains the domain $\varinjlim \mathfrak{s}(V_j)$ given in Definition 3.1).

3.4. Definition. If W and V are finite-dimensional linear subspaces of H then denote by $\mathfrak{s}(W, V)$ the Schwartz space of functions from W into $\Lambda^*(V) \otimes \mathbb{C}$. If $W' \subset W$ and $V' \subset V$ then the map $f \mapsto f \cdot \xi_{W''}$, where $W'' = W \ominus W'$, defines an isometry of $\mathfrak{s}(W', V')$ into $\mathfrak{s}(W, V)$, just as in Definition 2.1. Observe that

$$\varinjlim_{W,V\subset H} \mathfrak{s}(W, V) = \varinjlim_{W\subset H} \mathfrak{s}(W),$$

since the set of pairs (V, V) is cofinal in the directed set of all pairs of finite-dimensional subspaces.

3.5. Definition. Let W be a finite-dimensional subspace of H_h and let V be any finite-dimensional subspace of H containing $h[W]$. Define an operator

$$h(B_W)\colon \mathfrak{s}(W, W) \to \mathfrak{s}(W, V)$$

by

$$h(B_W) = \sum_{i=1}^{m} \alpha \bar{c}(hw_i)\frac{\partial}{\partial x_i} + c(hw_i)x_i,$$

where $w_1, \ldots, w_m$ is any orthonormal basis for W (the choice does not affect the operator) and $x_1, \ldots, x_m$ are dual coordinates on W. As in Lemma 2.7 and Definition 2.8, the formula

$$\tilde{h}(B)\xi = h(B_W)\xi, \qquad \xi \in \mathfrak{s}(W)$$

unambiguously defines a symmetric operator

$$\mathfrak{s}(H_h) = \varinjlim_{W\subset H_h} \mathfrak{s}(W) \xrightarrow{\tilde{h}(B)} \varinjlim_{W,V\subset H} \mathfrak{s}(W, V) = \mathfrak{s}(H).$$

3.6. Lemma. *If h is a diagonalizable self-adjoint operator on H then the restriction of the symmetric operator $\tilde{h}(B)\colon \mathfrak{s}(H_h) \to \mathfrak{s}(H)$ to the direct limit $\varinjlim \mathfrak{s}(V_j) \subset \mathfrak{s}(H_h)$ of Definition 3.1 is the operator $h(B)$.*

Proof. If V_j is the direct sum of eigenspaces $W_1 \oplus \cdots \oplus W_j$ for h, then by choosing an orthonormal basis for V compatible with this decomposition it is clear that

$$h(B_{V_j}) = \lambda_1 B_{W_1} + \cdots + \lambda_j B_{W_j}.$$

The lemma follows immediately from this. $\square$

Thus if h is diagonalizable then the operator $\tilde{h}(B)$ is a symmetric extension of the essentially self-adjoint operator $h(B)$ (and consequently has the same closure). In view of this, henceforth we shall drop the tilde, and write $h(B)$ in place of $\tilde{h}(B)$.

3.7. Lemma. *If h is a bounded self-adjoint operator then $\|h(B)\xi\| \leq \|h\| \cdot \|B\xi\|$, for every vector $\xi \in \mathfrak{s}(H)$.*

Proof. Suppose first that h is diagonalizable, with eigenspaces W_j, and eigenvalues λ_j. Let $V_j = W_1 \oplus \cdots \oplus W_j$. On the subspace $\varinjlim \mathfrak{s}(V_j)$ the operator $h(B)^2$ is the sum of the infinite series

$$h(B)^2 = \lambda_1^2 B_{W_1}^2 + \lambda_2^2 B_{W_2}^2 + \cdots ,$$

while of course

$$B^2 = B_{W_1}^2 + B_{W_2}^2 + \cdots .$$

Since $\lambda_j^2 \leq \|h\|^2$ for all j, it follows easily that $\|h(B)\xi\| \leq \|h\|\|B\xi\|$ if $\xi \in \varinjlim \mathfrak{s}(V_j)$. The inequality shows that the domain of the closure of B is contained in the domain of the closure of $h(B)$. Furthermore, by a continuity argument, the inequality extends to the domain of the closure of B. Since B is essentially self-adjoint on $\varinjlim \mathfrak{s}(V_j)$, the domain of its closure certainly contains $\mathfrak{s}(H)$, and so the lemma is proved for diagonalizable operators.

A general bounded self-adjoint operator is a sum $h_1 + h_2$ of bounded, self-adjoint and diagonalizable operators, and we can even ensure that h_2 has norm no more than any preassigned $\epsilon > 0$. This follows, for instance, from the Weyl-von Neumann Theorem on diagonalizing self-adjoint operators modulo compact operators. Since it is clear from Definition 3.5 that $h(B) = h_1(B) + h_2(B)$ on $\mathfrak{s}(H)$, it follows that $\|h(B)\xi\| \leq (\|h_1\| + \epsilon)\|B\xi\|$, and since ϵ is arbitrarily small, the lemma follows. □

3.8. Lemma. *Let h_1 and h_2 be two self-adjoint operators on H which differ by a bounded operator, and let $H_h \subset H$ be their common domain. Assume that $h_1^2 \geq 1$ and $h_2^2 \geq 1$. Then*

$$\|h_1(B)\xi - h_2(B)\xi\| \leq \|h_1 - h_2\|\|B\xi\|,$$

for every ξ in $\mathfrak{s}(H_h)$.

Proof. The difference $h_1(B) - h_2(B)$ is the operator $h(B)$, where $h = h_1 - h_2$. According to Lemma 3.3, $\mathfrak{s}(H_h) \subset \mathfrak{s}(H)$. So this lemma follows from the previous one. □

The following technical result will be called upon in Sect. 5.

3.9. Lemma. *Let h_1 and h_2 be two positive, self-adjoint operators with compact resolvent which differ by a bounded operator, and set*

$$h_{1,\alpha} = 1 + \alpha h_1, \qquad B_{1,\alpha} = h_{1,\alpha}(B)$$
$$h_{2,\alpha} = 1 + \alpha h_2, \qquad B_{2,\alpha} = h_{2,\alpha}(B).$$

Then $\lim_{\alpha \to 0} \|f(B_{1,\alpha}) - f(B_{2,\alpha})\| = 0$, *for any $f \in C_0(\mathbb{R})$.*

Remark. When we use this lemma in Sect. 5 we shall want to think of $B_{j,\alpha}$ as an unbounded operator on the Hilbert space $\mathcal{H}_\alpha(H)$. The proof below is valid whether or not the Hilbert space $\mathcal{H}(H)$ varies with $\alpha > 0$.

Proof of the Lemma. By the Stone-Weierstrass Theorem, it suffices to prove the lemma for the function $f(x) = (x+i)^{-1}$. By Lemma 3.8, if $\xi \in \mathfrak{s}(H_h)$ then

$$\|B_{1,\alpha}\xi - B_{2,\alpha}\xi\| \leq \alpha \|h_1 - h_2\| \|B\xi\|.$$

If η belongs to the dense subspace $(B_{1,\alpha} + i)\mathfrak{s}(H_h) \subset \mathcal{H}(H)$ then

$$(B_{2,\alpha} + i)^{-1}\eta - (B_{1,\alpha} + i)^{-1}\eta = (B_{2,\alpha} + i)^{-1}(B_{1,\alpha} - B_{2,\alpha})(B_{1,\alpha} + i)^{-1}\eta,$$

so if $\xi = (B_{1,\alpha} + i)^{-1}\eta \in \mathfrak{s}(H_h)$ then

$$\left\|(B_{2,\alpha} + i)^{-1}\eta - (B_{1,\alpha} + i)^{-1}\eta\right\| \leq \|B_{2,\alpha}\xi - B_{1,\alpha}\xi\| \leq \alpha \|h_1 - h_2\| \|B\xi\|.$$

But since $h_{1,\alpha}^2 \geq 1$, it follows from Lemma 3.3 that $\|B\xi\| \leq \|B_{1,\alpha}\xi\|$, and so

$$\|B\xi\|^2 \leq \|B_{1,\alpha}\xi\|^2 < \|B_{1,\alpha}\xi\|^2 + \|\xi\|^2 = \|(B_{1,\alpha} + i)\xi\|^2 = \|\eta\|^2.$$

Therefore,

$$\left\|(B_{2,\alpha} + i)^{-1}\eta - (B_{1,\alpha} + i)^{-1}\eta\right\| \leq \alpha \|h_1 - h_2\| \|\eta\|,$$

which proves the lemma. □

Actually, we shall need a slight strengthening of the Lemma 3.9:

3.10. Lemma. *With the notation of Lemma 3.9,*

$$\lim_{\alpha \to 0} \sup \left\{ \left\| f(s^{-1}B_{1,\alpha}) - f(s^{-1}B_{2,\alpha}) \right\| \,:\, s > 0 \right\} = 0$$

for any $f \in C_0(\mathbb{R})$.

Proof. The previous proof may be repeated verbatim, but with $B_{1,\alpha}$ replaced by $s^{-1}B_{1,\alpha}$, and similarly for $B_{2,\alpha}$ and B. □

To illustrate the purpose of Lemma 3.9, let us note the following simple consequence:

3.11. Proposition. *Suppose that a group G acts on H by linear isometries. Let h be a positive self-adjoint operator on H and suppose that $g(h) - h$ is a bounded operator, for every $g \in G$. If $B_\alpha = h_\alpha(B)$, as in Lemma 3.9, then*

$$\lim_{\alpha \to 0} \| f(B_\alpha) - g(f(B_\alpha)) \| = 0,$$

for every $g \in G$ and $f \in C_0(\mathbb{R})$.

Thus despite the fact that $h(B)$ is not equivariant, the operators $h_\alpha(B)$ constitute a sort of 'asymptotically equivariant' family of operators.

Proof of the Proposition. Set $h = h_1$ and $g(h) = h_2$, and define $B_{1,\alpha}$ and $B_{2,\alpha}$ as in Lemma 3.9. Then since B is G-equivariant, $g(f(B_{1,\alpha}) = f(B_{2,\alpha})$. So the proposition follows immediately from Lemma 3.9. □

Remark. As in 3.10, it also follows that $\lim_{\alpha \to 0} \| f(s^{-1} B_\alpha) - g(f(s^{-1} B_\alpha)) \| = 0$, uniformly for $s > 0$. This will be used in Sect. 6.

This concludes that part of our discussion of the functional calculus $h \mapsto h(B)$ which is needed for this paper. But because it may be of interest elsewhere, we shall conclude this section with a further description of $h(B)$. Note first that for any subspace $E \subset H$ we can define the Bott-Dirac operator $B_E \colon \mathfrak{s}(H) \to \mathfrak{s}(H)$ to be the limit, as in Definition 2.8, of operators B_V as V ranges over the finite-dimensional subspaces of E. For two orthogonal subspaces E' and E'' in H, one has $B_{E'+E''} = B_{E'} + B_{E''}$ and $B^2_{E'+E''} = B^2_{E'} + B^2_{E''}$.

3.12. Definition. Let h be a self-adjoint operator on H. Using the Spectral Theorem, let us write it in the integral form

$$h = \int_{-\infty}^{+\infty} t \, dP_t$$

where dP_t is a projection-valued measure on $\mathbb{R}$. Thus h is the limit of Riemann sums $\sum_k t_k (P_{t_{k+1}} - P_{t_k})$. We define an operator $h(B)$ on $\mathcal{H}_\alpha$ as the limit of the Riemann sums $\sum_k t_k B_k$ where B_k is the operator B_E for the subspace $E = Im(P_{t_{k+1}} - P_{t_k})$.

We leave to the reader the task of making precise and verifying the convergence of the Riemann sums. Symbolically one can write:

$$h = \int_{-\infty}^{+\infty} t \, dP_t, \qquad h(B) = \int_{-\infty}^{+\infty} t B_{dP_t H}.$$

4. The C^*-algebra of a Hilbert space

In this section we shall review the construction in [15] of a certain non-commutative C^*-algebra associated to a real Hilbert space H. It plays the role of the algebra of continuous functions on the cotangent space of H which vanish at infinity (note that if H is infinite-dimensional then there are no non-zero, continuous functions on H which vanish at infinity, in the ordinary sense of the term). The algebra described here differs very slightly from the one introduced in [15]. In fact, we will use both the algebra introduced in [15] and a new one which will be now defined.

Let V be a finite-dimensional affine subspace of H and, as in Sect. 2, let V_0 be the associated vector subspace of H.

4.1. Definition. Denote by $\mathcal{L}(V)$ the C^*-algebra of linear operators on the finite-dimensional complex Hilbert space $\Lambda^*(V_0) \otimes \mathbb{C}$. Observe that $\Lambda^*(V_0) \otimes \mathbb{C}$ is a $\mathbb{Z}/2$-graded Hilbert space, and so $\mathcal{L}(V)$ is a $\mathbb{Z}/2$-graded C^*-algebra. Denote by $\mathit{Cliff}(V)$ the subalgebra of $\mathcal{L}(V)$ generated by the Clifford multiplication operators $c(v)$ (defined in 2.4) for all $v \in V$.

4.2. Definition. Denote by $\mathcal{C}(V)$ the $\mathbb{Z}/2$-graded C^*-algebra $C_0(V_0 \times V, \mathcal{L}(V))$ of continuous, $\mathcal{L}(V)$-valued functions on $V_0 \times V$ which vanish at infinity, with the $\mathbb{Z}/2$-grading induced from $\mathcal{L}(V)$. Denote by $\tilde{\mathcal{C}}(V)$ the $\mathbb{Z}/2$-graded C^*-algebra $C_0(V, \mathit{Cliff}(V))$ of continuous, $\mathit{Cliff}(V)$-valued functions on V which vanish at infinity, with the $\mathbb{Z}/2$-grading induced from $\mathit{Cliff}(V)$.

The algebra $\tilde{\mathcal{C}}(V)$ was used in [15]; it is the same as the algebra called $C_\tau(V)$ in [22]. Here we will mainly use $\mathcal{C}(V)$ (which is “twice bigger” than $\tilde{\mathcal{C}}(V)$) because it is better suited to the construction we will carry out in the next section. Note that by a well-known isomorphism, $\mathcal{L}(V)$ identifies with the complex Clifford algebra of the Euclidean space $V_0 \times V_0$ – this links both algebras.

4.3. Definition. Denote by $\mathcal{S} = C_0(\mathbb{R})$ the C^*-algebra of continuous complex-valued functions on $\mathbb{R}$ which vanish at infinity. We shall consider $\mathcal{S}$ as $\mathbb{Z}/2$-graded, according to even and odd functions. If A is any $\mathbb{Z}/2$-graded C^*-algebra then denote by $\mathcal{S}A$ the graded tensor product $\mathcal{S}\hat{\otimes}A$. In particular, let $\mathcal{S}\mathcal{C}(V) = \mathcal{S}\hat{\otimes}\mathcal{C}(V)$ and $\mathcal{S}\tilde{\mathcal{C}}(V) = \mathcal{S}\hat{\otimes}\tilde{\mathcal{C}}(V)$. We will also use the following notation:

$$\mathcal{A}(V) = \mathcal{S}\mathcal{C}(V), \qquad \tilde{\mathcal{A}}(V) = \mathcal{S}\tilde{\mathcal{C}}(V).$$

Remark. One should think of the operation of graded tensor product with $\mathcal{S}$ as encoding the grading information from $\mathcal{C}(V)$ in the ungraded C^*-algebra $\mathcal{A}(V)$. When we come to consider KK-theoretic invariants in Sect. 7, we shall consider $\mathcal{A}(V)$ and $\tilde{\mathcal{A}}(V)$ as *ungraded* C^*-algebras (in other words we shall ignore their $\mathbb{Z}/2$-grading). In Sect. 6, where E-theoretic invariants will be discussed, one can consider these C^*-algebras either as graded or as ungraded – both points of view are possible. But since the gradings will not play any role at all the reader may perfectly well ignore them throughout the paper.

Now let V' be an affine subspace of V. Associated to the inclusion $V' \subset V$ we are going to define a natural $*$-homomorphism $\mathcal{A}(V') \to \mathcal{A}(V)$. Having done so we will be able to define

$$\mathcal{A}(H) = \varinjlim_{V \subset H} \mathcal{A}(V),$$

where the C^*-algebra direct limit is over the directed system of all finite-dimensional affine subspaces of H.

4.4. Definition. Suppose that W is a finite-dimensional *linear* subspace of H. The *Bott operator* $\mathfrak{B}_W$ for $\mathcal{C}(W)$ is the function $\mathfrak{B}_W\colon W \times W \to \mathcal{L}(W)$ whose value at (w_1, w_2) is the operator

$$i\bar{c}(w_1) + c(w_2)\colon \Lambda^*(W) \otimes \mathbb{C} \to \Lambda^*(W) \otimes \mathbb{C}$$

(the Clifford multiplication operators $c(w_1)$ and $\bar{c}(w_2)$ were introduced in Definition 2.4). The Bott operator is a grading-degree one, essentially self-adjoint, unbounded multiplier of $\mathcal{C}(W)$, with domain the compactly supported functions in $\mathcal{C}(W)$. Using it, define a $*$-homomorphism $\mathcal{A}(0) \to \mathcal{A}(W)$ by the formula

$$f \mapsto f(X \hat{\otimes} 1 + 1 \hat{\otimes} \mathfrak{B}_W),$$

where $f \in \mathcal{S} = \mathcal{A}(0)$. Here X denotes the operator of multiplication by the function x on $\mathbb{R}$, viewed as a degree one, essentially self-adjoint, unbounded multiplier of $\mathcal{S} = C_0(\mathbb{R})$ with domain the compactly supported functions in $\mathcal{S}$ (see [15]). Similarly, in the case of $\tilde{\mathcal{A}}(W)$, we can use the Bott operator $\mathfrak{B}_W$ on *Cliff*(W) given by the Clifford multiplication $c(w)$ (which is the second variable part of the previous Bott operator) in order to define (by the same formula) a $*$-homomorphism $\tilde{\mathcal{A}}(0) \to \tilde{\mathcal{A}}(W)$. (See again [15].)

Suppose now that V and V' are finite-dimensional affine subspaces of H with $V' \subset V$. There is a canonical inclusion $\mathcal{L}(V') \to \mathcal{L}(V)$ under which the Clifford operators $c(v') \in \mathcal{L}(V')$ are mapped to the corresponding Clifford operators $c(v') \in \mathcal{L}(V)$. So if we denote by $p\colon V \to V'$ the orthogonal projection then by composing with p we can transform any function $V' \to \mathcal{L}(V')$ to a function $V \to \mathcal{L}(V')$, and hence to a function $V \to \mathcal{L}(V)$. This gives a $*$-homomorphism from $\mathcal{C}(V')$ into the multiplier algebra of $\mathcal{C}(V)$ (but not into $\mathcal{C}(V)$ itself, since the transformed functions will not generally vanish at infinity).

Let W be the orthogonal complement of V_0' in V_0. There is similarly a canonical inclusion $\mathcal{L}(W) \to \mathcal{L}(V)$, and also a 'projection' $q\colon V \to W$ defined by $q(v) = v - p(v)$. Using these, the Bott operator for W may be viewed as an unbounded multiplier of $\mathcal{C}(V)$, and the operator $X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_W$ as an unbounded multiplier of $\mathcal{A}(V)$.

We can now define a $*$-homomorphism $\mathcal{A}(V') \to \mathcal{A}(V)$ by the formula:

$$\mathcal{A}(V') = \mathcal{S}\hat{\otimes}\mathcal{C}(V') \ni f\hat{\otimes}h \mapsto f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_W) \cdot h \in \mathcal{A}(V).$$

Note that although generally neither of the multipliers $f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_W)$ or h belongs to $\mathcal{A}(V)$, the product always does.

The map $\mathcal{A}(V') \to \mathcal{A}(V)$ may also be described as follows. The canonical inclusions $\mathcal{L}(V') \to \mathcal{L}(V)$ and $\mathcal{L}(W) \to \mathcal{L}(V)$ induce an isomorphism

$$\mathcal{L}(V) \cong \mathcal{L}(W)\hat{\otimes}\mathcal{L}(V'),$$

and consequent isomorphisms

$$\mathcal{C}(V) \cong \mathcal{C}(W)\hat{\otimes}\mathcal{C}(V')$$
$$\mathcal{A}(V) \cong \mathcal{A}(W)\hat{\otimes}\mathcal{C}(V').$$

We can now form a $*$-homomorphism

$$\mathcal{A}(V') \cong \mathcal{A}(0)\hat{\otimes}\mathcal{C}(V') \to \mathcal{A}(W)\hat{\otimes}\mathcal{C}(V') \cong \mathcal{A}(V)$$

by tensoring the Bott homomorphism of Definition 4.4 with the identity on $\mathcal{C}(V')$.

Quite similarly, there exists also a $*$-homomorphism $\tilde{\mathcal{A}}(V') \to \tilde{\mathcal{A}}(V)$ (see [15]).

4.5. Lemma. *The $*$-homomorphisms $\mathcal{A}(V') \to \mathcal{A}(V)$ we have just defined are transitive with respect to a triple of inclusions $V'' \subset V' \subset V$.*

Proof. This is proved in [15, Proposition 3.2] for inclusions of linear subspaces; the proof for affine inclusions is just the same. □

4.6. Definition. Define the C^*-algebras

$$\mathcal{A}(H) = \varinjlim_{V\subset H} \mathcal{A}(V), \qquad \tilde{\mathcal{A}}(H) = \varinjlim_{V\subset H} \tilde{\mathcal{A}}(V),$$

where the inductive limit is taken over the directed set of all finite-dimensional affine subspaces of H.

As in Lemma 2.3, if H' is a dense subspace of H then the inclusion

$$\varinjlim_{V\subset H'} \mathcal{A}(V) \hookrightarrow \varinjlim_{V\subset H} \mathcal{A}(V)$$

is an isomorphism of C^*-algebras. In particular, if $V_1 \subset V_2 \subset \cdots$ is an increasing family of finite-dimensional affine subspaces of H, whose union is dense in H, then the canonical isometric inclusion of $\varinjlim \mathcal{A}(V_n)$ into $\mathcal{A}(H)$ is a unitary isomorphism. The same is true for $\tilde{\mathcal{A}}(H)$.

We may regard V as a subspace of $V_0 \times V$ via the embedding $v \mapsto (0, v)$. In the same way as in Definition 4.4, we define a $*$-homomorphism $\tilde{\mathcal{A}}(0) \to \tilde{\mathcal{A}}(V_0)$ using the Bott operator on V_0 which is the Clifford multiplication operator $i\bar{c}(v)$ (i.e. the first variable part of the Bott operator of Definition 4.4). Now repeating the construction following Definition 4.4, we get an embedding $\tilde{\mathcal{A}}(V) \subset \mathcal{A}(V)$ for any affine subspace $V \subset H$. The transitivity of the construction (similar to Lemma 4.5) guarantees that we can pass to an inductive limit:

4.7. Lemma. *The above procedure defines an embedding* $\tilde{\mathcal{A}}(H) \subset \mathcal{A}(H)$. □

Suppose now that a second-countable, locally compact group G acts continuously by affine isometries on the real Hilbert space H. Let us write the action of G on H in the form:

$$g \cdot v = \pi(g)v + \kappa(g)$$

where π is a linear orthogonal representation of G on H and κ is a one-cocycle on G with values in H. Thus κ is a continuous function from G into H such that $\kappa(g_1 g_2) = \pi(g_1)\kappa(g_2) + \kappa(g_1)$.

The C^*-algebras $\mathcal{A}(H)$ and $\tilde{\mathcal{A}}(H)$ carry a natural action of our group G, and equipped with this action, they are G-C^*-algebra – that is, the action is continuous.

4.8. Definition. The affine isometric action of G on H is *metrically proper* if $\lim_{g\to\infty} \|\kappa(g)\| = \infty$.

We recall from [11,14,22] that a G-C^*-algebra A is *proper* if there is a locally compact, second-countable, proper G-space Z and an equivariant $*$-homomorphism from $C_0(Z)$ into the center of the multiplier algebra of A such that $C_0(Z) \cdot A$ is dense in A.

4.9. Proposition. *If G acts metrically properly on H then $\mathcal{A}(H)$ and $\tilde{\mathcal{A}}(H)$ are proper G-C^*-algebras.*

Proof. The following elegant argument, which much improves our original proof, is due to G. Skandalis. We will give the proof for $\mathcal{A}(H)$, the proof for $\tilde{\mathcal{A}}(H)$ is similar.

For any non-zero, finite-dimensional affine subspace $V \subset H$, the center $\mathcal{Z}(V)$ of the C^*-algebra $\mathcal{A}(V)$ is the algebra

$$\mathcal{Z}(V) = \mathcal{S}^{ev} \otimes C_0(V_0 \times V),$$

where $\mathcal{S}^{ev}$ is the subalgebra of all even functions in $\mathcal{S}$. This algebra is isomorphic to the algebra of continuous functions, vanishing at infinity, on the locally compact space $[0,\infty) \times V_0 \times V$. The maps of our inductive system, $\mathcal{A}(V') \to \mathcal{A}(V)$, corresponding to embeddings $V' \subset V$, carry these subalgebras into each other: $\mathcal{Z}(V') \to \mathcal{Z}(V)$. So we can form the direct limit $\mathcal{Z}(H)$. It has the property that $\mathcal{Z}(H) \cdot \mathcal{A}(H)$ is dense in $\mathcal{A}(H)$. Its Gelfand spectrum is the locally compact space $Z = [0,\infty) \times H \times H$, where Z is given the weakest topology for which the projection to $H \times H$ is weakly continuous and the function

$$Z = [0,\infty) \times H \times H \ni (t, v_1, v_2) \mapsto t^2 + \|v_1\|^2 + \|v_2\|^2 \in \mathbb{R}$$

is continuous. If G acts metrically properly on H then the induced action on the locally compact space Z (which is the given affine action of G on the second factor H and the linear part of this action on the first) is proper, in the ordinary sense of the term. □

5. A continuous field of C^*-algebras associated to H

In Sect. 2 we fixed a scalar $\alpha > 0$ and constructed from the real Hilbert space H a complex Hilbert space of differential forms $\mathcal{H}(H)$. Let us now make the dependance of the complex Hilbert space $\mathcal{H}(H)$ on α explicit by writing $\mathcal{H}_\alpha(H)$.

The collection of all the Hilbert spaces $\mathcal{H}_\alpha(H)$, for $\alpha > 0$, constitutes a continuous field in a fairly obvious fashion: the space of continuous sections is generated by the continuous functions of $\alpha > 0$ with values in $\mathcal{H}(V)$, as V varies through the finite-dimensional affine subspaces of H. Associated to the continuous field of Hilbert spaces $\{\mathcal{H}_\alpha(H)\}_{0<\alpha<\infty}$ is the continuous field of elementary $\mathbb{Z}/2$-graded C^*-algebras $\{\mathcal{K}(\mathcal{H}_\alpha(H))\}_{0<\alpha<\infty}$.

5.1. Definition. Denote by $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$ the continuous field of C^*-algebras obtained by tensoring the elementary field $\{\mathcal{K}(\mathcal{H}_\alpha(H))\}_{0<\alpha<\infty}$ with the $\mathbb{Z}/2$-graded C^*-algebra $\mathcal{S} = C_0(\mathbb{R})$ of Definition 4.3. Thus

$$\mathcal{A}_\alpha(H) = \mathcal{S}\mathcal{K}(\mathcal{H}_\alpha(H)) = \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}_\alpha(H)) \qquad (0 < \alpha < \infty).$$

In addition, denote by $\mathcal{A}_0(H)$ the C^*-algebra $\mathcal{A}(H)$ of Sect. 4.

Remark. We refer the reader to the article of Kirchberg and Wassermann [24] for a detailed discussion of tensor products and continuous fields of C^*-algebras. Kirchberg and Wassermann treat ungraded tensor products, whereas in our case the graded tensor product is used. But the elementary facts we need – for instance that the tensor product of a continuous field by a nuclear C^*-algebra is again a continuous field – are not affected by the grading.

The aim of this section is to extend the continuous field $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$ to a continuous field of C^*-algebras $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ over the closed half line $[0,\infty)$.[2] Here is how we shall do it. First we shall consider and solve the same problem in finite dimensions, defining for each finite-dimensional affine subspace V of H a continuous field $\{\mathcal{A}_\alpha(V)\}_{0\le\alpha<\infty}$. Next we shall fix a positive, self-adjoint operator h on H, with compact resolvent, and for each finite-dimensional affine space V lying within the domain of h we shall use the functional calculus of Sect. 3 to define embeddings $\mathcal{A}_\alpha(V)\to\mathcal{A}_\alpha(H)$ for all $\alpha\ge 0$. A continuous section of $\{\mathcal{A}_\alpha(V)\}_{0\le\alpha<\infty}$ will then determine a section of $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$, and we shall deem that the sections obtained in this way – which we shall call the *basic* sections of $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ – generate all continuous sections of the field.

Because our construction depends on the compact-resolvent operator h, the field $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ will not be canonical. However if a second-countable group G acts continuously on H by affine isometries then by choosing h carefully we will obtain from our construction a G-continuous field $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$, meaning that G acts continuously on each fiber algebra, and that the continuous sections of our field are transformed under G to continuous sections. In addition the field $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ will be *completely* canonical at the level of homotopy.

We begin by stating a well-known result in the realm of finite-dimensional spaces. Let V be a finite-dimensional affine subspace of H and for $\alpha\ge 0$ let $C^*_\alpha(V_0, C_0(V))$ be the crossed product C^*-algebra associated to the action of the vector group V_0 on the space V by the action $v_0\cdot v=\alpha v_0+v$.

5.2. Proposition. *The crossed product algebras $C^*_\alpha(V_0, C_0(V))$ constitute the fibers of a continuous field of C^*-algebras over $[0,\infty)$ in the following way: each Schwartz function $V_0\times V\to\mathbb{C}$ defines an element of each crossed product algebra $C^*_\alpha(V_0, C_0(V))$, and the sections so-obtained generate all the continuous sections of $\{C^*_\alpha(V_0, C_0(V))\}_{0\le\alpha<\infty}$.* □

Remark. By 'generate' we mean that the continuous sections are precisely those which are locally approximable by the above 'constant' sections, in the same sense that a function is continuous if and only if it is locally approximable by constant functions.

If $\alpha=0$ then $C^*_\alpha(V_0, C_0(V))\cong C_0(V_0\times V)$ via the Gelfand isomorphism

$$C^*(V_0)\cong C_0(\widehat{V_0})\cong C_0(V_0),$$

where V_0 is identified with its Pontrjagin dual by means of the pairing $(v_1,v_2)\mapsto\exp(i\langle v_1,v_2\rangle)$. If $\alpha>0$ then $C^*_\alpha(V_0, C_0(V))\cong\mathcal{K}(L^2(V))$, via

[2] It is also possible [16] to extend the field to $\alpha=\infty$, setting $\mathcal{A}_\infty(H)=\mathcal{A}(0)$, but we shall not use this here.

the obvious representation of the crossed product on $L^2(V)$. So Proposition 5.2 provides us with a continuous field of the following sort:

$$\begin{cases} C_0(V_0 \times V), & \alpha = 0, \\ \mathcal{K}(L^2(V)), & \alpha > 0. \end{cases}$$

5.3. Example. Suppose that f is a Schwartz function on V and let e be the Gaussian $e(v) = e^{-\|v\|^2}$ on V_0. Then by Fourier analysis, the prescription

$$\begin{cases} e \otimes f \in C_0(V_0 \times V), & \alpha = 0, \\ e^{-\alpha^2 \Delta} M_f \in \mathcal{K}(L^2(V)), & \alpha > 0, \end{cases}$$

where Δ is the Laplace operator on V and M_f is the operator of pointwise multiplication by f, defines a continuous section, in fact a generating 'constant' section, of the above continuous field.

Recall now that $\mathcal{L}(V)$ denotes the C^*-algebra of endomorphisms of the finite-dimensional Hilbert space $\Lambda^* V_0 \otimes \mathbb{C}$. Tensoring the above continuous field with $\mathcal{L}(V)$, we obtain a new continuous field

$$\mathcal{C}_\alpha(V) = \begin{cases} \mathcal{C}(V), & \alpha = 0, \\ \mathcal{K}(\mathcal{H}(V)), & \alpha > 0. \end{cases}$$

By further tensoring with $\mathcal{S} = C_0(\mathbb{R})$ we obtain a continuous field

$$\mathcal{A}_\alpha(V) = \begin{cases} \mathcal{A}(V), & \alpha = 0, \\ \mathcal{S}\mathcal{K}(\mathcal{H}(V)), & \alpha > 0. \end{cases}$$

Suppose now that $V' \subset V$. We are going to define an embedding of continuous fields

$$\{\mathcal{A}_\alpha(V')\}_{0 \le \alpha < \infty} \to \{\mathcal{A}_\alpha(V)\}_{0 \le \alpha < \infty}.$$

As usual, denote by W the orthogonal complement of V_0' in V_0, so that

$$V = W + V'.$$

A pair $(f_W, f_{V'})$ of functions, one on W and one on V', determines a function on V by the formula $f_V(w+v') = f_W(w) f_{V'}(v')$. This construction, together with the isomorphism $\mathcal{L}(V) \cong \mathcal{L}(W) \hat{\otimes} \mathcal{L}(V')$ produces an isomorphism

$$\mathcal{H}(V) \cong \mathcal{H}(W) \hat{\otimes} \mathcal{H}(V')$$

(compare Sect. 4). Thus

$$\mathcal{K}(\mathcal{H}(V)) \cong \mathcal{K}(\mathcal{H}(W)) \hat{\otimes} \mathcal{K}(\mathcal{H}(V')).$$

Bearing in mind what was shown in Sect. 4, we see that there are isomorphisms of continuous fields

$$\{\mathcal{A}_\alpha(V)\}_{0\le\alpha<\infty} \cong \{\mathcal{A}_\alpha(W)\hat{\otimes}\mathcal{C}_\alpha(V')\}_{0\le\alpha<\infty}$$

and of course

$$\{\mathcal{A}_\alpha(V')\}_{0\le\alpha<\infty} \cong \{\mathcal{S}\hat{\otimes}\mathcal{C}_\alpha(V')\}_{0\le\alpha<\infty}$$

for all α. So to define the required embedding of continuous fields we need only embed the constant field with fiber $\mathcal{S}$ into the field $\{\mathcal{A}_\alpha(W)\}_{0\le\alpha<\infty}$. This is done as follows:

5.4. Lemma. *Let $f \in \mathcal{S}$. The prescription*

$$f \mapsto \begin{cases} f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_W), & \alpha = 0, \\ f(X\hat{\otimes}1 + 1\hat{\otimes}B_W), & \alpha > 0 \end{cases}$$

defines a continuous section of the field $\{\mathcal{A}_\alpha(W)\}_{0\le\alpha<\infty}$.

Remark. Of course, $\mathfrak{B}_W$ is the Bott operator from Sect. 4 while B_W is the Bott-Dirac operator from Sect. 2 (whose definition, recall, depends on $\alpha > 0$).

Proof. It suffices to prove the lemma for generators of the C^*-algebra $\mathcal{S} = C_0(\mathbb{R})$, for which we take the functions $f(x) = e^{-x^2}$ and $f(x) = xe^{-x^2}$. For $f(x) = e^{-x^2}$ the prescription in the statement of the lemma produces the section

$$\begin{cases} e^{-x^2}\hat{\otimes}e^{-\mathfrak{B}_W^2}, & \alpha = 0, \\ e^{-x^2}\hat{\otimes}e^{-B_W^2}, & \alpha > 0. \end{cases}$$

But $\mathfrak{B}_W^2$ is the scalar function $\|w_1\|^2 + \|w_2\|^2$ on $W \oplus W$, while as we noted in Sect. 3,

$$B_W^2 = \alpha^2\Delta + \|w\|^2 + \alpha T,$$

where T is a bounded operator. It follows that there is an asymptotic equivalence

$$\exp\left(-B_W^2\right) \sim \exp(-\alpha^2\Delta - \|w\|^2),$$

as $\alpha \to 0$. From here on we shall follow Appendix B of [15]. Mehler's formula asserts that

$$\exp(-\alpha^2\Delta - \|w\|^2) = \exp(-\beta\|w\|^2)\exp(-\gamma\Delta)\exp(-\beta\|w\|^2),$$

where $\beta = \alpha^2/2 + O(\alpha^4)$ and $\gamma = \alpha^2 + O(\alpha^4)$. It follows that there is an asymptotic equivalence

$$\exp\left(-B_W^2\right) \sim \exp\left(-\frac{\alpha^2}{2}\|w\|^2\right)\exp(-\alpha^2\Delta)\exp\left(-\frac{\alpha^2}{2}\|w\|^2\right),$$

as $\alpha \to 0$, and so the Fourier analysis calculation cited in Example 5.3 shows that our section is continuous, as required.

The argument for the function $f(x) = xe^{-x^2}$ is another direct calculation, following Appendix B of [15] again. $\square$

To summarize what we have shown: the embeddings $\mathcal{S} \to \mathcal{A}_\alpha(W)$ of Lemma 5.4 produce an embedding of continuous fields

$$\left\{\mathcal{A}_\alpha(V')\right\}_{0\le\alpha<\infty} \cong \left\{\mathcal{S}\hat{\otimes}\mathcal{C}_\alpha(V')\right\}_{0\le\alpha<\infty} \longrightarrow \left\{\mathcal{A}_\alpha(W)\hat{\otimes}\mathcal{C}_\alpha(V')\right\}_{0\le\alpha<\infty} \cong \left\{\mathcal{A}_\alpha(V)\right\}_{0\le\alpha<\infty}.$$

Let us proceed now to the construction of embeddings

$$\mathcal{A}_\alpha(V) \longrightarrow \mathcal{A}_\alpha(H) \qquad (0 \le \alpha < \infty).$$

Let V be a finite-dimensional affine subspace of H and let $V^\perp$ be the orthogonal complement of vector subspace $V_0 \subset H$. If $V \subset V_1$, and if $W_1 \subset V^\perp$ is the orthogonal complement of V in V_1, then we have already seen that there is an isomorphism of Hilbert spaces $\mathcal{H}(V_1) = \mathcal{H}(W_1)\hat{\otimes}\mathcal{H}(V)$. Passing to the direct limit we obtain an isomorphism

$$\mathcal{H}_\alpha(H) \cong \mathcal{H}_\alpha(V^\perp)\hat{\otimes}\mathcal{H}(V),$$

where on the right hand side $\mathcal{H}_\alpha(V^\perp)$ denotes the construction of Sect. 2, applied to the real Hilbert space $V^\perp$. It follows that

$$\mathcal{K}(\mathcal{H}_\alpha(H)) \cong \mathcal{K}(\mathcal{H}_\alpha(V^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(V)),$$

for every $\alpha > 0$. Since it is also clear from Sect. 4 that

$$\mathcal{A}(H) \cong \mathcal{A}(V^\perp)\hat{\otimes}\mathcal{C}(V),$$

we obtain, for all $\alpha \ge 0$, isomorphisms

$$\mathcal{A}_\alpha(H) \cong \mathcal{A}_\alpha(V^\perp)\hat{\otimes}\mathcal{C}_\alpha(V)$$

analogous to those in the finite-dimensional case. Since

$$\mathcal{A}_\alpha(V) \cong \mathcal{S}\hat{\otimes}\mathcal{C}_\alpha(V),$$

it is natural to attempt to define embeddings $\mathcal{A}_\alpha(V) \to \mathcal{A}_\alpha(H)$ by means of embeddings $\mathcal{S} \to \mathcal{A}_\alpha(V^\perp)$, using for the latter the formula

$$f \mapsto \begin{cases} f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_{V^\perp}), & \alpha = 0, \\ f(X\hat{\otimes}1 + 1\hat{\otimes}B_{V^\perp}), & \alpha > 0 \end{cases}$$

as in the finite-dimensional case, where $f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_{V^\perp})$ now means the element of $\mathcal{A}(V^\perp)$ associated to $f \in \mathcal{S} = \mathcal{A}(0)$ by the canonical inclusion $\mathcal{A}(0) \to \mathcal{A}(V^\perp)$. Unfortunately the Bott-Dirac operator $B_{V^\perp}$ does not have compact resolvent, and so for $\alpha > 0$ the formula does not define an element of $\mathcal{A}_\alpha(H) = \mathcal{S}\mathcal{K}(\mathcal{H}_\alpha(H))$. To remedy this problem we are going to invoke the non-commutative functional calculus of Sect. 3.

Fix a positive, self-adjoint operator h on H with compact resolvent. Having done so, *we shall from here on only consider finite-dimensional affine subspaces V which belong the domain of h*. We shall denote by h^V the compression of h to the real Hilbert space $V^\perp$, and we shall denote by $h^V(B)$ the operator obtained from h^V by applying the functional calculus of Sect. 3 to the Bott-Dirac operator of $V^\perp$. As in Sect. 3 we shall write $h^V_\alpha(B)$ for the functional calculus operator associated to $h^V_\alpha = 1 + \alpha h^V$. This is a self-adjoint, compact-resolvent operator on the Hilbert space $\mathcal{H}_\alpha(V^\perp)$.

Using the above, we define embeddings

$$\mathcal{A}_\alpha(V) = \mathcal{S}\hat{\otimes}\mathcal{C}_\alpha(V) \to \mathcal{A}_\alpha(V^\perp)\hat{\otimes}\mathcal{C}_\alpha(V) \cong \mathcal{A}_\alpha(H)$$

by the formula

$$f\hat{\otimes}T_\alpha \mapsto \begin{cases} f(X\hat{\otimes}1 + 1\hat{\otimes}\mathfrak{B}_{V^\perp})\hat{\otimes}T_\alpha, & \alpha = 0, \\ f\big(X\hat{\otimes}1 + 1\hat{\otimes}h^\alpha_V(B)\big)\hat{\otimes}T_\alpha, & \alpha > 0. \end{cases}$$

We shall call a section of the family of algebras $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ *basic* if it is obtained from a continuous section of a field $\{\mathcal{A}_\alpha(V)\}_{0\le\alpha<\infty}$ using the above embedding $\mathcal{A}_\alpha(V) \to \mathcal{A}_\alpha(H)$. We shall further say that the basic section is *affiliated* to $V \subset H$.

Because the above embeddings are isometric, the pointwise norm of any basic section is a continuous function of $\alpha \ge 0$.

If f is supported in $[-\alpha N, \alpha N]$ and if the compressed operator h^V is bounded below by N (as it will be if the finite-dimensional affine subspace $V \subset H$ is suitably large) then the above embedding takes the element $f\hat{\otimes}T_\alpha \in \mathcal{S}\mathcal{K}(\mathcal{H}(V))$ to $f\hat{\otimes}P^{V^\perp}_\alpha\hat{\otimes}T_\alpha$, where $P^{V^\perp}_\alpha$ is the orthogonal projection onto the unit vector $\xi \in \mathcal{H}_\alpha(V^\perp)$ described in Definition 2.2. Since, as V varies, the operators of the form $P^{V^\perp}_\alpha\hat{\otimes}T_\alpha$ are dense in $\mathcal{K}(\mathcal{H}(H))$, it follows that every continuous section of the field $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$ which vanishes as $\alpha \to 0$ is a uniform limit, over compact subsets of $(0,\infty)$, of basic sections which vanish at $\alpha = 0$.

In what follows, let us refer to a continuous section of the field $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$ which vanishes as $\alpha \to 0$ as a *null* section of the family of algebras $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$. The following calculation is the final step we must take before we are able to give the family $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$ the structure of a continuous field.

5.5. Proposition. *Modulo null sections, the sum or product of any two basic sections is a basic section.*

Let us take this for granted, for a moment, and complete our main construction.

5.6. Definition. A section of the family of algebras $\{\mathcal{A}_\alpha(H)\}_{0\leq\alpha<\infty}$ is *continuous* if it is a uniform limit, over compact subsets of $[0,\infty)$, of basic sections.

The continuous sections do indeed constitute the continuous sections for a continuous field: the pointwise norm of any continuous section is a continuous function (because, as noted above, this is true for the basic sections); the continuous sections form an algebra over the continuous functions on $[0,\infty)$ (by Proposition 5.5 and the preceding discussion about null sections); and every element of every algebra $\mathcal{A}_\alpha(H)$ is the value at α of some continuous section.

Proof of Proposition 5.5. Since the finite-dimensional affine subspaces V of the domain of h form a directed set under inclusion, and since the sum and product of any two basic sections which are affiliated to a single finite-dimensional affine subspace are obviously basic, it suffices to show that if $V \subset V_1 \subset \operatorname{domain}(h)$ then any basic section which is affiliated to V is equal to a basic section affiliated to V_1, modulo a null section. In addition, it suffices to prove this for a basic section affiliated to an 'elementary' section of $\{A_\alpha(V)\}_{0\leq\alpha<\infty}$ of the type $f\hat{\otimes}T_\alpha$. We have previously constructed an embedding of $\{A_\alpha(V)\}_{0\leq\alpha<\infty}$ into $\{A_\alpha(V_1)\}_{0\leq\alpha<\infty}$, and if we apply first this embedding to the section $f\hat{\otimes}T_\alpha$, and then the embedding of $\{A_\alpha(V_1)\}_{0\leq\alpha<\infty}$ into $\{A_\alpha(H)\}_{0\leq\alpha<\infty}$, we obtain the section

$$f\big(X\hat{\otimes}1 + 1\hat{\otimes}h_\alpha^{V_1}(B)\big)\hat{\otimes}T_\alpha \in \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}(V^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(V)).$$

In the formula the operator h^{V_1} (which is, to begin with, the compression of h to $V_1^\perp \subset V^\perp$) is regarded as an operator on $V^\perp$ by setting h^{V_1} equal to zero on the orthogonal complement of $V_1^\perp$ in $V^\perp$. On the other hand, the basic section associated directly to $f\hat{\otimes}T_\alpha$ is

$$f\big(X\hat{\otimes}1 + 1\hat{\otimes}h_\alpha^{V}(B)\big)\hat{\otimes}T_\alpha \in \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}(V^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(V)).$$

Comparing the two formulas, and bearing in mind that the operators h^V and h^{V_1} on $V^\perp$ differ by a bounded operator, we see that the proposition follows from Lemma 3.9. $\square$

Suppose now that a second-countable, locally compact group G acts on the real Hilbert space H by a continuous, affine and isometric action. Then G also acts on the continuous field $\{\mathcal{K}(\mathcal{H}_\alpha(H))\}_{0<\alpha<\infty}$, and hence on $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$, in the sense that each $g \in G$ transforms each continuous section of $\{\mathcal{A}_\alpha(H)\}_{0<\alpha<\infty}$ to another continuous section. Of course, G also acts on the algebra $\mathcal{A}_0(H) = \mathcal{A}(H)$ of Sect. 4. Let us now address the question of whether G acts on the continuous field $\{\mathcal{A}_\alpha(H)\}_{0\leq\alpha<\infty}$ in Definition 5.6.

5.7. Lemma. *Let G be a second countable, locally compact group and suppose that G acts continuously on a real, separable Hilbert space H by affine isometries $g \cdot v = \pi(g)v + \kappa(g)$. There exists a positive, self-adjoint operator h on H with compact resolvent such that $\kappa(g) \in$ domain (h) and $\pi(g)h - h\pi(g)$ is bounded, for any $g \in G$. In particular, the domain of h is G-invariant.*

Remark. We shall say that an operator h as in the lemma is *adapted* to the affine action of G on h.

Proof. Let $V_1, V_2, V_3, \dots$ be any increasing sequence of finite-dimensional subspaces of H whose union is dense in H. Let $K_1 \subset K_2 \subset \dots \subset G$ be an exhaustive system of compact subsets of G. We can define inductively a subsequence $V_{n_1} \subset V_{n_2} \subset \dots \subset H$ such that the orthogonal projections $P_{n_1}, P_{n_2}, \dots$ onto these subspaces have the properties that

$$\|(1 - P_{n_j})\kappa(K_j)\| \leq 2^{-j}$$

and

$$\|(1 - P_{n_{j+1}})\pi(K_j)P_{n_j}\| \leq 2^{-j}.$$

Put $h = \sum_{j=0}^{\infty}(1 - P_{n_j})$ (with $P_{n_0} = 0$). It is clear that $\kappa(G)$ belongs to the domain of this operator. If $Q_{n_j} = P_{n_j} - P_{n_{j-1}}$ then $h = \sum_j nQ_{n_j}$, which clearly has compact resolvent. The operator $h\pi(g) - \pi(g)h$ can be written in the form: $\sum_{k>j}(k-j)(Q_{n_k}\pi(g)Q_{n_j} - Q_{n_j}\pi(g)Q_{n_k})$. Estimating separately the two parts of this sum corresponding to $k = j+1$ and $k > j+1$ we find that $h\pi(g) - \pi(g)h$ is bounded. This implies that domain (h) is invariant under the linear part π of the G-action. But since $\kappa(G) \subset$ domain (h), the last assertion follows. □

5.8. Proposition. *Let G be a second countable, locally compact group which acts continuously on a real, separable Hilbert space H by affine isometries. If h is a positive, self-adjoint, compact-resolvent operator on the Hilbert space H which is adapted to the action of G then the field $\{\mathcal{A}_\alpha(H)\}_{0\leq\alpha<\infty}$ constructed from h is G-continuous.*

Remark. One might in addition ask whether G acts continuously on the C^*-algebra of continuous sections of $\{\mathcal{A}_\alpha(H)\}_{0\leq\alpha<\infty}$ which vanish at $+\infty$. This is so, and is a consequence of the Baire category theorem. See Theorem 1.1.4 of [28].

Proof of the Proposition. It suffices to show that the image of a basic section under an affine isometry g is, modulo null sections, again a basic section. But if we apply g to a basic section of the form

$$f\big(X\hat{\otimes}1 + 1\hat{\otimes}h_\alpha^V(B)\big)\hat{\otimes}T_\alpha \in \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}(V^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(V)).$$

we obtain the section

$$f\big(X\hat{\otimes}1 + 1\hat{\otimes}\tilde{h}_\alpha^{\tilde{V}}(B)\big)\hat{\otimes}g(T_\alpha) \in \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}(\tilde{V}^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(\tilde{V})),$$

where $\tilde{V} = gV$ and $\tilde{h} = \pi(g)h\pi(g^{-1})$. Observe that since h is adapted to the action of G on H, the affine space $\tilde{V}$ belongs to the domain of h and the operator $\tilde{h}$ is a bounded perturbation of h. So it follows from Lemma 3.9 that the latter section is equal to the basic section

$$f\big(X\hat{\otimes}1 + 1\hat{\otimes}h_\alpha^{\tilde{V}}(B)\big)\hat{\otimes}g(T_\alpha) \in \mathcal{S}\hat{\otimes}\mathcal{K}(\mathcal{H}(\tilde{V}^\perp))\hat{\otimes}\mathcal{K}(\mathcal{H}(\tilde{V})),$$

modulo null sections. □

The dependence of the field $\{\mathcal{A}_\alpha(H)\}_{0\leq\alpha<\infty}$ on the operator h is not important because of the following:

5.9. Proposition. *Let G be a second countable, locally compact group and suppose that G acts continuously on a real, separable Hilbert space H by linear isometries. If h_0 and h_1 are two positive, self-adjoint operators on H with compact resolvent, and if both are adapted to the action of G on H, then there exists a G-equivariant, continuous field of C^*-algebras $\{\mathcal{A}_{\alpha,t}(H)\}$ over $[0,\infty)\times[0,1]$ whose restrictions to $0 \in [0,1]$ and $1 \in [0,1]$ are the continuous fields constructed, as in Definition 5.6 from h_0 and h_1, respectively.*

Proof. Let us say that operators h_0 and h_1 are *compatible* if there is such a field. Note that compatibility is an equivalence relation.

If h_0 and h_1 have a common eigenbasis then the linear path h_t of operators between h_0 and h_1, thought of as a path of operators defined on the algebraic span of the common eigenbasis for h_0 and h_1, consists of essentially self-adjoint operators. Indeed the path determines a single essentially self-adjoint, compact resolvent operator k on the Hilbert module $H \otimes C[0,1]$ over $C[0,1]$ which is adapted to the affine action of G on $H \otimes C[0,1]$. Applying the construction of Definition 5.6 to each operator in the family $\{h_t\}$ (or alternatively to k in the context of Hilbert modules) we see that h_0 and h_1 are compatible.

In the general case, fix eigenbases for H associated to the operators h_0 and h_1, and choose an orthogonal operator T mapping the eigenbasis for h_0 bijectively to that of h_1. Since the orthogonal group of a Hilbert space is connected (in the strong, or even the norm topology), there is a path joining T to the identity operator. It follows that there exists a continuous family of orthonormal bases for H, parametrized by $t \in [0,1]$, which is at $t = 0$ the eigenbasis for h_0 and at $t = 1$ the eigenbasis for h_1. Let us regard this family of orthonormal bases as an orthonormal basis for the Hilbert module $H\otimes C[0,1]$. Starting with the subspaces V_n spanned by the first n-elements of our eigenbasis for $H \otimes C[0,1]$, the same construction as in the proof of Lemma 5.7 gives a self-adjoint, compact resolvent operator k on this Hilbert module which is adapted to the action of G on $H \otimes C[0,1]$. Applying the construction of Definition 5.6 to k we see that k_0 and k_1 are compatible. But the operator k, restricted to $t = 0$, commutes with h_0 while the operator k, restricted to $t = 1$, commutes with h_1. It follows that h_0 is compatible with h_1, as required. □

6. The Dirac and dual Dirac elements in *E*-theory

In this section we shall use ideas from the theory of asymptotic morphisms to construct ‘Dirac’ and ‘dual Dirac’ elements in E-theory. We shall then be able to prove the E-theoretic version of the Baum-Connes conjecture, at least for *discrete* groups which act properly and isometrically on Hilbert space, and for the assembly map $\mu_{\max}$. To fully prove the results stated in Sect. 1 we shall need to translate our constructions from E-theory to KK-theory. Because this translation is in places rather delicate we shall need to carry out the present E-theoretic calculations in a somewhat more refined context than would otherwise be necessary.

Let G be a second countable, locally compact topological group and let A and B be separable G-C^*-algebras.

6.1. Definition. (See [5], [11], [28].) An *equivariant asymptotic morphism* from A to B is a family of functions $\{\varphi_t\}_{t\in[1,\infty)} : A \to B$ such that:

- $t \mapsto \varphi_t(a)$ is bounded and norm-continuous on $[1,\infty)$, for any $a \in A$;
- $(g, a) \mapsto g(\varphi_t(a))$ is norm-continuous in $g \in G$ and $a \in A$, uniformly in $t \in [1,\infty)$; and
- for every $a, a_1, a_2 \in A$, $g \in G$ and $\alpha \in \mathbb{C}$,

$$\lim_{t\to\infty} \left\{ \begin{array}{c} \varphi_t(a_1a_2) - \varphi_t(a_1)\varphi_t(a_2) \\ \varphi_t(a_1 + a_2) - \varphi_t(a_1) - \varphi_t(a_2) \\ \varphi_t(\alpha a) - \alpha\varphi_t(a) \\ \varphi_t(a^*) - \varphi_t(a)^* \\ \varphi_t(g(a)) - g(\varphi_t(a)) \end{array} \right\} = 0.$$

We shall usually indicate an asymptotic morphism as above by the notation $\varphi\colon A \twoheadrightarrow B$. The apparently restrictive uniform continuity condition in the definition is really quite innocent: see Proposition 1.1.3 of [28].

Obviously, an equivariant $*$-homomorphism $\varphi\colon A \to B$ determines an equivariant asymptotic morphism for which $\varphi_t = \varphi$, for all t. Slightly less trivial is the assertion that an asymptotically G-equivariant family of $*$-homomorphisms $\varphi_t\colon A \to B$, where by ‘asymptotically G-equivariant’ we mean that $\varphi_t(g(a)) - g(\varphi_t(a)) \to 0$ pointwise, as $t \to \infty$, is an equivariant asymptotic morphism from A to B. Uniform continuity in a and g follows from Theorem 1.1.4 of [28].

One of the two main constructions in the theory of asymptotic morphisms associates to any extension of separable G-C^*-algebras

$$0 \to J \to E \to A \to 0$$

an asymptotic morphism $\varphi\colon \Sigma A \twoheadrightarrow J$, where $\Sigma A = C_0(0,1) \otimes A$. Recall first the following notion:

6.2. Definition. A *continuous quasicentral approximate unit* for a separable G-C^*-algebra E containing an ideal J is a continuous family $\{u_t\}_{1\le t<\infty}$ of positive elements in the unit ball of J for which

- $\lim_{t\to\infty} \|u_t x - x\| = 0$, for every $x \in J$;
- $\lim_{t\to\infty} \|u_t x - x u_t\| = 0$ for every $x \in E$; and
- $\lim_{t\to\infty} \|g(u_t) - u_t\| = 0$ uniformly on compact subsets of G.

See [11], Lemma 6.3 or [28], Lemma 1.2.1, for the existence of such families in the case of separable C^*-algebras. The following lemma describes the asymptotic morphism associated to an extension:

6.3. Lemma. (See [6], Lemma 10; [11], Proposition 6.5; [28], Lemma 1.2.2.) *Given an extension*

$$0 \to J \to E \to A \to 0$$

of separable G-C^*-algebras, let $s : A \to E$ be an arbitrary cross-section (not necessarily linear, or equivariant, or even continuous) and $\{u_t\}$ a continuous quasicentral approximate unit for E. There exists an asymptotic morphism $\varphi : \Sigma A \twoheadrightarrow J$ such that $\varphi_t(f \otimes a)$ is asymptotically equivalent to $f(u_t)s(a)$, for every $f \in \Sigma = C_0(0,1)$ and every $a \in A$. □

We will call this asymptotic morphism $\varphi\colon \Sigma A \twoheadrightarrow J$ the *central* asymptotic morphism associated to the above extension. Of course it depends on the choice of continuous quasicentral approximate unit $\{u_t\}$, and to be accurate, even given $\{u_t\}$, Lemma 6.3 only determines the central asymptotic morphism $\varphi\colon \Sigma A \twoheadrightarrow J$ up to asymptotic equivalence. But its homotopy class is well-defined, where homotopy is defined via asymptotic morphisms $A \twoheadrightarrow B[0,1]$.

A commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & E_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_2 & \longrightarrow & E_2 & \longrightarrow & A_2 & \longrightarrow & 0
\end{array}$$

in the category of separable G-C^*-algebras and equivariant $*$-homomorphisms gives rise to a homotopy-commutative diagram

$$\begin{array}{ccc}
\Sigma A_1 & \twoheadrightarrow & J_1 \\
\downarrow & & \downarrow \\
\Sigma A_2 & \twoheadrightarrow & J_2
\end{array}$$

of asymptotic morphisms and $*$-homomorphisms. See [28], 1.2.10 and 1.2.11; or [11], 6.8. The construction of the central asymptotic morphism is normalized in the following way: the central asymptotic morphism

$\Sigma A \to \Sigma A$ associated to the extension

$$0 \to C_0(0,1) \otimes A \to C_0[0,1) \otimes A \to A \to 0$$

is (homotopic to) the identity morphism.

The other main construction in the theory of asymptotic morphisms is the composition of asymptotic morphisms, at the level of homotopy. For this we refer the reader to [6], [11] or [28].

The following definition is slightly more subtle than is needed for E-theory; but it will be important when we come to relate some subsequent constructions to KK-theory.

6.4. Definition. Let A and B be separable G-C^*-algebras. Define a commutative semigroup $\{A, B\}_G$ of equivariant asymptotic morphisms as follows. Representing cycles are equivariant asymptotic morphisms $A \to\!\!\!\to \mathcal{K}(\mathcal{E})$ where $\mathcal{E}$ is a separable G-Hilbert module over B. The equivalence relation is homotopy: two morphisms are homotopic if there is a separable G-Hilbert module $\tilde{\mathcal{E}}$ over $B \otimes C[0,1]$ and an equivariant asymptotic morphism $A \to\!\!\!\to \mathcal{K}(\tilde{\mathcal{E}})$ whose restrictions to the end points of the interval $[0, 1]$ are the two initial morphisms. The sum is defined using the sum of Hilbert modules.

We shall be particularly interested in the case where A is replaced by ΣA, or indeed by $\Sigma^k A$, where as above $\Sigma A = C_0(0,1) \otimes A$ and we define $\Sigma^k A = \Sigma\Sigma^{k-1} A$. As long as $k \geq 1$, the semigroup $\{\Sigma^k A, B\}_G$ is a group: the inverse is defined by mapping one of the tensor multiples Σ into itself with the opposite orientation: $(0,1) \to (0,1) : x \mapsto 1 - x$.

We note that the presence of possibly non-trivial Hilbert modules in Definition 6.4 makes some aspects of the analysis of $\{A, B\}_G$ rather delicate. In particular, while any equivariant asymptotic morphism $A_1 \to\!\!\!\to A_2$ of separable algebras induces a map $\{A_2, B\}_G \to \{A_1, B\}_G$ by composition of asymptotic morphisms, it is generally more complicated to compose elements in say $\{A, B\}_G$ and $\{B, C\}_G$. We shall not attempt to do so here.

Returning to extensions and central asymptotic morphisms, if the ideal J is the C^*-algebra of compact operators on a separable Hilbert B-module then we shall consider the central asymptotic morphism associated to the extension

$$0 \to J \to E \to A \to 0$$

as an element of the group $\{\Sigma A, B\}_G$, and we will call this element the *central invariant* of the extension.

We recall now a few of the features of the equivariant E-theory groups $E_G(A, B)$ defined in [11]. These groups constitute the morphism groups in an additive category, whose objects are the separable G-C^*-algebras. More precisely:

$$E_G(A, B) = \{\Sigma A \otimes \mathcal{K}(\mathcal{H}), \Sigma B \otimes \mathcal{K}(\mathcal{H})\}_G,$$

where $\mathcal{H}$ is the standard G-Hilbert space $L^2(G) \otimes \mathcal{H}_0$. In view of the stabilization theorem for Hilbert modules, the presence of the factor $\mathcal{K}(\mathcal{H})$ in the right hand side of this formula means that all Hilbert modules which appear in Definition 6.4 may be taken to be trivial here (in other words, the same group will result whether or not we restrict to trivial modules $A \otimes \mathcal{H}$).

The associative composition law

$$E_G(A, B) \otimes E_G(B, C) \to E_G(A, C)$$

is given by composition of asymptotic morphisms (the additional factors of $\mathcal{K}(\mathcal{H})$ involved in the definition of $E_G(A, B)$ simplify the general problem of composing asymptotic morphisms at the level of $\{\ ,\ \}_G$-classes). This pairing is the E-theoretic analogue of a similar composition law in KK-theory.

There is a map from $\{A, B\}_G$ into $E_G(A, B)$, given by tensoring asymptotic morphisms with the identity maps on Σ and on $\mathcal{K}(\mathcal{H})$. This process is compatible with composition, to the extent that the composition of morphisms producing elements in $\{A, B\}_G$ and $\{B, C\}_G$ is defined.

By means of the central invariant, every short exact sequence of separable G-C^*-algebras

$$0 \to J \to E \to A \to 0$$

determines a functorial morphism $\Sigma A \to J$ in the equivariant E-theory category.

The C^*-algebra $\Sigma^2 = C_0(0, 1) \otimes C_0(0, 1)$ (with trivial G action) is isomorphic in the equivariant E-theory category to $\mathbb{C}$. This is a form of *Bott periodicity.* There is a tensor product functor in E-theory, compatible with the maximal tensor product $\otimes_{\max}$ on separable C^*-algebras. So it follows from Bott periodicity that $\Sigma^2 A \cong A$ in E-theory, for all A.

The functor on the category of separable G-C^*-algebras and equivariant $*$-homomorphisms which associates to a G-C^*-algebra A the K-theory of the full crossed product $C^*(G, A)$ factors through the E-theory category (thus for example if A_1 and A_2 are isomorphic in the E-theory category then $C^*(G, A_1)$ and $C^*(G, A_2)$ have isomorphic K-theory groups). This makes E-theory, like KK-theory, a powerful tool for calculating the K-theory of group C^*-algebras. The following result is proved in [11] (where a full description of the Baum-Connes assembly map may also be found):

6.5. Theorem. *If G is a discrete group and if the identity morphism $\mathbb{C} \to \mathbb{C}$ factors in the equivariant E-theory category through a proper G-C^*-algebra, then for any separable G-C^*-algebra A the Baum-Connes assembly map*

$$\mu_{\max}\colon E_G(\mathcal{E}G, A) \to K_*(C^*(G, A))$$

is an isomorphism. If $C^*_{\mathrm{red}}(G)$ is an exact C^*-algebra then the same hypotheses imply that the assembly map

$$\mu_{\mathrm{red}}\colon E_G(\mathcal{E}G, A) \to K_*\big(C^*_{\mathrm{red}}(G, A)\big)$$

is also an isomorphism. □

Remark. We recall that a C^*-algebra D is *exact* if tensoring by D, using the minimal C^*-algebra tensor product, preserves short exact sequences. It is known that if G is a discrete subgroup of a connected Lie group then $C^*_{\mathrm{red}}(G)$ is exact. See [25].

We shall now apply the above information to prove the isomorphism of $\mu_{\max}$ (and of μ_{red} in the exact case) for any discrete group which admits a metrically proper affine isometric action on a Hilbert space. The case of non-discrete, or non-exact, groups will be the subject of the remaining sections.[3]

Let G be a second-countable, locally compact group acting properly and isometrically on a real Hilbert space H.

6.6. Definition. The *dual Dirac element* $\tilde{\beta} \in \{\mathcal{S}, \tilde{\mathcal{A}}(H)\}_G$ is the class in the group $\{\mathcal{S}, \tilde{\mathcal{A}}(H)\}_G$ of the asymptotically G-equivariant family of $*$-homomorphisms $\varphi_t\colon \mathcal{S} = \tilde{\mathcal{A}}(0) \to \tilde{\mathcal{A}}(H)$ defined by composing the embedding $\varphi\colon \tilde{\mathcal{A}}(0) \to \tilde{\mathcal{A}}(H)$ associated to the inclusion $0 \to H$ with the $*$-homomorphisms $\mathcal{S} \to \mathcal{S}$ induced from the family of contractions $x \mapsto t^{-1}x$ on $\mathbb{R}$. Thus $\varphi_t(f) = \varphi(f_t)$, where $f_t(x) = f(t^{-1}x)$. Exactly the same definition applies also to the algebra $\mathcal{A}(H)$ and gives an element $\beta \in \{\mathcal{S}, \mathcal{A}(H)\}_G$ which we will also call the *dual Dirac element*. Note that the element β is the image of $\tilde{\beta}$ under the natural embedding $\tilde{\mathcal{A}}(H) \to \mathcal{A}(H)$ constructed in Lemma 4.7. We shall sometimes also use the term 'dual-Dirac element' to refer to the asymptotic morphism, as opposed to its homotopy class.

Let h be a compact-resolvent, positive self-adjoint operator on the Hilbert space H which is adapted to the action of G on H, as in Lemma 5.7. As in Sect. 5, associated to h there is a continuous field $\{\mathcal{A}_\alpha(H)\}_{0\le\alpha<\infty}$. Let us denote by $\mathcal{F}_h(H)$ the C^*-algebra of continuous sections of the field $\{\mathcal{A}_\alpha(H)\}$ which vanish at $\alpha = \infty$. Evaluation at $\alpha = 0$ produces an extension of G-C^*-algebras

$$0 \to \mathcal{K}(\mathcal{S}\hat{\otimes}\mathcal{E}) \to \mathcal{F}_h(H) \to \mathcal{A}(H) \to 0,$$

in which $\mathcal{E}$ denotes the Hilbert $C_0(0, \infty)$-module of sections of the field of Hilbert spaces $\{\mathcal{H}_\alpha(H)\}$ which vanish both at $\alpha = 0$ and $\alpha = \infty$, and $\mathcal{S}\hat{\otimes}\mathcal{E}$

[3] The case of non-discrete groups could probably be handled by appropriately extending the results of [29] from KK-theory to E-theory. The case of non-exact groups could be handled within E-theory as in [14], but as explained in the introduction we shall obtain an improved result in the coming sections.

is the corresponding Hilbert module over $\mathcal{S}\Sigma = \mathcal{S} \otimes C_0(0, \infty)$. (We identify here $\Sigma = C_0(0, 1)$ with $C_0(0, \infty)$ via an orientation preserving continuous monotone map $(0, 1) \to (0, \infty)$.)

6.7. Definition. The *Dirac element* $\alpha \in \{\Sigma\mathcal{A}(H), \mathcal{S}\Sigma\}_G$ is the central invariant of the extension

$$0 \to \mathcal{K}(\mathcal{S}\hat{\otimes}\mathcal{E}) \to \mathcal{F}_h(H) \to \mathcal{A}(H) \to 0$$

just described. The *Dirac element* $\tilde{\alpha} \in \{\Sigma\tilde{\mathcal{A}}(H), \mathcal{S}\Sigma\}_G$ is the restriction of α to the subalgebra $\tilde{\mathcal{A}}(H)$ of $\mathcal{A}(H)$. (We shall sometimes also use the term 'Dirac element' to refer to the asymptotic morphism rather than to its homotopy class.)

We noted in Proposition 5.9 that the homotopy class of the above extension does not depend on h. As a result, two different choices for h produce the same Dirac element.

The Dirac and dual Dirac elements determine classes $\alpha \in E_G(\Sigma\mathcal{A}(H), \mathcal{S}\Sigma)$ and $\Sigma\beta \in E_G(\Sigma\mathcal{S}, \Sigma\mathcal{A}(H))$ (the latter by tensoring the dual Dirac asymptotic morphism by an additional copy of Σ). We are going to prove the following result:

6.8. Theorem. *The E-theory composition*

$$\Sigma\mathcal{S} \xrightarrow{\Sigma\beta} \Sigma\mathcal{A}(H) \xrightarrow{\alpha} \mathcal{S}\Sigma$$

of the Dirac and dual Dirac morphisms is equal in E-theory to the transposition isomorphism $\tau\colon \Sigma\mathcal{S} \to \mathcal{S}\Sigma$. The same is also true for $\tilde{\mathcal{A}}(H)$ in place of $\mathcal{A}(H)$.

Bearing in mind Proposition 4.9 and Theorem 6.5, we will obtain the following immediate consequence:

6.9. Corollary. *Let G be a countable discrete group which admits a metrically proper, affine-isometric action on a Hilbert space. For any separable G-C^*-algebra A the Baum-Connes assembly map*

$$\mu_{\max}\colon E_G(\mathcal{E}G, A) \to K_*(C^*(G, A))$$

is an isomorphism. If $C^*_{\mathrm{red}}(G)$ is an exact C^*-algebra then the same hypotheses imply that the assembly map

$$\mu_{\mathrm{red}}\colon E_G(\mathcal{E}G, A) \to K_*\big(C^*_{\mathrm{red}}(G, A)\big)$$

is also an isomorphism. □

Actually, with the requirements of the coming sections in mind, we shall prove a more precise version of Theorem 6.8:

6.10. Theorem. *The composition of asymptotic morphisms*

$$\alpha \cdot \Sigma\beta : \Sigma\mathcal{S} \twoheadrightarrow \Sigma\mathcal{A}(H) \twoheadrightarrow \mathcal{K}(\mathcal{S}\mathcal{E})$$

is equal to the homotopy class of the transposition map $1 : \Sigma\mathcal{S} \to \mathcal{S}\Sigma$ *in the group* $\{\Sigma\mathcal{S}, \mathcal{S}\Sigma\}_G$.

Remark. This theorem certainly implies Theorem 6.8.

Proof of Theorem 6.10. In view of the observations made in Definitions 6.6 and 6.7, it is enough to prove the assertion for $\mathcal{A}(H)$. The assertion for $\tilde{\mathcal{A}}(H)$ will follow.

Write the affine, isometric action of G as

$$g \cdot v = \pi(g)v + \kappa(g),$$

where π is a linear, orthogonal representation of G on H. In order to prove that the composition $\alpha \cdot \Sigma\beta$ is equal to 1, let us consider a family of affine G-actions

$$g_s \cdot v = \pi(g)v + s\kappa(g),$$

parametrized by $s \in [0, 1]$. Observe that when $s = 1$ we recover the initial affine action, whereas when $s = 0$ the action is *linear.* Denote by $\mathcal{A}(H)_s$ the C^*-algebra $\mathcal{A} = \mathcal{A}(H)$, but with the scaled G-action $(g, a) \mapsto g_s(a)$. The algebras $\mathcal{A}(H)_s$ form a continuous field of G-C^*-algebras over the unit interval, and we shall denote by $\mathcal{A}(H)[0, 1]$ the G-C^*-algebra of continuous sections. In a similar way, form a continuous field of G-C^*-algebras $\mathcal{K}(\mathcal{S}\mathcal{E})_s = \mathcal{K}(\mathcal{S}\mathcal{E})$ (we are abbreviating $\mathcal{S}\hat{\otimes}\mathcal{E}$ to $\mathcal{S}\mathcal{E}$) and denote by $\mathcal{K}(\mathcal{S}\mathcal{E})[0, 1]$ the G-C^*-algebra of continuous sections. There are obvious $[0, 1]$-parametrized versions of the dual Dirac and Dirac elements, and their composition

$$\Sigma\mathcal{S} \twoheadrightarrow \Sigma\mathcal{A}[0, 1] \twoheadrightarrow \mathcal{K}(\mathcal{S}\mathcal{E})[0, 1]$$

provides a homotopy between the composition $\alpha \cdot \Sigma\beta$ for the initial affine action of G and the same composition $\alpha \cdot \Sigma\beta$ for the linear one. This means that in order to prove that the composition $\alpha \cdot \Sigma\beta$ is equal to 1, it is enough to consider the case of a linear action of G on H.

In the case of a linear action the natural embedding $\mathcal{S} \to \mathcal{A}(H)$ defined by the identification $\mathcal{S} \simeq \mathcal{A}(0)$ is G-equivariant. Let us consider an extension of G-algebras

$$0 \to \mathcal{K}(\mathcal{S}\mathcal{E}) \to \mathcal{F}_h^{\mathcal{S}} \to \mathcal{S} \to 0$$

which is the pull-back of the basic extension

$$0 \to \mathcal{K}(\mathcal{S}\mathcal{E}) \to \mathcal{F}_h(H) \to \mathcal{A}(H) \to 0$$

along the embedding $\mathcal{S} \to \mathcal{A}(H)$. By functoriality of the central invariant, the composition $\alpha \cdot \Sigma\beta$ coincides with the central invariant $\Sigma\mathcal{S} \twoheadrightarrow \mathcal{K}(\mathcal{S}\mathcal{E})$ of the pull-back extension. So it is enough to prove that the central invariant of the latter extension is the element τ in $\{\Sigma\mathcal{S}, \mathcal{S}\Sigma\}_G$.

Recall now the operation of sum for extensions of G-C^*-algebras (see for instance [20], Sect. 7). When we have two extensions,

$$0 \to \mathcal{K}(\mathcal{E}_i) \to D_i \to A \to 0 \qquad (i = 1, 2),$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert modules over B, the sum of the two extensions is the extension

$$0 \to \mathcal{K}(\mathcal{E}_1 \oplus \mathcal{E}_2) \to D_\oplus \to A \to 0,$$

where $D_\oplus$ is an algebra of matrices of the type: $\begin{pmatrix} d_1 & k_{12} \\ k_{21} & d_2 \end{pmatrix}$, with $d_i \in D_i$ and $k_{ij} \in \mathcal{K}(\mathcal{E}_i, \mathcal{E}_j)$, such that d_1 and d_2 are mapped to the same element of A (and the map of $D_\oplus \to A$ is defined by assigning just this element to the above matrix). The central invariant of a sum of two extensions is equal to the sum of central invariants.

In the present case, the pull-back extension that we are trying to analyze is the sum of the unit extension

$$0 \to \mathcal{S} \otimes C_0(0, \infty) \to \mathcal{S} \otimes C_0[0, \infty) \to \mathcal{S} \to 0,$$

whose central invariant is τ, and the extension

$$0 \to P\mathcal{K}(\mathcal{S}\mathcal{E})P \to P\mathcal{F}_h^{\mathcal{S}}P \to \mathcal{S} \to 0,$$

where P is the orthogonal projection onto the orthogonal complement of the kernel of the Bott-Dirac operator. To complete the proof of the theorem we must show that the central invariant of this extension is zero.

If $f \in \mathcal{S}$ and $s > 0$ then define $f_{\alpha,s} \in \mathcal{S}\mathcal{K}(\mathcal{H}_\alpha(H))$ by the formula

$$f_{\alpha,s} = f\big(X\hat{\otimes}1 + 1\hat{\otimes}s^{-1}h_\alpha(B)\big).$$

For a given s, the $f_{\alpha,s}$ determine an element in $\mathcal{F}_h^{\mathcal{S}}$, and indeed of $P\mathcal{F}_h^{\mathcal{S}}P$ by compression. If $\mathcal{D}_h^{\mathcal{S}}$ is the C^*-algebra generated by the functions

$$(0, 1] \ni s \mapsto f_s \in P\mathcal{F}_h^{\mathcal{S}}P$$

together with the functions from $(0, 1]$ into $\mathcal{K}(\mathcal{S}\mathcal{E})$ which vanish at 0, then we obtain an equivariant extension

$$0 \to C_0(0, 1] \otimes P\mathcal{K}(\mathcal{S}\mathcal{E})P \to \mathcal{D}_h^{\mathcal{S}} \to \mathcal{S} \to 0$$

(equivariance of the extension follows from Lemma 3.10 and the remark following 3.11). Restricting to $s = 1$ gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(0,1] \otimes P\mathcal{K}(\mathcal{S}\mathcal{E})P & \longrightarrow & \mathcal{D}_h^{\mathcal{S}} & \longrightarrow & \mathcal{S} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{=} & & \\
0 & \longrightarrow & P\mathcal{K}(\mathcal{S}\mathcal{E})P & \longrightarrow & P\mathcal{F}_h^{\mathcal{S}}P & \longrightarrow & \mathcal{S} & \longrightarrow & 0
\end{array},$$

so the required triviality of the central invariant for the bottom row follows from functoriality of the central invariant and homotopy invariance. $\square$

One can also calculate the composition of the Dirac and dual Dirac elements in the reverse order. We will not use this calculation in the proof of the main results of this paper. For this reason, we give here only a sketch of the proof of the following theorem.

6.11. Theorem. *The composition* $(\beta \otimes 1) \cdot \alpha$ *of asymptotic morphisms:*

$$\Sigma\mathcal{A}(H) \to\!\!\to \mathcal{S}\mathcal{K}(\mathcal{E}) \to\!\!\to \mathcal{A}(H)\hat{\otimes}\mathcal{K}(\mathcal{E}),$$

is equal to the homotopy class of the transposition map in $\{\Sigma\mathcal{A}(H), \mathcal{A}(H) \otimes \Sigma\}_G$. *The same is also true for* $\tilde{\mathcal{A}}(H)$ *in place of* $\mathcal{A}(H)$.

Sketch of proof. We will give the proof for the algebra $\mathcal{A}(H)$. The proof for $\tilde{\mathcal{A}}(H)$ is similar. The argument is based on the existence of a natural extension of C^*-algebras:

$$0 \to \mathcal{K}(\mathcal{A}(H)\hat{\otimes}\mathcal{E}) \to \mathcal{F}_h(H \times H) \to \mathcal{A}(H \times H) \to 0.$$

To produce this extension, one needs to change a little bit the construction described in Sect. 5. We start with the continuous field of C^*-algebras $\mathcal{A}(H)\hat{\otimes}\mathcal{K}(\mathcal{H}_\alpha(H))$. We construct embeddings $\mathcal{A}(V)\hat{\otimes}\mathcal{C}_\alpha(V) \to \mathcal{A}(H)\hat{\otimes}\mathcal{K}(\mathcal{H}_\alpha(H))$ in the same way as was done in the definition of a basic section but taking into account one more tensor variable which comes from $\mathcal{A}(V)$. The rest of the construction is the same as in Sect. 5. Let us denote the central invariant of the resulting extension by ζ.

Consider now an embedding: $H \to H \times H$ given by $h \mapsto (0, h)$. This embedding *is not equivariant* but it gives rise to a natural asymptotic morphism $\varphi_t : \mathcal{A}(H) \to \mathcal{A}(H \times H)$ defined by

$$\varphi_t(f\hat{\otimes}T) = f\big(X\hat{\otimes}1 + t^{-1}(1\hat{\otimes}\mathfrak{B}_H)\big)\hat{\otimes}T$$

where $\mathfrak{B}_H$ is the Bott element for the first copy of H. It is easy to show that the composition $\zeta \cdot \{\varphi_t\}$ is homotopic to the composition $(\beta \otimes 1) \cdot \alpha$.

We can now use the rotation trick of Atiyah [2]. Let us apply the rotation homotopy of $H \times H$ given by the usual 2×2 matrix. It interchanges the two variables in $H \times H$. The embedding $H \to H \times H$ becomes now $h \mapsto (h, 0)$. So the asymptotic morphism φ_t transforms into the asymptotic morphism φ'_t given essentially by the same formula but in which $\mathfrak{B}_H$ is already the Bott element for the second copy of H.

We can now apply the same argument as in the proof of the previous theorem. Over the second variable H, it is possible to make a homotopy of the group action to the linear one. After that, our embedding $\mathcal{A}(H) \to \mathcal{A}(H \times H)$ will become G-equivariant. Moreover, the extension we have constructed at the beginning of the proof will become the sum of two extensions. The rest of the proof is similar to the proof of Theorem 6.10.

□

7. Asymptotic morphisms and *KK*-theory

In this section and the next we shall carry out the project of transferring the arguments of the previous section from E-theory to KK-theory (our basic reference for KK-theory is [22]). We will work in the category of trivially graded C^*-algebras. Our aim is to be able to apply to a-T-menable groups the following result of J.-L. Tu ([29], Theorem 2.2; see also [5] and the proof of Theorem 14.1 in [11]), which strengthens Theorem 6.5:

7.1. Theorem. *Let G be a second countable, locally compact and Hausdorff topological group, and let A be a separable, proper G-C^*-algebra. If there are elements $x \in KK^G(\mathbb{C}, A)$ and $y \in KK^G(A, \mathbb{C})$ such that $x \otimes_A y = 1_{\mathbb{C}}$ in $KK^G(\mathbb{C}, \mathbb{C})$ then for every separable G-C^*-algebra B the Baum-Connes assembly map*

$$\mu_{\mathrm{red}}\colon KK^G(\mathcal{E}G, B) \to K_*\big(C^*_{\mathrm{red}}(G, B)\big)$$

is an isomorphism. In addition, G is K-amenable.

Suppose that G acts isometrically on a real Hilbert space H. In Sect. 8 we will construct a 'dual Dirac element' $b \in KK_1^G(\mathbb{C}, \mathcal{A}(H))$. Under the additional hypothesis that the action of G on H is *proper* we shall also construct a 'Dirac element' $d \in KK_1^G(\mathcal{A}(H), \mathbb{C})$. Of course, we should like to show that $b \otimes_{\mathcal{A}(H)} d = 1$ in $KK^G(\mathbb{C}, \mathbb{C})$. Unfortunately the argument we used in Sect. 6 to settle the analogous point in E-theory involved not only the given proper action on H but also the associated linear action, and the natural homotopy between them. Since the linear action is certainly not proper (except in the trivial case where G is compact) the E-theory argument does not readily transfer to KK-theory.

The essential problem here is that while there is a natural transformation

$$KK^G(A, B) \to E_G(A, B),$$

there is in general no natural transformation the other way round. But as a partial substitute we shall define and analyze homomorphisms

$$\rho\colon \{\Sigma^2, B\}_G \to KK_0^G(\mathbb{C}, B)$$

and

$$\rho\colon \{\Sigma, B\}_G \to KK_1^G(\mathbb{C}, B).$$

Using them we shall eventually be able to reduce the calculation $b \otimes_{\mathcal{A}(H)} d = 1$ to the analogous asymptotic morphism calculation, which we already did in Theorem 6.10.

We will start by studying maps in the reverse direction and reformulating the construction of the central asymptotic morphism associated to an extension (see Lemma 6.3). As we have already said above, we will assume that all considered C^*-algebras are trivially graded.

7.2. Definition. Let A and B be separable G-C^*-algebras.

(a) Define a homomorphism

$$\eta : KK_1^G(A, B) \to \{\Sigma A, B\}_G$$

as follows. View $KK_1^G(A, B)$ as the group of homotopy classes of pairs $(\mathcal{E}, P)$, where $\mathcal{E}$ is a separable G-Hilbert module over B, $\varphi\colon A \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism, and $P \in \mathcal{L}(\mathcal{E})$ is an operator for which

(i) $\varphi(a)(P^* - P)$, $\varphi(a)(P^2 - P)$, $\varphi(a)(g(P) - P)$ and $\varphi(a)P - P\varphi(a)$ are operators in $\mathcal{K}(\mathcal{E})$, for all $a \in A$ and all $g \in G$; and
(ii) the operators $\varphi(a)(g(P) - P)$ are norm-continuous in g.

Let $\{u_t\}$ be an approximate unit for $\mathcal{K}(\mathcal{E})$ which is quasicentral with respect to A, the action of G, and the operator P. Define an equivariant asymptotic morphism

$$\varphi_t : \Sigma A \twoheadrightarrow \mathcal{K}(\mathcal{E})$$

by

$$\varphi_t\colon f \otimes a \mapsto f(u_t)\varphi(a)P \qquad t \geq 1,$$

and put $\eta(\mathcal{E}, P) = \{\varphi_t\}$.

(b) Define a homomorphism

$$\eta : KK_0^G(A, B) \to \{\Sigma^2 A, B\}_G$$

as follows. View $KK_0^G(A, B)$ as the group of homotopy classes of pairs $(\mathcal{E}, F)$, where as above $\mathcal{E}$ is a separable G-Hilbert module over B and $\varphi\colon A \to \mathcal{L}(\mathcal{E})$ is a $*$-homomorphism, but where $F \in \mathcal{L}(\mathcal{E})$ is an operator for which

(i) $\varphi(a)(F^*F - 1)$, $\varphi(a)(FF^* - 1)$, $\varphi(a)(g(F) - F)$ and $\varphi(a)F - F\varphi(a)$ are operators in $\mathcal{K}(\mathcal{E})$, for all $a \in A$ and all $g \in G$; and
(ii) the operators $\varphi(a)(g(F) - F)$ are norm-continuous in g.

Let $\{u_t\}$ be an approximate unit for $\mathcal{K}(\mathcal{E})$ which is quasicentral with respect A, the action of G, and the operator F, and define an asymptotic morphism

$$\varphi_t : \Sigma^2 A \twoheadrightarrow \mathcal{K}(\mathcal{E})$$

by

$$\varphi_t\colon f_1 \otimes f_2 \otimes a \mapsto f_1(u_t)f_2(F)\varphi(a), \qquad t \geq 1.$$

Here the second copy of Σ is identified with the algebra of continuous functions on the unit circle vanishing at the point 1 and $f_2(F)$ means that we take $f_2(F)$ in the Calkin algebra $\mathcal{L}(\mathcal{E})/\mathcal{K}(\mathcal{E})$ and then lift it arbitrarily to $\mathcal{L}(\mathcal{E})$. Put $\eta(\mathcal{E}, F) = \{\varphi_t\}$.

The following simple lemma relates the two homomorphisms just defined:

7.3. Lemma. *When $A = \Sigma A_1$, the homomorphism η on $KK_1^G(A, B)$ coincides up to sign with the homomorphism η on $KK_0^G(A_1, B)$, after $KK_0^G(A_1, B)$ and $KK_1^G(A, B)$ are identified by Bott Periodicity.* □

Remark. We note that the periodicity isomorphisms in KK-theory are unique, up to sign, so there is no need to specify them in the above assertion.

We now introduce the promised homomorphisms ρ which associate KK-theory classes to asymptotic morphisms.

7.4. Definition. Let B be a separable G-C^*-algebra.

(a) Define a homomorphism

$$\rho : \{\Sigma, B\}_G \to KK_0^G(\mathbb{C}, B \otimes C_0(0,\infty)) \cong KK_1^G(\mathbb{C}, B)$$

as follows. Let $\mathcal{E}$ be a separable G-Hilbert module over B and let $\varphi_t : \Sigma \twoheadrightarrow \mathcal{K}(\mathcal{E})$ be an equivariant asymptotic morphism. For $t \leq 1$, define $\varphi_t = t\varphi_1$ and then extend φ_t unitally to the unitalized algebra $\widetilde{\Sigma}$ to obtain a unital map:

$$\{\varphi_t\} : \tilde{\Sigma} \to \mathcal{L}(\mathcal{E} \otimes C_0(0,\infty)).$$

Set $V_t = \varphi_t(v)$, where v is the generator of the algebra $\widetilde{\Sigma}$, which we consider as the algebra of continuous functions on the unit circle. The element $V = \{V_t\} \in \mathcal{L}(\mathcal{E} \otimes C_0(0,\infty))$ satisfies the conditions:

$$VV^* - 1,\ V^*V - 1,\ g(V) - V \in \mathcal{K}(\mathcal{E} \otimes C_0(0,\infty)), \qquad \forall g \in G,$$

and we define $\rho(\{\varphi_t\})$ to be the corresponding element of $KK_0^G(\mathbb{C}, B \otimes C_0(0,\infty))$.

(b) Similarly, we define a homomorphism

$$\rho : \{\Sigma^2, B\}_G \to KK_1^G(\mathbb{C}, B \otimes C_0(0,\infty)) \cong KK_0^G(\mathbb{C}, B)$$

as follows. Let $\varphi_t : \Sigma^2 \twoheadrightarrow \mathcal{K}(\mathcal{E})$ be an equivariant asymptotic morphism. For $t \leq 1$ define $\varphi_t = t\varphi_1$; extend φ_t unitally to unitalized algebra $\widetilde{\Sigma^2}$; and then pass to matrices to obtain a unital map

$$\{\varphi_t\} : M_2(\widetilde{\Sigma^2}) \to M_2(\mathcal{L}(\mathcal{E} \otimes C_0(0,\infty))).$$

Set $P_t = \varphi_t(p)$, where the element

$$p = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & z \\ \bar{z} & |z|^2 \end{pmatrix} \in M_2(\widetilde{\Sigma^2})$$

is the canonical rank-one projection, the algebra $\widetilde{\Sigma^2}$ being identified with the algebra of continuous functions on the Riemann sphere and z being the complex coordinate on the Riemann sphere. The element $P = \{P_t\} \in M_2(\mathcal{L}(\mathcal{E} \otimes C_0(0, \infty)))$ satisfies the conditions

$$P^* - P,\ P^2 - P,\ g(P) - P \in \mathcal{K}(\mathcal{E} \otimes C_0(0, \infty)), \qquad \forall g \in G,$$

and we define $\rho(\{\varphi_t\})$ to be the corresponding element of $KK_1^G(\mathbb{C}, B \otimes C_0(0, \infty))$.

7.5. Lemma. *In the case when $A = \mathbb{C}$, the compositions $\rho \circ \eta$ give periodicity isomorphisms*

$$KK_1^G(\mathbb{C}, B) \underset{\cong}{\longrightarrow} KK_0^G(\mathbb{C}, B \otimes C_0(0, \infty))$$

and

$$KK_0^G(\mathbb{C}, B) \underset{\cong}{\longrightarrow} KK_1^G(\mathbb{C}, B \otimes C_0(0, \infty)).$$

Proof. Let us prove the second assertion, which is the more complicated of the two.

We begin by noting that Definition 7.2(b) asks us to identify Σ^2 with the C_0-functions on the product $(0, 1) \times S_0^1$, where S_0^1 denotes the unit circle with the point 1 removed. The map $(r, v) \mapsto rv$ takes the cartesian product $(0, 1) \times S_0^1$ homeomorphically to the space X comprised of the open unit disk in $\mathbb{C}$ with the interval $[0, 1]$ removed. As in Definition 7.4 we identify the one-point compactification $\widetilde{X}$ of this space with the Riemann sphere via some homeomorphism (it does not yet matter which one). Now the one-point compactification $\widetilde{D}$ of the open unit disk maps onto $\widetilde{X}$ by collapsing $[0, 1]$ to a point. This map is a homotopy equivalence. The map

$$\widetilde{D} \to \widetilde{X} \xrightarrow{p} M_2(\mathbb{C}),$$

where p is as in Definition 7.4, is a degree ± 1 map from $\widetilde{D}$ onto the space of rank one projections, which is a 2-sphere.

It is easy to see that this map is homotopic to the following projection-valued map

$$\widetilde{D} \ni (r, v) \mapsto \begin{pmatrix} r & \sqrt{r - r^2}\,\bar{v} \\ \sqrt{r - r^2}\,v & 1 - r \end{pmatrix}.$$

Indeed, we may identify the Riemann sphere with $\widetilde{D}$ using the map given by the formulas: $r = |z|^2/(1+|z|^2),\ v = z/|z|$. Then the above two projection-valued maps will coincide up to conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Bearing in mind the above homotopy, the composition $\rho \circ \eta$ takes a cycle $(F, \mathcal{E})$ for the group $KK_0^G(\mathbb{C}, B)$ to the class of the cycle represented by the family

$$\begin{pmatrix} u_t & \sqrt{u_t - u_t^2}\,F^* \\ \sqrt{u_t - u_t^2}\,F & 1 - u_t \end{pmatrix} \qquad t \geq 0.$$

Here we extend the definition of u_t from $t \geq 1$ to $t \geq 0$ by setting $u_t = tu_1$ for $t \leq 1$. (We are assuming here for simplicity that the given representation of $\mathbb{C}$ on $\mathcal{E}$ – which is part of the data for a class in $KK_0^G(\mathbb{C}, B)$ that we omitted – takes 1 to the identity operator.) If now we define

$$w_t = \begin{cases} t, & t \leq 1 \\ 1, & t \geq 1 \end{cases}$$

then the straight line homotopy from u_t to w_t gives a homotopy of KK-cycles to the family

$$\begin{pmatrix} w_t & \sqrt{w_t - w_t^2} F^* \\ \sqrt{w_t - w_t^2} F & 1 - w_t \end{pmatrix} \qquad t \geq 0.$$

But this is just the K-theory product of the class in $KK_0^G(\mathbb{C}, B)$ we began with and a generator of $KK_1^G(\mathbb{C}, C_0(0, \infty))$. □

The following lemma is another easy exercise:

7.6. Lemma. *The homomorphisms ρ on $\{\Sigma, B\}_G$ and ρ on $\{\Sigma^2, \Sigma B\}_G$ agree up to sign after one identifies $KK_0^G(\mathbb{C}, B \otimes C_0(0, \infty))$ with $KK_1^G(\mathbb{C}, \Sigma B \otimes C_0(0, \infty))$ and after one maps $\{\Sigma, B\}_G$ into $\{\Sigma^2, \Sigma B\}_G$ using tensor product with Σ.* □

Now let $\mathcal{E}_1$ be a Hilbert B-module and let $\psi = \{\psi_t\} : A \twoheadrightarrow \mathcal{K}(\mathcal{E}_1)$ be an equivariant asymptotic morphism. Assuming that A is a separable and *proper* G-C^*-algebra we can define an associated homomorphism

$$\psi_* : K_*^G(A) \to K_*^G(B)$$

(for brevity we have introduced here the usual notation $K_*^G(A) = KK_*^G(\mathbb{C}, A)$). Since the two cases of K_0^G and K_1^G are similar, we will consider only the case of K_1^G.

As in the previous section, it will be useful to denote by $\mathcal{H}$ the Hilbert space direct sum of countably many copies of the regular representation of G: thus $\mathcal{H} = \oplus_{n=1}^{\infty} L^2(G)$.

7.7. Definition. Assume that A is a separable G-C^*-algebra. Suppose first that an element of $K_1^G(A)$ has the following special form: it is represented by a pair $(\mathcal{E}, T)$, where $\mathcal{E} = A \otimes \mathcal{H}$. Applying to this pair the homomorphism η we get an equivariant asymptotic morphism $\varphi_t : \Sigma \twoheadrightarrow A \otimes \mathcal{K}(\mathcal{H})$. Now take the tensor product of the initial asymptotic morphism ψ_t with the identity morphism on $\mathcal{K}(\mathcal{H})$ to get

$$\psi_t \otimes 1 : A \otimes \mathcal{K}(\mathcal{H}) \twoheadrightarrow \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{H}).$$

Applying the homomorphism ρ to the composition

$$\{(\psi_t \otimes 1)\} \cdot \{1 \otimes \varphi_t\} : \Sigma \twoheadrightarrow \mathcal{K}(\mathcal{E}_1) \otimes \mathcal{K}(\mathcal{H})$$

we get an element of $KK_0^G(\mathbb{C}, B \otimes C_0(0,\infty)) \cong K_1^G(B)$. Call it $\psi_*(\mathcal{E}, T)$. Now assume that A is a proper G-C^*-algebra. Then according to the Equivariant Stabilization Theorem (cf. [26], 2.9; [23], 5.3),

$$\mathcal{E} \oplus (A \otimes \mathcal{H}) \simeq A \otimes \mathcal{H},$$

so any element of $K_1^G(A)$ is represented by a pair $(\mathcal{E}, T)$, where $\mathcal{E} = A \otimes \mathcal{H}$. This allows us to define the required homomorphism ψ_* on the whole group $K_1^G(A)$.

We will need below the following simple modification of this construction. Let $\psi = \{\psi_t\} : \Sigma A \twoheadrightarrow \mathcal{K}(\mathcal{E}_1)$ be an equivariant asymptotic morphism. Then one can define an associated homomorphism $\psi_* : K_1^G(A) \to K_0^G(B)$ as the following composition:

$$KK_1^G(\mathbb{C}, A) \xrightarrow{\eta} \{\Sigma, A\}_G \xrightarrow{\text{suspension}}$$

$$\{\Sigma^2, \Sigma A\}_G \xrightarrow[\text{with } \psi]{\text{composition}} \{\Sigma^2, B\}_G \xrightarrow{\rho} KK_1^G(\mathbb{C}, B \otimes C_0(0,\infty)).$$

The hypothesis that A is proper allows one to view $\{\Sigma, A\}_G$ as homotopy classes of equivariant asymptotic morphisms from Σ to $A \otimes \mathcal{K}(\mathcal{H})$ (no nontrivial Hilbert modules $\mathcal{E}$ need be considered), and this allows us to carry out the composition of asymptotic morphisms in the middle of the sequence above.

7.8. Theorem. *Assume that A is a proper G-algebra. Let $x \in K_1^G(A)$, $y \in KK_1^G(A, B)$, and let $\psi = \eta(y)$ an asymptotic morphism $\Sigma A \twoheadrightarrow \mathcal{K}(\mathcal{E}_1)$, where $\mathcal{E}_1$ is a Hilbert B-module. Then the elements $\psi_*(x) \in KK_1^G(\mathbb{C}, B \otimes C_0(0,\infty))$ and $x \otimes_A y \in KK_0^G(\mathbb{C}, B)$ are equal, up to sign, after the two KK-groups are identified by Bott Periodicity.*

Proof. To simplify notation we will assume that x is represented by an operator $R \in \mathcal{M}(A)$ (i.e. the Hilbert module $\mathcal{E}$ is A itself and not $A \otimes \mathcal{H}$ as in Definition 7.7). This does not change the argument. Let $y = (\mathcal{E}_1, T)$, and assume in addition that the action of A on $\mathcal{E}_1$ is non-degenerate. (This is always possible – see [22], Lemma 2.8). It means that we can extend the homomorphism $A \to \mathcal{L}(\mathcal{E}_1)$ to $\mathcal{M}(A)$.

Again to simplify notation, let us denote by the same letter R the image of the operator $R \in \mathcal{M}(A)$ in $\mathcal{L}(\mathcal{E}_1)$. Set $P = (R+1)/2$, $Q = (T+1)/2$. Let $\{u_t\}$ and $\{v_t\}$ be the approximate units in A and $\mathcal{K}(\mathcal{E}_1)$ respectively used in Definition 7.2 (for the cases of $K_1^G(A)$ and $KK_1^G(A, B)$ respectively). Then the product $x \otimes_A y \in K_*^G(B)$ can be written in the form $MR + iNT$ where $i = \sqrt{-1}$ and M, N are the operators used in the definition of the KK-product (see [22], Theorem 2.11). The operators M and N are G-continuous operators with the following properties:

(i) $M \geq 0$, $N \geq 0$, and $M^2 + N^2 = 1$.
(ii) $M(P^2 - P), M(P^* - P), M(g(P) - P), g(M) - M, N[P, Q], N(Q^2 - Q)$, $N(Q^* - Q)$, and $N(g(Q) - Q)$ are elements of $\mathcal{K}(\mathcal{E}_1)$, for all $g \in G$.
(iii) The functions $g \mapsto M(g(P) - P)$ and $g \mapsto N(g(Q) - Q)$ are norm-continuous on G.

On the other hand, the composition $\eta(y) \cdot \eta(x)$ can be written as

$$\Sigma^2 \twoheadrightarrow \Sigma A \twoheadrightarrow \mathcal{K}(\mathcal{E}_1) : f_1 \otimes f_2 \mapsto f_1(v_{s(t)}) Q f_2(u_t) P,$$

where we denote also by the same letter u_t the image of u_t in $\mathcal{L}(\mathcal{E}_1)$. Here $s(t)$ denotes a reparametrization of the sort which is necessary in order to define the composition of two asymptotic morphisms.

To compare $\rho(\eta(y) \cdot \eta(x))$ with $x \otimes_A y \in K_*^G(B)$, we first change $\eta(y) \cdot \eta(x)$ by a certain homotopy. Namely, we replace $f_2(u_t)$ by $f_2(1 - M)$ using the homotopy: $r(1 - M) + (1 - r)u_t,\ 0 \leq r \leq 1$. After that, we can forget about the reparametrization $s(t)$, and the composition of our asymptotic morphisms takes the form

$$\Sigma^2 \twoheadrightarrow \Sigma A \twoheadrightarrow \mathcal{K}(\mathcal{E}_1) : f_1 \otimes f_2 \mapsto f_1(v_t) f_2(1 - M) PQ.$$

This latter asymptotic morphism has the form of $\eta(z)$, where $z \in K_0^G(B)$ is given by the operator $F = (\exp(2\pi i(1 - M)) - 1)PQ + 1 \in \mathcal{L}(\mathcal{E}_1)$. In view of Lemma 7.5, it remains to prove that F and $MR + iNT$ represent the same element in $K_0^G(B)$.

Put $X = MP$, $Y = NQ$. These two operators commute modulo $\mathcal{K}(\mathcal{E}_1)$. It follows from the properties of the operators M, N listed above that $XY(X^2 + Y^2 - 1) \in \mathcal{K}(\mathcal{E}_1)$. This means that the map $C([0, 1]^2) \to \mathcal{L}(\mathcal{E}_1)/\mathcal{K}(\mathcal{E}_1)$ given by the formula: $f \mapsto f(X, Y)$ factors through the quotient algebra $C([0, 1]^2)/J$, where J is the ideal generated by $xy(x^2 + y^2 - 1)$. The quotient is the algebra of continuous functions on the contour which bounds that quarter of the unit circle which lies in the first quadrant of the (x, y)-plane.

First, we will find operators homotopic to $MR + iNT$ and F which are functions of X, Y. We have: $MR + iNT = 2(X + iY) - (M + iN)$. We can multiply the latter operator by $M - iN$ (which is homotopic to 1) and get:

$$(M - iN)(2X + 2iY) - 1 = 2X^2 + 2Y^2 - 2iX(1 - X^2)^{1/2} + 2iY(1 - Y^2)^{1/2} - 1$$

modulo $\mathcal{K}(\mathcal{E}_1)$. For the operator F we find:

$$(\exp(-2\pi i M) - 1)PQ + 1 = (\exp(-2\pi i X) - 1)(1 - X^2)^{-1} Y^2 + 1$$

modulo $\mathcal{K}(\mathcal{E}_1)$. Now it remains to note that the two maps from our quarter-circle contour to the unit circle given by the formulas:

$$(x, y) \mapsto 2x^2 + 2y^2 - 2ix(1 - x^2)^{1/2} + 2iy(1 - y^2)^{1/2} - 1$$

and

$$(x, y) \mapsto (\exp(-2\pi i x) - 1)(1 - x^2)^{-1} y^2 + 1$$

are homotopic. This is done by checking that the winding numbers are the same; the simple calculation is omitted. □

8. Proper actions: Dirac and dual Dirac elements in *KK*-theory

This section contains the construction of the Dirac and the dual Dirac elements in KK-theory (the Dirac element will be constructed only under the assumption that the group G acts on the Hilbert space H properly). We will then use the relationship between asymptotic morphisms and KK-theory which was explained in the previous section to compute the product of the Dirac and dual Dirac elements. Since in the previous section we considered only trivially graded algebras, in the present section we will continue to regard all C^*-algebras (including $\mathcal{A}(H)$) as trivially graded.

8.1. Theorem. *Let A be a separable G-$C_0(X)$-algebra, where X is a proper G-space. Let $\mathcal{E}_0$ be a countably generated G-Hilbert module over a C^*-algebra D, and suppose the action of G on D is trivial. Let*

$$0 \to \mathcal{K}(\mathcal{E}_0) \to \mathcal{F} \to A \to 0$$

be an equivariant extension of G-C^*-algebras and let $\psi_0 : A \to \mathcal{L}(\mathcal{E}_0)/\mathcal{K}(\mathcal{E}_0)$ be the Busby homomorphism corresponding to this extension.[4] *Assume that there is a (not necessarily equivariant) completely positive lifting $\sigma' \colon A \to \mathcal{L}(\mathcal{E}_0)$ of the Busby homomorphism ψ_0. Then there is a countably generated G-Hilbert module $\mathcal{E}_1$ over D, a G-covariant representation $\varphi_1 : A \to \mathcal{L}(\mathcal{E}_1)$ and a self-adjoint operator $P \in \mathcal{L}(\mathcal{E}_1)$ such that*

(i) *$\varphi_1(a)(P^2 - P)$, $\varphi_1(a)(g(P) - P)$ and $\varphi_1(a)P - P\varphi_1(a)$ are operators in $\mathcal{K}(\mathcal{E}_1)$, for all $a \in A$ and all $g \in G$; and the operators $\varphi_1(a)(g(P) - P)$ are norm-continuous in g;*

(ii) *the homomorphism $\psi_1 \colon A \to \mathcal{L}(\mathcal{E}_1)/\mathcal{K}(\mathcal{E}_1)$ given by $\psi_1(a) = P\varphi_1(a)$ is homotopy equivalent to the homomorphism ψ_0 in the sense that there is a G-Hilbert module $\{\mathcal{E}_t\}_{t\in[0,1]}$ over $D[0,1]$ and a homomorphism $\psi \colon A \to \mathcal{L}(\{\mathcal{E}_t\})$ whose restrictions to 0 and 1 are just ψ_0 and ψ_1.*

Proof. As usual, one has first to construct a suitable completely positive lifting and then use the Stinespring construction. We do not know if there always exists under these assumptions an equivariant completely positive lifting. But such a lifting can always be constructed for a dense subalgebra A_c of A, and we start by doing this.

Let A_c be the subalgebra of compactly supported elements in A (in the sense of support in X, – see [22], 3.2). We denote by c a cut-off function for X – a non-negative, bounded, continuous function on X such that the intersection of the support of c with any G-compact subset of X is compact, and $\int_G g(c)dg = 1$. We define now a lifting $\sigma \colon A_c \to \mathcal{L}(\mathcal{E}_0)$ as follows:

$$\sigma(a) = \int_G g(\sigma'(cg^{-1}(a)))dg.$$

[4] Associated to the extension is a $*$-homomorphism from $\mathcal{F}$ to $\mathcal{L}(\mathcal{E}_0)$; recall that the *Busby homomorphism* is the induced map from A into the quotient $\mathcal{L}(\mathcal{E}_0)/\mathcal{K}(\mathcal{E}_0)$.

Note that the integral converges for any compactly supported a because the integrand vanishes off a compact subset of G. The map σ is an equivariant lifting of ψ_0 on A_c and it is completely positive in a suitable sense (see below). However, it may be unbounded and so it cannot be used directly in the Stinespring construction.

Denote by G_d the group G equipped with the discrete topology and denote by B_{alg} the algebraic crossed product of G_d and A_c. Thus the elements of B_{alg} are finite sums $\sum_i a_i g_i$ where $a_i \in A_c,\ g_i \in G$; the product is given by $a_1 g_1 \cdot a_2 g_2 = a_1 g_1(a_2) g_1 g_2$; the involution by $(ag)^* = g^{-1}(a^*)g^{-1}$. We extend the map σ defined above to B_{alg} in the obvious way: $\sigma(ag) = \sigma(a)g$. In this formula, the symbol g on the right denotes the unitary operator on $\mathcal{E}_0$ implementing the action of the element g of the group G_d. It easy to check that the map σ so defined is completely positive in the sense that σ applied to a matrix with entries $((a_i g_i)^*(a_j g_j))$ gives a positive matrix over $\mathcal{L}(\mathcal{E}_0)$.

Denote by B the crossed product C^*-algebra $C^*(G_d, A)$. Let $\tilde{B}$ be the C^*-algebra B with unit adjoined. If $u \in A_c$ is self-adjoint then the formula $b \mapsto ubu$ defines a completely positive map $\kappa_u \colon B_{alg} \to B_{alg}$, and composing κ_u with σ we get a completely positive map $\sigma\kappa_u \colon B_{alg} \to \mathcal{L}(\mathcal{E}_0)$. Since B_{alg} is a dense subalgebra in B, and since $\sigma\kappa_u$ is bounded in the norm of B, it extends to a continuous map that we will denote $\sigma_u \colon B \to \mathcal{L}(\mathcal{E}_0)$.

Now we will apply an infinite inductive procedure borrowed from Lemma 3.1 and Theorem 6 of [1], and we will also make use of Remark 2.5 of [8]. If $\{u_i\}$ is a quasicentral approximate unit in A consisting of elements of A_c (the word "quasicentral" here means quasicentral with respect to the action of G) then define *unital* completely positive maps $\sigma_i \colon \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$ by the formula

$$\sigma_i(b) = \tilde{u}_i^{-1/2} \sigma_{u_i}(b) \tilde{u}_i^{-1/2},$$

where $b \in B$ and $\tilde{u}_i = \sup(1, \sigma(u_i))$. It is clear that $\psi(\sigma_i(b)) = u_i^{1/2} b u_i^{1/2}$ for every $b \in B$. Our general aim is to modify the maps $\sigma_i \colon \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$ so as to obtain a sequence of unital completely positive maps $\tilde{\sigma}_i \colon \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$ which converge pointwise to a completely positive map $\tilde{\sigma}$ that is properly suited to a subsequent Stinespring construction. First we will give the general form of the modification procedure. Then by making appropriate choices of approximate units we will prove convergence of the modified maps.

Let $\{v_i\}$ be an approximate unit in $\mathcal{K}(\mathcal{E}_0)$ which is quasicentral with respect to the algebra $\mathcal{F}$ and the action of G, and define unital maps $\tilde{\sigma}_i \colon \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$ inductively by setting $\tilde{\sigma}_1 = \sigma_1$ and then defining

$$\tilde{\sigma}_n(b) = v_n^{1/2} \tilde{\sigma}_{n-1}(b) v_n^{1/2} + (1 - v_n)^{1/2} \sigma_n(b) (1 - v_n)^{1/2}$$

for $b \in \tilde{B}$. Notice that

$$\begin{aligned} \|\tilde{\sigma}_n(b) - \tilde{\sigma}_{n-1}(b)\| \leq {} & \left\| v_n^{1/2} \tilde{\sigma}_{n-1}(b) v_n^{1/2} - \tilde{\sigma}_{n-1}(b) v_n \right\| \\ & + \left\| (1 - v_n)^{1/2} \sigma_n(b) (1 - v_n)^{1/2} - \sigma_n(b)(1 - v_n) \right\| \\ & + \|(\sigma_n(b) - \tilde{\sigma}_{n-1}(b))(1 - v_n)\|. \end{aligned}$$

So in order to prove pointwise convergence we make the following choices of the approximate units $\{v_i\}$ and $\{u_i\}$. Let X be a compact subset of A whose linear span is dense in A. Also let $K_1 \subset K_2 \subset \dots$ be an exhaustive family of compact subsets of G. Choose (in an inductive fashion) v_n so that when b belongs to the set $\{\sum_m a_m g_m \mid a_m \in X, g_m \in K_n\}$ the first two summands on the right side of the above inequality are small. Concerning the third summand, we know at least that for $b \in B$,

$$\psi(\sigma_n(b) - \tilde{\sigma}_{n-1}(b)) = u_n^{1/2} b u_n^{1/2} - u_{n-1}^{1/2} b u_{n-1}^{1/2}.$$

The norm of this element is small (for all b belonging to the same set as above) if the approximate unit $\{u_i\}$ is chosen appropriately. Finally, note that for any $f \in \mathcal{F}$, the norms $\|f(1 - v_n)\|$ converge to $\|\psi(f)\|$ as n goes to infinity. So the third summand on the right hand side of our inequality will also be small if v_n has been chosen appropriately.

We have constructed a pointwise convergent sequence of completely positive unital maps $\tilde{\sigma}_n : \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$. Denote the limit by $\tilde{\sigma} : \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$. The map $\tilde{\sigma}$ possesses an important property which easily follows from the construction given above:

(∗) for any $b \in B$, the functions $g \mapsto \tilde{\sigma}(bg) - \tilde{\sigma}(b)g$ and $g \mapsto \tilde{\sigma}(gb) - g\tilde{\sigma}(b)$ are norm-continuous on G.

This completes the first part of the proof: the construction of a unital completely positive lifting.

Consider now the algebraic tensor product $\tilde{B} \otimes_{\mathbb{C}} \mathcal{E}_0$. Using the completely positive map $\tilde{\sigma} : \tilde{B} \to \mathcal{L}(\mathcal{E}_0)$, define a D-inner product on $\tilde{B} \otimes_{\mathbb{C}} \mathcal{E}_0$ by

$$(b_1 \otimes x_1, b_2 \otimes x_2) = (x_1, \tilde{\sigma}(b_1^* b_2) x_2).$$

By applying the argument given in the proof of the Stinespring Theorem (Theorem 3) of [19] we obtain a Hilbert module $\tilde{\mathcal{E}}$ over D and a representation $\tilde{\varphi} : \tilde{B} \to \mathcal{L}(\tilde{\mathcal{E}})$ (the representation $\tilde{\varphi}$ comes from the action of $\tilde{B}$ on $\tilde{B} \otimes \mathcal{E}_0$ defined by left multiplication). Our initial Hilbert module $\mathcal{E}_0$ is a direct summand of $\tilde{\mathcal{E}}$, and the compression of $\tilde{\varphi} : \tilde{B} \to \mathcal{L}(\tilde{\mathcal{E}})$ to $\mathcal{E}_0$ is $\tilde{\sigma}$. We denote the orthogonal projection of $\tilde{\mathcal{E}}$ onto $\mathcal{E}_0$ by Q_0. Since the compression of $\tilde{\varphi} : \tilde{B} \to \mathcal{L}(\tilde{\mathcal{E}})$ to $\mathcal{E}_0$ is multiplicative modulo $\mathcal{K}(\mathcal{E}_0)$, it is easy to deduce (see, for instance, the proof of Lemma 1 of Sect. 7 in [20]) that Q_0 commutes modulo $\mathcal{K}(\tilde{\mathcal{E}})$ with $\tilde{\varphi}(\tilde{B})$.

Unfortunately, the Hilbert module $\tilde{\mathcal{E}}$ will not serve as the Hilbert module $\mathcal{E}_1$ in the statement of the theorem we are trying to prove because there may be no G-action on $\tilde{\mathcal{E}}$ (the G-action corresponding to the homomorphism $\tilde{\varphi}$ may be degenerate). A well defined G-action exists only on a submodule $B \cdot \tilde{\mathcal{E}} \simeq B \otimes_{\tilde{B}} \tilde{\mathcal{E}}$ of $\tilde{\mathcal{E}}$ which we will call $\mathcal{E}_1$. This G-action comes from the action of G on B by left multiplication. It is easy to check using property (∗) above and the definition of the inner product on $\mathcal{E}_1$ that this G-action is continuous. The continuity of the G-action together with the separability of the algebra A imply that $\mathcal{E}_1$ is countably generated. We denote by φ_1 the homomorphism $A \to \mathcal{L}(\mathcal{E}_1)$ defined by the action of A on $\mathcal{E}_1$ given by left multiplication.

Now we have to construct the operator $P \in \mathcal{L}(\mathcal{E}_1)$. For this we note that the C^*-algebra B has a countable approximate unit (namely the approximate unit $\{u_i\}$ that was already used above), so by the Stabilization Theorem (Theorem 2 of [19]) the Hilbert $\tilde{B}$-module B is a direct summand of $\tilde{B} \otimes \ell^2$. This means that the Hilbert D-module $\mathcal{E}_1 = B \otimes_{\tilde{B}} \tilde{\mathcal{E}}$ is a direct summand of $(\tilde{B} \otimes \ell^2) \otimes_{\tilde{B}} \tilde{\mathcal{E}} \simeq \tilde{\mathcal{E}} \otimes \ell^2$. We denote by $Q_1 \in \mathcal{L}(\tilde{\mathcal{E}} \otimes \ell^2)$ the orthogonal projection onto $\mathcal{E}_1$ and define $P \in \mathcal{L}(\mathcal{E}_1)$ as the compression $Q_1(Q_0 \otimes 1)Q_1$ of $Q_0 \otimes 1 \in \mathcal{L}(\tilde{\mathcal{E}} \otimes \ell^2)$ to $\mathcal{E}_1$.

In order to check condition (i) in the statement of the Theorem, we note that although the projections $Q_0 \otimes 1$ and $Q_1 \in \mathcal{L}(\tilde{\mathcal{E}} \otimes \ell^2)$ do not commute, one has for any $b \in B$, $(\tilde{\varphi} \otimes 1)(b) \cdot [Q_1, Q_0 \otimes 1] \in \mathcal{K}(\tilde{\mathcal{E}} \otimes \ell^2)$. This easily implies all assertions of condition (i) except the last one: norm-continuity. However, as was pointed out in [28] (see the remark made after the proof of Theorem 1.1.4), the norm-continuity of operators $\varphi_1(a)(g(P) - P)$ follows from the other assertions of condition (i). (One has to apply Theorem 1.1.4 of [28] to the natural extension of C^*-algebras defined by the Busby homomorphism ψ_1 of condition (ii) in the statement of our Theorem 8.1.)

Coming finally to the last assertion (condition (ii)) of the theorem, we first point out that there is a natural homotopy of Hilbert $\tilde{B}$-modules joining $\tilde{B}$ and B. This is given by $Z(B, \tilde{B}) = \{f\colon [0,1] \to \tilde{B} \mid f((1) \in B\}$. Using the Stabilization Theorem, we get a homotopy of projections in $\mathcal{L}(\tilde{B}[0,1] \otimes \ell^2)$ whose image is just the above homotopy of Hilbert modules $Z(B, \tilde{B})$. Taking $\otimes_{\tilde{B}[0,1]}\tilde{\mathcal{E}}$, we get a homotopy of projections $S_t \in \mathcal{L}(\tilde{\mathcal{E}}[0,1] \otimes \ell^2)$ joining a coordinate projection of $\tilde{\mathcal{E}} \otimes \ell^2$ onto a direct summand $\tilde{\mathcal{E}}$ with the projection Q_1 onto $\mathcal{E}_1$. Now consider the compression $P_t = (Q_0 \otimes 1)S_t(Q_0 \otimes 1) \in \mathcal{L}(\mathcal{E}_0 \otimes \ell^2)$ of this homotopy of projections.

Note that the homomorphism $\tilde{\varphi} \otimes 1\colon A \to \mathcal{L}(\tilde{\mathcal{E}} \otimes \ell^2)$, after being compressed first by the projection $Q_0 \otimes 1$ and then multiplied by the homotopy of operators P_t, becomes a homotopy of homomorphisms of A into the Calkin algebra of the Hilbert module $\mathcal{E}_0 \otimes \ell^2$. At the point $t = 0$ we obtain the homomorphism $\psi_0 \oplus 0$ and at the point $t = 1$ we obtain the homomorphism $\psi_1 \oplus 0$. The assertion in condition (ii) follows. $\square$

We can define now the Dirac and the dual Dirac elements in KK-theory. For the construction of the Dirac element we will have to assume that the action of the group G on the Hilbert space H is proper. The construction of the dual Dirac element does not require this assumption. The properness of G-action on H implies that the $G - C^*$-algebra $\mathcal{A}(H)$ is proper (Proposition 4.9). Also this algebra is nuclear as the inductive limit of nuclear algebras. This means that the extension of C^*-algebras

$$0 \to \mathcal{K}(\mathcal{S}\hat{\otimes}\mathcal{E}) \to \mathcal{F}_h(H) \to \mathcal{A}(H) \to 0$$

described before Definition 6.7 admits a completely positive cross-section.

8.2. Definition. Let the action of G on H be proper. Denote by d the element of $KK_1^G(\mathcal{A}(H), \mathcal{S}\Sigma) \simeq KK_1^G(\mathcal{A}(H), \mathbb{C})$ defined by the pair $(\mathcal{E}_1, P)$ constructed by means of Theorem 8.1 for the extension $0 \to \mathcal{K}(\mathcal{S}\hat{\otimes}\mathcal{E}) \to \mathcal{F}_h(H) \to \mathcal{A}(H) \to 0$. Also denote by $\tilde{d}$ the restriction of this element to $\tilde{\mathcal{A}}(H)$. Call these elements the Dirac elements.

To define suitable dual Dirac elements we use the $*$-homomorphism from $\tilde{\mathcal{A}}(0)$ into $\tilde{\mathcal{A}}(H)$ which is associated to the inclusion of the zero subspace into H. Using the functional calculus, this map may be presented in the form $f \mapsto f(\mathfrak{B})$, where $\mathfrak{B}$ is a uniquely determined unbounded multiplier of $\tilde{\mathcal{A}}(H)$, which we shall call the *unbounded Bott element* for $\tilde{\mathcal{A}}(H)$. There is a similar element for $\mathcal{A}(H)$. Compare Definition 4.4, which gives concrete formulas for these Bott elements in finite dimensions.

8.3. Definition. Let $\mathfrak{B}$ be the unbounded Bott element for the algebra $\tilde{\mathcal{A}}(H)$. Denote by $\tilde{b}$ the element of $KK_1^G(\mathbb{C}, \tilde{\mathcal{A}}(H))$ defined by the operator $\mathfrak{B}(1 + \mathfrak{B}^2)^{-1/2} \in \mathcal{M}(\tilde{\mathcal{A}}(H))$, and denote by b the image of this element in $KK_1^G(\mathbb{C}, \mathcal{A}(H))$. (The element b is defined by the operator $\mathfrak{B}(1+\mathfrak{B}^2)^{-1/2} \in \mathcal{M}(\mathcal{A}(H))$ where $\mathfrak{B}$ is the unbounded Bott element for the algebra $\mathcal{A}(H)$.) Call these elements the dual Dirac elements.

8.4. Lemma. *The Dirac element d in KK-theory and the Dirac asymptotic morphism α of Definition 6.7 are related by the equalities: $\eta(d) = \alpha$, $\eta(\tilde{d}) = \tilde{\alpha}$, where η is the homomorphism of Definition 7.2. The dual Dirac element in KK-theory and the dual Dirac asymptotic morphism of Definition 6.6 are related by the equalities: $\eta(b) = \beta$, $\eta(\tilde{b}) = \tilde{\beta}$.*

Proof. Only the equalities for the dual Dirac elements require proof. The calculation in this case is a standard one, and proceeds as follows. Let us write

$$P_t = 2t^{-1}\mathfrak{B}(1 + t^{-2}\mathfrak{B}^2)^{-1/2} - 1.$$

Then $\eta(b)$ is represented by the asymptotic morphism $C_0(0, 1) \to \mathcal{A}(H)$ defined by the formula

$$f \mapsto f(u_t)P_1.$$

Here u_t is a suitable quasicentral approximate unit, as in Definition 7.2. On the other hand the dual Dirac asymptotic morphism $C_0(\mathbb{R}) \to \mathcal{A}(H)$ is given by the formula

$$h \mapsto h(t^{-1}\mathfrak{B}).$$

If we identify $(0, 1)$ and $\mathbb{R}$ via the correspondence $x \leftrightarrow 2x(1 + x^2)^{-1/2} - 1$ then the dual Dirac asymptotic morphism becomes the asymptotic morphism $C_0(0, 1) \to \mathcal{A}(H)$ given by the formula

$$f \mapsto f(P_t).$$

To show that this is homotopic to the asymptotic morphism $f \mapsto f(u_t)P_1$ we first note that for $f \in C_0(0,1)$ the families $f(u_t)P_1$ and $f((1-u_t)^{1/2}P_1(1-u_t)^{1/2})$ are asymptotic to one another. Indeed both families determine asymptotic morphisms on $C_0[0,1)$ and they agree (up to asymptotic equivalence) on the generating function $f(x) = 1-x$. Next, as long as $s(t)$ is a sufficiently slowly growing function of t the families $f((1-u_t)^{1/2}P_1(1-u_t)^{1/2})$ and $f((1-u_t)^{1/2}P_{s(t)}(1-u_t)^{1/2})$ are asymptotic. Finally, connecting u_t to 0 by the straight line homotopy gives a homotopy from $f((1-u_t)^{1/2}P_{s(t)}(1-u_t)^{1/2})$ to $f(P_{s(t)})$. A reparametrization now completes the proof. □

8.5. Theorem. *Assume that the action of G on H is proper. Then $b \otimes_{\mathcal{A}(H)} d = 1 \in KK^G(\mathbb{C}, \mathbb{C})$ and $\tilde{b} \otimes_{\tilde{\mathcal{A}}(H)} \tilde{d} = 1 \in KK^G(\mathbb{C}, \mathbb{C})$.*

Proof. In view of the previous Lemma, the assertion follows from Theorems 6.10, 7.8 and Lemma 7.5. □

One can also calculate the product of the Dirac and the dual Dirac elements in the reverse order. This calculation is not used in the proof of the main results of this paper. It is based on Theorem 6.11 and the isomorphism of equivariant E and KK-theories in the case when the first argument is a proper algebra (see [23]). We omit the proof but give only the statement of this result.

8.6. Theorem. *Assume that the action of G on H is proper. Then $d \otimes_{\mathbb{C}} b = 1_{\mathcal{A}(H)} \in KK^G(\mathcal{A}(H), \mathcal{A}(H))$ and $\tilde{d} \otimes_{\mathbb{C}} \tilde{b} = 1_{\tilde{\mathcal{A}}(H)} \in KK^G(\tilde{\mathcal{A}}(H), \tilde{\mathcal{A}}(H))$.* □

9. Consequences

9.1. Theorem. *Any locally compact group G acting properly, isometrically on a Hilbert space satisfies the Baum-Connes conjecture with coefficients in any separable G-algebra.*

Proof. The assertion follows from Proposition 4.9, Theorem 8.5 and Theorem 7.1. □

This theorem has the following corollary:

9.2. Corollary. All second-countable locally compact amenable groups satisfy the Baum-Connes conjecture with coefficients.

Proof. It is proved in [4] that any countable amenable group acts properly, isometrically on a Hilbert space. It is a simple matter to extend the argument of [4] to second-countable locally compact amenable groups. □

The following result which was first proved in [17] can now be obtained from Theorem 8.5. It is interesting to point out that unlike [17], we deduce this result here without using representation theory of Lie groups at all. We recall from [22], 5.7, that for any Lie group G there exists a canonical idempotent element γ_G in the ring $KK^G(\mathbb{C}, \mathbb{C})$.

9.3. Theorem. *The element* $\gamma_G \in KK^G(\mathbb{C}, \mathbb{C})$ *for the group* $G = SU(n, 1)$ *is equal to* 1.

Proof. The following characterization (or rather definition) of the element γ_G for a locally compact group G appeared as a remark in one of the earliest versions of [23]. It has become part of folklore now.[5] The element $\gamma_G \in KK^G(\mathbb{C}, \mathbb{C})$, when it exists for a locally compact group G, is uniquely characterized by the following two properties:

1. $\gamma_G = x \otimes_A y$, where $x \in K^G_*(A)$, $y \in K^*_G(A)$ for some proper $G - C^*$-algebra A.
2. $p^*(\gamma_G) = 1$ in $RKK^G(\mathcal{E}G;\ \mathbb{C}, \mathbb{C})$, where p is the map of $\mathcal{E}G$ to a point.

The proof of uniqueness is a simple exercise. In our case we take $A = \mathcal{A}(H)$, $x = b$, and $y = d$. □

Finally it also follows from Proposition 4.9, Theorem 8.5 and Theorem 7.1 that:

9.4. Theorem. *Any locally compact group acting properly, isometrically on a Hilbert space is K-amenable.* □

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[5] The reader is warned that a slightly different definition is used in [11].

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