Clay Mathematics Proceedings

The Residue Index Theorem of Connes and Moscovici

Nigel Higson

Contents

1. Introduction

Several years ago Alain Connes and Henri Moscovici discovered a quite general "local" index formula in noncommutative geometry [**12**] which, when applied to Dirac-type operators on compact manifolds, amounts to an interesting combination of two quite different approaches to index theory.

Atiyah and Bott noted that the index of an elliptic operator D may be expressed as a complex residue

$$\operatorname{Index}(D) = \operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Trace}(\varepsilon(I+\Delta)^{-s})\big),$$

where $\Delta = D^2$ (see [**1**]). Rather surprisingly, the residue may be computed, at least in principle, as the integral of an explicit expression involving the coefficients of D, the metric g, and the derivatives of these functions. However the formulas can be very complicated.

In a different direction, Atiyah and Singer developed the crucial link between index theory and K-theory. They showed, for example, that an elliptic operator D on M determines a class

$$[D] \in K_0(M)$$

in the K-homology of M (see [**2**] for one account of this). As it turned out, this was a major advance: when combined with the Bott periodicity theorem, the construction of $[D]$ leads quite directly to a proof of the index theorem.

When specialized to the case of elliptic operators on manifolds, the index formula of Connes and Moscovici associates to an elliptic operator D on M a cocycle for the group $HCP^*(C^\infty(M))$, the periodic cyclic cohomology of the algebra of smooth functions on M. In this respect the Connes-Moscovici formula calls to mind the construction of Atiyah and Singer, since cyclic cohomology is related to K-homology by a Chern character isomorphism. But the actual formula for the Connes-Moscovici cocycle involves only residues of zeta-type functions associated to D. In this respect it calls to mind the Atiyah-Bott formula.

The proper context for the Connes-Moscovici index formula is the noncommutative geometry of Connes [**7**], and in particular the theory of spectral triples. Connes and Moscovici have developed at length a particular case of the index formula which is relevant to the transverse geometry of foliations [**12, 13**]. This work, which involves elaborate use of Hopf algebras, has attracted considerable attention (see the survey articles [**8**] and [**26**] for overviews). At the same time, other instances of the index formula are beginning to be developed (see for example [**9**], which among other things gives a good account of the meaning of the term "local" in noncommutative geometry).

The original proof of the Connes-Moscovici formula, which is somewhat involved, reduces the local index formula to prior work on the transgression of the Chern character, and is therefore is actually spread over several papers [**12, 11, 10**]. Roughly speaking, the residues of zeta functions which appear in the formula are related by the Mellin transform to invariants attached to the heat semigroup $e^{-t\Delta}$. The heat semigroup figures prominently in the theory of the JLO cocycle in cyclic theory, and so previous work on this subject can now be brought to bear on the local index formula.

The main purpose of these notes is to present, in a self-contained way, a new and perhaps more accessible proof of the local index formula. But for the benefit of those who are just becoming acquainted with Connes' noncommutative geometry, we have also tried to provide some context for the formula by reviewing at the beginning of the notes some antecedent ideas in cyclic and Hochschild cohomology.

As for the proof of the theorem itself, in contrast to the orginal proof of Connes and Moscovici, we shall work directly with the complex powers Δ^{-z}. Our strategy is to find an elementary quantity $\langle a^0, [D, a^1], \ldots, [D, a^p]\rangle_z$ (see Definition 4.12), a sort of multiple zeta function, which is meromorphic in the argument z, and whose residue at $z = -\frac{p}{2}$ is the complicated combination of residues which appears in the Connes-Moscovici cocycle. The proof of the index formula can then be organized in a fairly conceptual way using the new quantities. The main steps are summarized in Theorems 5.5, 5.6, 7.1 and 7.12.

The "elementary quantity" $\langle a^0, [D, a^1], \ldots, [D, a^p]\rangle_z$ was obtained by emulating some computations of Quillen [**23**] on the structure of Chern character cocycles in cyclic theory. Quillen constructed a natural "connection form" Θ in a differential graded cochain algebra, along with a "curvature form" $K = d\Theta + \Theta^2$, for which the quantities

$$\Gamma(z)\,\mathrm{Trace}\big(K^{-z}\big) = \frac{\Gamma(z)}{2\pi i}\,\mathrm{Trace}\bigg(\int \lambda^{-z}(\lambda - K)^{-1}\,d\lambda\bigg)$$

have components $\langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$. Taking residues at $z = -\frac{p}{2}$ we get (at least formally)

$$\operatorname{Trace}\big(K^{\frac{p}{2}}\big) = \operatorname{Res}_{z=-\frac{p}{2}} \langle 1, [D, a^1], \ldots, [D, a^p]\rangle_z$$

Now, in the context of vector bundles with curvature form K, the pth component of the Chern character is a constant times $\operatorname{Trace}(K^{\frac{p}{2}})$. As a result, it is natural to guess that our elementary quantities $\langle \cdots \rangle_z$ are related to the Chern character and index theory, after taking residues. All this will be explained in a little more detail at the end of the notes, in Appendix B. Appendix A explains the relation between the Connes-Moscovici cocycle and the JLO cocycle, which was one of the orginal objects of Quillen's study and which, as we noted above, played an important in the original approach to the index formula.

A final appendix presents a proof of Connes' Hochschild class formula. This is essentially a back-formation from the proof of the local index formula presented here. (Connes' Hochschild formula is introduced in Section 3 as motivation for the development of the local index formula.)

Obviously the whole of the present work is strongly influenced by the work of Connes and Moscovici. Moreover in several places the computations which follow are very similar ones they have carried out in their own work. I am very grateful to both of them for their encouragement and support. I also thank members of Penn State's Geometric Functional Analysis Seminar, especially Raphaël Ponge, for their advice, and for patiently listening to early versions of this work.

2. The Cyclic Chern Character

In this section we shall establish some notation and terminology related to Fredholm index theory and cyclic cohomology. For obvious reasons we shall follow Connes' approach to cyclic cohomology, which is described for example in his book [**7**, Chapter 3]. Along the way we shall make explicit choices of normalization constants.

2.1. Fredholm Index Problems. A linear operator $T\colon V \to W$ from one vector space to another is *Fredholm* its kernel and cokernel are finite-dimensional, in which case the *index* of T is defined to be

$$\operatorname{Index}(T) = \dim\ker(T) - \dim\operatorname{coker}(T).$$

The index of a Fredholm operator has some important stability properties, which make it feasible in many circumstances to attempt a computation of the index even if computations of the kernel and cokernel, or even their dimensions, are beyond reach.

First, if $F\colon V \to W$ is any finite-rank operator then $T + F$ is also Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T + F)$. Second, if V and W are Hilbert spaces then the set of all bounded Fredholm operators from V to W is an open subset of the set of all bounded operators in the operator norm-topology, and moreover the index function is locally constant. In addition, if $K\colon V \to W$ is any compact operator between Hilbert spaces (which is to say that K is a norm-limit of finite-rank operators), then $T + K$ is Fredholm, and moreover $\operatorname{Index}(T) = \operatorname{Index}(T + K)$. In fact, an important theorem of Atkinson asserts that a bounded linear operator between Hilbert spaces is Fredholm if and only if it is invertible modulo compact operators. See for example [**15**].

The following situation occurs frequently in geometric problems which make contact with Fredholm index theory. One is presented with an associative algebra A of bounded operators on a Hilbert space H, and one is given a bounded self-adjoint operator $F\colon H \to H$ with the property that $F^2 = 1$, and for which, for every $a \in A$, the operator $[F, a] = Fa - aF$ is *compact.* This setup (or a small modification of it) was first studied by Atiyah [**2**], who made the following observation related to index theory and K-theory. Since $F^2 = 1$ the operator $P = \frac{1}{2}(P+1)$ is a projection on H (it is the orthogonal projection onto the $+1$ eigenspace of F). If u is any invertible element of A then the operator $PuP\colon PH \to PH$ is Fredholm. This is because the operator $Pu^{-1}P\colon PH \to PH$ is an inverse, modulo compact operators, and so Atkinson's theorem, cited above, applies.

A bit more generally, if $U = [u_{ij}]$ is an $n \times n$ invertible matrix over A then the matrix $PUP = [Pu_{ij}P]$, regarded as an operator on the direct sum of n copies of PH, is a Fredholm operator (for basically the same reason). Now the invertible matrices over A constitute generators for the (algebraic) K-theory group $K_1^{\mathrm{alg}}(A)$ (see [**22**] for details[1]). It is not hard to see that Atiyah's index construction gives rise to a homomorphism of groups

$$\mathrm{Index}_F\colon K_1^{\mathrm{alg}}(A) \to \mathbb{Z}.$$

If A is a reasonable[2] topological algebra, for instance a Banach algebra, so that *topological* K-groups are defined, then the index construction even descends to a homomorphism

$$\mathrm{Index}_F\colon K_1^{\mathrm{top}}(A) \to \mathbb{Z}.$$

In short, the data consisting of A and F together provides a supply of Fredholm operators, and one can investigate in various examples the possibility of determining the indices of these Fredholm operators.

2.1. Example. Let A be the algebra of smooth, complex-valued functions on the unit circle S^1, H is the Hilbert space $L^2(S^1)$, and F is the *Hilbert transform* on the circle, which maps the trigonometric function $\exp(2\pi inx)$ to $\exp(2\pi inx)$ when $n \geq 0$ and to $-\exp(2\pi inx)$ when $n < 0$. To see that the operators $[F, a]$ are compact, one can first make an explicit computation in the case where a is a trigonometric monomial $a(x) = \exp(2\pi inx)$, with the result that $[F, a]$ is in fact a finite-rank operator. The general case follows by approximating a general $a \in A$ by a trigonometric polynomial. In this example one has the famous index formula

$$\mathrm{Index}(PuP) = -\frac{1}{2\pi i}\int_{S^1} u^{-1}du.$$

The right hand side is (minus) the winding number of the function $u\colon S^1 \to \mathbb{C}\setminus\{0\}$. (There is also a simple generalization to matrices $U = [u_{ij}]$.) The topological K_1-group here is $\mathbb{Z}$, and the index homomorphism is an isomorphism.

These notes are concerned with formulas for the Fredholm indices which arise from certain instances of Atiyah's construction. We are going to write down a bit more carefully the basic data for the construction, and then add a first additional hypothesis to narrow the scope of the problem just a little.

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[1] Except to provide some background context, we shall not use K-theory in these notes.

[2] See the appendix of [**3**] for a discussion of some types of reasonable topological algebra.

2.2. Definition. Let A be an associative algebra over $\mathbb{C}$. An *odd Fredholm module* over A is a triple consisting of:

(a) a Hilbert space H,
(b) a representation of A as bounded operators on H, and
(c) a self-adjoint operator $F\colon H \to H$ such that $F^2 = 1$ and such that $[F, \pi(a)]$ is a compact operator, for every $a \in A$.

An *even Fredholm module* over A consists of the same data as above, together with a self-adjoint operator $\varepsilon\colon H \to H$ such that $\varepsilon^2 = 1$, such that ε commutes with each operator $\pi(a)$, and such that ε anticommutes with F.

Since ε is self-adjoint and since $\varepsilon^2 = 1$, the Hilbert space H decomposes as an orthogonal direct sum $H = H_0 \oplus H_1$ in such a way that $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The additional hypothesis imply that

$$F = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \quad \text{and} \quad \pi(a) = \begin{pmatrix} \pi_0(a) & 0 \\ 0 & \pi_1(a) \end{pmatrix}.$$

Even Fredholm modules often arise from geometric problems on even-dimensional manifolds — hence the terminology. They are actually closer to Atiyah's original constructions in [**2**] than are the odd Fredholm modules.

Associated to an even Fredholm module there is the following index construction. If p is an idempotent element of A then the operator

$$\pi_1(p) F \pi_0(p)\colon \pi_0(p)H_0 \to \pi_1(p)H_1$$

is Fredholm, since $\pi_0(p)F\pi_1(p)\colon \pi_1(p)H_1 \to \pi_0(p)H_0$ is an inverse, modulo compact operators. This construction passes easily to matrices, and we obtain a homomorphism

$$\mathrm{Index}_F\colon K_0^{\mathrm{alg}}(A) \to \mathbb{Z}.$$

which is the counterpart of the index homomorphism we previously constructed in the odd case.

2.3. Definition. A Fredholm module over A is *finitely summable* if there is some $d \geq 0$ such that for every integer $n \geq d$ every product of commutators

$$[F, \pi(a^0)][F, \pi(a^1)] \cdots [F, \pi(a^n)]$$

is a trace-class operator. (See [**25**] for a discussion of trace class operators.)

2.4. Example. The Fredholm module presented in Example 2.1 is finitely summable: one can take $d = 1$.

We are going to determine formulas in multi-linear algebra for the indices of Fredholm operators associated to finitely summable Fredholm modules.

2.2. Cyclic Cocycles.

2.5. Definition. A $(p+1)$-linear functional $\phi\colon A^{p+1} \to \mathbb{C}$ is said to be *cyclic* if

$$\phi(a^0, a^1, \ldots, a^p) = (-1)^p \phi(a^p, a^0, \ldots, a^{p-1}),$$

for all $a^0, \ldots, a^p$ in A.

2.6. Definition. The *coboundary* of a $(p+1)$-linear functional $\phi\colon A^{p+1} \to \mathbb{C}$ is the $(p+2)$-linear functional

$$b\phi(a^0, \dots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \dots, a^j a^{j+1}, \dots, a^{p+1}) \\ + (-1)^{p+1} \phi(a^{p+1}a^0, \dots, a^p)$$

A $(p+1)$-multilinear functional ϕ is a *p-cocycle* if $b\phi = 0$.

It is easy to check that the coboundary of any coboundary is zero, or in other words $b^2 = 0$. Thus every coboundary is a cocycle and as a result we can form what are called the *Hochschild cohomology groups* of A: the pth Hochschild group is the quotient of the p-cocycles by the p-cocycles which are coboundaries. We will return to these groups in Section 3, but for the purposes of index theory we are much more interested in the special properties of *cyclic* cocycles.

2.7. Theorem (Connes). *Let ϕ be a $(p+1)$-linear functional on which is both cyclic and a cocycle.*

(a) *If p is odd, and if u is an invertible element of A then the quantity*

$$\langle \phi, u \rangle = \text{constant} \cdot \phi(u^{-1}, u, \dots, u^{-1}, u)$$

depends only on the class of u in the abelianization of $GL_1(A)$, and defines a homomorphism from the abelianization into $\mathbb{C}$.

(b) *If p is even and if e is an idempotent element of A then the quantity*

$$\langle \phi, e \rangle = \text{constant} \cdot \phi(e, e, \dots, e)$$

depends only on the equivalence class[3] of e. If e_1 and e_2 are orthogonal, in the sense that $e_1 e_2 = e_2 e_1 = 0$, then

$$\langle \phi, e_1 + e_2 \rangle = \langle \phi, e_1 \rangle + \langle \phi, e_2 \rangle. \qquad \square$$

2.8. Remark. We have inserted as yet unspecified constants into the formulas for the pairings $\langle\, , \rangle$. As we shall see, they are needed to make the pairings for varying p consistent with one another, The constants will be made explicit in Theorem 2.27.

2.9. Example. The simplest non-trivial instances of the theorem occur when $p = 1$ or $p = 2$. For $p = 1$ the explicit conditions on ϕ are

$$\begin{cases} \phi(a^0, a^1) = -\phi(a^1, a^0) \\ \phi(a^0a^1, a^2) - \phi(a^0, a^1a^2) + \phi(a^2a^0, a^1) = 0, \end{cases}$$

while for $p = 2$ the conditions are

$$\begin{cases} \phi(a^0, a^1, a^2) = \phi(a^2, a^0, a^1) \\ \phi(a^0a^1, a^2, a^3) - \phi(a^0, a^1a^2, a^3) + \phi(a^0, a^1, a^2a^3) - \phi(a^3a^0, a^1, a^2) = 0. \end{cases}$$

The reader who has not done so before ought to try to tackle the theorem for his or herself in these cases before consulting Connes' paper [**4**].

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[3] Two idempotents e and f are *equivalent* if there are elements x and y of A such that $e = xy$ and $f = yx$. If A is for example a matrix algebra then two idempotent matrices are equivalent if and only if their ranges have the same dimension.

The pairings $\langle\ ,\ \rangle$ defined by the theorem extend easily to invertible and idempotent matrices, and thereby define homomorphisms

$$\langle \phi, \ \rangle \colon K_1^{\mathrm{alg}}(A) \to \mathbb{C} \qquad p \text{ odd}$$
$$\langle \phi, \ \rangle \colon K_0^{\mathrm{alg}}(A) \to \mathbb{C} \qquad p \text{ even}$$

The question now arises, can the index homomorphisms constructed in the previous section be recovered as instances of the above homomorphisms, for suitable cyclic cocycles ϕ? This was answered by Connes, as follows:

2.10. Theorem. *Let (A, H, F) be a finitely summable, odd Fredholm module and let $n = 2k+1$ be an odd integer such that for all $a^0, \ldots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\phi\big(a^0, \ldots, a^n\big) = \frac{1}{2} \operatorname{Trace}\big(F[F, a^0][F, a^1] \ldots [F, a^n]\big)$$

defines a cyclic n-cocycle on A. If u is an invertible element of A then

$$\phi(u, u^{-1}, \ldots, u, u^{-1}) = (-1)^{k+1} 2^{2k+1} \operatorname{Index}(PuP \colon PH \to PH),$$

where $P = \frac{1}{2}(F+1)$. □

2.11. Theorem. *Let (A, H, F) be a finitely summable, even Fredholm module, and let $n = 2k$ be an even integer such that for all $a^0, \ldots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\phi\big(a^0, \ldots, a^n\big) = \frac{1}{2} \operatorname{Trace}\big(\varepsilon F[F, a^0][F, a^1] \ldots [F, a^n]\big)$$

defines a cyclic n-cocycle on A. If e is an idempotent element of A then

$$\phi(e, e, \ldots, e) = (-1)^k \operatorname{Index}(eFe \colon eH_0 \to H_1).$$

□

The proofs of these results may be found in [**4**] or [**7**, IV.1] (but in the next section we shall at least verify that the formulas do indeed define cyclic cocycles).

2.3. Cyclic Cohomology. Throughout this section we shall assume that A is an associative algebra over $\mathbb{C}$ with a mutlilicative identity 1. The definitions for algebras with an identity are a little different and will be considered later.

It is a remarkable fact that if ϕ is a cyclic multi-linear functional then so is its coboundary $b\phi$.[4] As a result of this we can form the *cyclic cohomology groups* of A:

2.12. Definition. The pth cyclic cohomology group of a complex algebra A is the quotient $HC^p(A)$ of the cyclic p-cocycles by the cyclic p-cocycles which are cyclic coboundaries.

But we are interested in a small modification of the cyclic cohomology groups, called the *periodic* cyclic cohomology groups of A. There are only two such groups — an even one and and odd one. The even periodic group $HCP^{\mathrm{even}}(A)$ in some sense combines all the $HC^{2k}(A)$ into one group, while the odd periodic group $HCP^{odd}(A)$ does the same for the $HC^{2k+1}(A)$. One reason for considering the

[4]Note that cyclicity for a $(p+1)$-linear functional ϕ has to do with invariance under the action of the cyclic group C_{p+1}, whereas cyclicity for $b\phi$ has to do with invariance under C_{p+2}, so to a certain extent b intertwines the actions of two different groups — this is what is so remarkable.

periodic groups is that Connes' construction of the cyclic cocycle associated to a Fredholm module produces not one cyclic cocycle, but one for each sufficiently large integer n of the correct parity. As we shall see, the periodic cyclic cohomology groups provide a framework within which these different cocycles can be compared with one another.

The definition of $HCP^{\mathrm{even}\,/\,\mathrm{odd}}(A)$ is, at first sight, a little strange, but after we look at some examples it will come to seem more natural.

2.13. Definition. Let A be an associative algebra over $\mathbb{C}$ with a multiplicative identity element 1. If p is a non-negative integer, then denote by $C^p(A)$ space of $(p+1)$-multi-linear maps ϕ from A into $\mathbb{C}$ wich have the property that if $a^j = 1$, for some $j \geq 1$, then $\phi(a^0, \ldots, a^p) = 0$. Define operators

$$b\colon C^p(A) \to C^{p+1}(A) \quad \text{and} \quad B\colon C^{p+1}(A) \to C^p(A),$$

by the formulas

$$b\phi(a^0, \ldots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{p+1}) \\ + (-1)^{p+1} \phi(a^{p+1} a^0, \ldots, a^p)$$

and

$$B\phi(a^0, \ldots, a^p) = \sum_{j=0}^{p} (-1)^{pj} \phi(1, a^j, a^{j+1}, \ldots, a^{j-1}).$$

2.14. Remark. The operator b is the same as the coboundary operator that we encountered in the previous section, except that we are now considering a slightly restricted class of multi-linear maps on which b is defined (we should not that a simple computations shows b to be well defined as a map from $C^p(A)$ into $C^{p+1}(A)$). In what follows, we could in fact work with *all* multi-linear functionals, rather than just those for which $\phi(a^0, \ldots, a^p) = 0$ when $a^j = 1$ for some $j \geq 1$ (although this would entail a small modification to the formula for the operator B; see [**21**]). The setup we are considering is a bit more standard, and allows for some slightly simpler formulas.

2.15. Lemma. *$b^2 = 0$, $B^2 = 0$ and $bB + Bb = 0$.* □

As a result of the lemma, we can assemble from the spaces $C^p(A)$ the following double complex, which is continued indefinitely to the left and to the top.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(A) \arrow[u, "b"] \arrow[r, "B"] & C^2(A) \arrow[u, "b"] \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(A) \arrow[u, "b"] \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(A) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & &
\end{tikzcd}
$$

2.16. Definition. The *periodic cyclic cohomology of A* is the cohomology of the totalization of this complex.

Thanks to the symmetry inherent in the complex, all even cohomology groups are the same, as are all the odd groups. As a result, one speaks of the *even* and *odd* periodic cyclic cohomology groups of A. A cocycle for the even group is a sequence

$$(\phi_0, \phi_2, \phi_4, \dots),$$

where $\phi_{2k} \in C^{2k}$, $\phi_{2k} = 0$ for all but finitely many k, and

$$b\phi_{2k} + B\phi_{2k+2} = 0$$

for all $k \geq 0$. Similarly a cocycle for the odd group is a sequence

$$(\phi_1, \phi_3, \phi_5, \dots),$$

where $\phi_{2k+1} \in C^{2k+1}$, $\phi_{2k+1} = 0$ for all but finitely many k, and

$$b\phi_{2k+1} + B\phi_{2k+3} = 0$$

for all $k \geq 0$ (and in addition $B\phi_1 = 0$).

2.17. Definition. We shall refer to cocycles of the above sort as *(b, B)-cocycles.* This will help us distinguish between these cocycles and the cyclic cocycles which we introduced in the last section.

Suppose now that ϕ_n is a cyclic n-cocycle, as in the last section, and suppose that ϕ_n has the property that $\phi(a^0, \dots, a^n) = 0$ when some a^j is equal to 1. Note that Connes' cocycles described in Theorems 2.10 and 2.11 have this property. By definition, $b\phi_n = 0$, and clearly $B\phi_n = 0$ too, since the definition of D involves the insertion of 1 as the first argument of ϕ_n. As a result, the sequence

$$(0, \dots, 0, \phi_n, 0, \dots),$$

obtained by placing ϕ_n in position n and 0 everywhere else, is a (b, B)-cocycle. In this was we shall from now on regard every cyclic cocycle as a (b, B)-cocycle.

2.18. Remark. It is known that every (b, B)-cocycle is cohomologous to a cyclic cocycle of some degree p (see [**21**]).

Let us now return to the cocycles which Connes constructed from a Fredholm module.

2.19. Theorem. *Let (A, H, F) be a finitely summable, odd Fredholm module and let n be an odd integer such that for all $a^0, \dots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\mathrm{ch}_n^F\big(a^0, \dots, a^n\big) = \frac{\Gamma(\frac{n}{2}+1)}{2 \cdot n!} \operatorname{Trace}\big(F[F, a^0][F, a^1] \dots [F, a^n]\big)$$

defines a cyclic n-cocycle on A whose periodic cyclic cohomology class is independent of n.

Proof. Define

$$\psi_{n+1}(a^0, \dots, a^{n+1}) = \frac{\Gamma(\frac{n}{2}+2)}{(n+2)!} \operatorname{Trace}\big(a^0 F[F, a^1][F, a^2] \dots [F, a^{n+1}]\big).$$

It is then straightforward to compute that $b\psi_{n+1} = -\mathrm{ch}_{n+2}^F$ while $B\psi_{n+1} = \mathrm{ch}_n^F$. Hence $\mathrm{ch}_n^F - \mathrm{ch}_{n+2}^F$ is a (b, B)-coboundary. □

2.20. Remarks. Obviously, the multiplicative factor $\frac{\Gamma(\frac{n}{2}+1)}{2n!}$ is chosen to guarantee that the class of ch_n^F in periodic cyclic cohomology is independent of n. Since $b^2 = 0$, the formula $b\psi_{n+1} = -\mathrm{ch}_{n+2}^F$ proves that ch_{n+2}^F is a cocycle (i.e. $b\,\mathrm{ch}_{n+2}^F = 0$). In addition, since it is clear from the definition of the operator B that the range of B consists entirely of cyclic multi-linear functionals, the formula $B\psi_{n+1} = \mathrm{ch}_n^F$ proves that ch_n^F is cyclic.

2.21. Theorem. *Let (A, H, F) be a finitely summable, even Fredholm module and let n be an odd integer such that for all $a^0, \dots, a^n$ in A the product $[F, a^0] \cdots [F, a^n]$ is a trace-class operator. The formula*

$$\mathrm{ch}_n^F\big(a^0, \dots, a^n\big) = \frac{\Gamma(\frac{n}{2}+1)}{2 \cdot n!} \operatorname{Trace}\big(\varepsilon F[F, a^0][F, a^1] \dots [F, a^n]\big)$$

defines a cyclic n-cocycle on A whose periodic cyclic cohomology class is independent of n.

Proof. Define

$$\psi_{n+1}(a^0, \dots, a^{n+1}) = \frac{\Gamma(\frac{n}{2}+2)}{(n+2)!} \operatorname{Trace}\big(a^0 F[F, a^1][F, a^2] \dots [F, a^{n+1}]\big)$$

and proceed as before. □

2.22. Definition. The cocycle ch_n^F defined in Theorem 2.19 or 2.21 is the *cyclic Chern character* of the odd or even Fredholm module (A, H, F).

2.4. Comparison with de Rham Theory. Let M be a smooth, closed manifold and denote by $C^\infty(M)$ the algebra of smooth, complex-valued functions on M. For $p \geq 0$ denote by Ω_p the space of p-dimensional de Rham currents (dual to the space Ω^p of smooth p-forms).

Each current $c \in \Omega_p$ determines a cochain $\phi_c \in C^p(A)$ for the algebra $C^\infty(M)$ by the formula

$$\phi_c(f^0, \dots, f^p) = \int_c f^0 df^1 \cdots df^p.$$

The following is a simple computation:

2.23. LEMMA. *If $c \in \Omega_p$ is any p-current on M then*

$$b\phi_c = 0 \quad \textit{and} \quad B\phi_c = p \cdot \phi_{d^* c},$$

where $d^* \colon \Omega_p \to \Omega_{p-1}$ is the operator which is adjoint to the de Rham differential. □

The lemma implies that if we assemble the spaces Ω_p into a bicomplex, as follows,

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "4d^*"] & \Omega_3 \arrow[u, "0"] \arrow[r, "3d^*"] & \Omega_2 \arrow[u, "0"] \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] \\
\cdots \arrow[r, "3d^*"] & \Omega_2 \arrow[u, "0"] \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & \\
\cdots \arrow[r, "2d^*"] & \Omega_1 \arrow[u, "0"] \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & & \\
\cdots \arrow[r, "d^*"] & \Omega_0 \arrow[u, "0"] & & &
\end{tikzcd}$$

then the construction $c \mapsto \phi_c$ defines a map from this bicomplex to the bicomplex which computes periodic cyclic cohomology of $A = C^\infty(M)$.

A fundamental result of Connes [**4**, Theorem 46] asserts that this map of complexes is an isomorphism on cohomology:

2.24. THEOREM. *The inclusion $c \mapsto \phi_c$ of the above double complex into the (b, B)-bicomplex induces isomorphisms*

$$HCP^{\mathrm{even}}_{\mathrm{cont}}(C^\infty(M)) \cong H_0(M) \oplus H_2(M) \oplus \cdots$$

and

$$HCP^{\mathrm{odd}}_{\mathrm{cont}}(C^\infty(M)) \cong H_1(M) \oplus H_3(M) \oplus \cdots$$

Here $HCP^*_{\mathrm{cont}}(C^\infty(M))$ denotes the periodic cyclic cohomology of M, computed from the bicomplex of continuous *multi-linear functionals on $C^\infty(M)$.* □

It follows that an even/odd (b, B)-cocycle for $C^\infty(M)$ is something very like a family of closed currents on M of even/odd degrees.

Connes' theorem is proved by first identifying the (continuous) Hochschild cohomology of the algebra $A = C^\infty(M)$. Lemma 2.23 shows that there is a map of complexes

$$\begin{tikzcd}
\Omega^0 \arrow[r, "0"] \arrow[d] & \Omega^1 \arrow[r, "0"] \arrow[d] & \cdots \arrow[r, "0"] & \Omega^n \arrow[r, "0"] \arrow[d] & 0 \arrow[r, "0"] \arrow[d] & \cdots \\
C^0(A) \arrow[r, "b"'] & C^1(A) \arrow[r, "b"'] & \cdots \arrow[r, "b"'] & C^n(A) \arrow[r, "b"'] & C^{n+1}(A) \arrow[r, "b"'] & \cdots
\end{tikzcd}$$

in which the vertical maps come from the construction $c \mapsto \phi_c$. The following result is known as the Hochschild Kostant Rosenberg theorem (see [**21**]), although this precise formulation is due to Connes [**4**].

2.25. Theorem. *The above map induces an isomorphism from Ω_p to the pth continuous Hochschild cohomology group $HH^p_{\text{cont}}(C^\infty(M))$.*

Let us conclude this section with a few brief remarks about non-periodic cyclic cohomology groups. We already noted that every cyclic p-cocycle determines a (B,b)-cocycle (even or odd, according to the parity of p). In view of the Hochschild Kostant Rosenberg theorem, and in view of the fact that every cyclic p-cocycle is in particular a Hochschild p-cocycle, so that if A is any algebra then there is a natural map from pth cyclic cohomology group $HC^p(A)$, as given in Definition 2.12, into the Hochschild group $HH^p(A)$, it might be thought that when $A = C^\infty(M)$ the cyclic p-cocycles correspond to the summand $H_p(M)$ in Theorem 2.24. But this is not exactly right. It cannot be right because if a (b,B)-cocycle is cohomologous to a cyclic p-cocycle, it may be shown that it is also cohomologous to a cyclic $(p+2)$-cocycle, and to a cyclic $(p+4)$-cocycle, and so on. So when $A = C^\infty(M)$ a single cyclic p-cocycle can encode information not just about closed p-currents, but also about closed $(p-2)$-currents, closed $(p-4)$-currents, and so on. The precise relation, again discovered by Connes, is as follows:

2.26. Theorem. *Denote by $Z_p(M)$ the set of closed de Rham k-currents on M. There are isomorphisms*

$$HC^{2k}_{\text{cont}}(C^\infty(M)) \cong H_0(M) \oplus H_2(M) \oplus \cdots \oplus H_{2k-2}(M) \oplus Z_{2k}(M)$$

and

$$HC^{2k+1}_{\text{cont}}(C^\infty(M)) \cong H_1(M) \oplus H_3(M) \oplus \cdots \oplus H_{2k-1}(M) \oplus Z_{2k+1}(M).$$

Here $HC^*_{\text{cont}}(C^\infty(M))$ denotes the cyclic cohomology of M, computed from the complex of continuous *cyclic multi-linear functionals on $C^\infty(M)$.* □

2.5. Pairings with K-Theory. The pairings described in Theorem 2.7 between cyclic cocycles and K-theory have the following counterparts in periodic cyclic theory.

2.27. Theorem (Connes). *Let A be an algebra with a multiplicative identity.*

(a) *If $\phi = (\phi_1, \phi_3, \dots)$ is an odd (b,B)-cocycle for A, and u is an invertible element of A, then the quantity*

$$\langle \phi, u \rangle = \frac{1}{\Gamma(\frac{1}{2})} \sum_{k=0}^{\infty} (-1)^{k+1} k! \phi_{2k+1}(u^{-1}, u, \dots, u^{-1}, u)$$

depends only on the class of u in the abelianization of $GL_1(A)$ and the periodic cyclic cohomology class of ϕ, and defines a homomorphism from the abelianization into $\mathbb{C}$.

(b) *If $\phi = (\phi_0, \phi_2, \dots)$ is an even (b,B)-cocycle for A, and e is an idempotent element of A, then the quantity*

$$\langle \phi, e \rangle = \phi_0(e) + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \phi_{2k}(e - \frac{1}{2}, e, e, \dots, e).$$

depends only on the equivalence class of e and the periodic cyclic cohomology class of ϕ. Moreover if e_1 and e_2 are orthogonal idempotents in A, then

$$\langle \phi, e_1 + e_2 \rangle = \langle \phi, e_1 \rangle + \langle \phi, e_2 \rangle.$$

Proof. See [**16**] for the odd case and [**17**] for the even case. □

2.28. Example. Putting together Theorem 2.10 with the formula (a) in Theorem 2.27, we see that if (A, H, F) is a finitely summable, odd Fredholm module, and if u is an invertible element of A, then

$$\langle \mathrm{ch}_n^F, u \rangle = \mathrm{Index}(PuP\colon PH \to PH),$$

where P is the idempotent $P = \frac{1}{2}(F+1)$, and the odd integer n is large enough that the Chern character is defined. Similarly, if (A, H, F) is an even Fredholm module and if e is an idempotent element of A then

$$\langle \mathrm{ch}_n^F, e \rangle = \mathrm{Index}(eFe\colon eH_0 \to eH_1),$$

for all even n which again are large enough that the Chern character is defined.

2.29. Remarks. The pairings described in Theorem 2.27 extend easily to the algebraic K-theory groups $K_1^{\mathrm{alg}}(A)$ and $K_0^{\mathrm{alg}}(A)$.

2.6. Improper Cocycles and Coefficients. We are going to describe to extensions of the notion of (b, B)-cocycle which will be useful in these notes.

If V is a complex vector space and p is a non-negative integer, then let us denote by $C^p(A, V)$ space of $(p+1)$-multi-linear maps ϕ from A into V for which $\phi(a^0, \ldots, a^p) = 0$ whenever $a^j = 1$ for some $j \geq 1$.

Define operators

$$b\colon C^p(A, V) \to C^{p+1}(A, V) \quad \text{and} \quad B\colon C^{p+1}(A, V) \to C^p(A, V),$$

by the same formulas we used previously:

$$\begin{aligned} b\phi(a^0, \ldots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \phi(a^0, \ldots, a^j a^{j+1}, \ldots, a^{p+1}) \\ + (-1)^{p+1} \phi(a^{p+1} a^0, \ldots, a^p) \end{aligned}$$

and

$$B\phi(a^0, \ldots, a^p) = \sum_{j=0}^{p} (-1)^{pj} \phi(1, a^j, a^{j+1}, \ldots, a^{j-1}).$$

Then assemble the double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
\cdots & \xrightarrow{B} & C^3(A,V) & \xrightarrow{B} & C^2(A,V) & \xrightarrow{B} & C^1(A,V) & \xrightarrow{B} & C^0(A,V) \\
 & & \uparrow b & & \uparrow b & & \uparrow b & & \\
\cdots & \xrightarrow{B} & C^2(A,V) & \xrightarrow{B} & C^1(A,V) & \xrightarrow{B} & C^0(A,V) & & \\
 & & \uparrow b & & \uparrow b & & & & \\
\cdots & \xrightarrow{B} & C^1(A,V) & \xrightarrow{B} & C^0(A,V) & & & & \\
 & & \uparrow b & & & & & & \\
\cdots & \xrightarrow{B} & C^0(A,V) & & & & & &
\end{array}$$

just as before.

2.30. Definition. The *periodic cyclic cohomology of A, with coefficients in V* is the cohomology of the totalization of this complex.

It is easy to check that the periodic cyclic cohomology of A with coefficients in V is just the space of homomorphisms into V from the periodic cyclic cohomology of A with coefficients in $\mathbb{C}$ (the latter is the "usual" periodic cyclic cohomology of A). Nevertheless the concept of coefficients will be a convenient one for us.

If we totalize the (b, B)-bicomplex (either the above one involving V or the previous one without V) by taking a direct *product* of cochain groups along the diagonals instead of a direct sum, then we obtain a complex with zero cohomology.

2.31. Definition. We shall refer to cocycles for this complex, consisting in the even case of sequences $(\phi_0, \phi_2, \phi_4, \dots)$, *all* of whose terms may be nonzero, as *improper* (b, B)-cocycles.

On its own, an improper periodic (b, B)-cocycle has no cohomological significance, but once again the concept will be a convenient one for us. For example in Section 5 we shall construct a fairly simple improper (b, B)-cocycle with coefficients in the space V of meromorphic functions on $\mathbb{C}$. By taking residues at $0 \in \mathbb{C}$ we shall obtain a linear map from V to $\mathbb{C}$, and applying this linear map to our cocycle we shall obtain in Section 5 a *proper* (b, B)-cocycle with coefficients in $\mathbb{C}$.

2.7. Nonunital Algebras. If the algebra A has no multiplicative unit then we define periodic cyclic cohomology as follows. Denote by $\widetilde{A}$ the algebra obtained by adjoining a unit to A and form the double complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots \\
\cdots \arrow[r, "B"] & C^3(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^2(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] \\
\cdots \arrow[r, "B"] & C^2(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & \\
\cdots \arrow[r, "B"] & C^1(\widetilde{A}) \arrow[u, "b"] \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & \\
\cdots \arrow[r, "B"] & C^0(A) \arrow[u, "b"] & & &
\end{tikzcd}$$

in which the spaces $C^p(\widetilde{A})$ are, as before, the $(p+1)$-multlinear functionals ϕ from $\widetilde{A}$ into $\mathbb{C}$ with the property that $\phi(a^0, \dots, a^p) = 0$ whenever $a^j = 1$ for some $j \geq 1$, and in which $C^0(A)$ is the space of linear functionals on A (not on $\widetilde{A}$). If we interpret $b\colon C^0(A) \to C^1(\widetilde{A})$ using the formula

$$b\phi(a^0, a^1) = \phi(a^0 a^1) - \phi(a^1 a^0) = \phi(a^0 a^1 - a^1 a^0)$$

then the differential is well defined, since the commutator $a^0a^1 - a^1a^0$ always belongs to A, even when $a^0, a^1 \in \widetilde{A}$.

2.32. Definition. The periodic cyclic cohomology of A is the cohomology of the totalization of the above complex. A (b, B)-cocycle for A is a cocycle for the above complex.

2.33. Remark. The periodic cyclic cohomology of a non-unital algebra A is isomorphic to the kernel of the restriction map from $HCP^*(\tilde{A})$ to $HCP^*(\mathbb{C})$.

2.34. Definition. By a cyclic p-cocycle on A we shall mean a cyclic cocycle ϕ on $\widetilde{A}$ for which $\phi(a^0, \dots, a^p) = 0$ whenever $a^j = 1$ for some j.

3. The Hochschild Character

The purpose of this section is to provide some motivation for the development of the local index formula in cyclic cohomology by describing an antecedent formula in Hochschild cohomology.

3.1. Spectral Triples. Examples of Fredholm modules arising from geometry often involve the following structure.

3.1. Definition. A *spectral triple* is a triple (A, H, D) consisting of:

(a) An associative algebra A of bounded operators on a Hilbert space H, and
(b) An unbounded self-adjoint operator D on H such that
 (i) for every $a \in A$ the operators $a(D \pm i)^{-1}$ are compact, and
 (ii) for every $a \in A$, the operator $[D, a]$ is defined on $\operatorname{dom} D$ and extends to a bounded operator on H.

3.2. Remark. In item (b) we require that D be self-adjoint in the sense of unbounded operator theory. This means that D is defined on some dense domain $\operatorname{dom} D \subseteq H$, that $\langle Du, v\rangle = \langle u, D\rangle$, for all $u, v \in \operatorname{dom} D$, and that the operators $(D \pm i)$ map $\operatorname{dom} H$ bijectively onto H. In item (ii) we require that each $a \in A$ map $\operatorname{dom} D$ into itself.

3.3. Example. Let $A = C^\infty(S^1)$, let $H = L^2(S^1)$ and let $D = \frac{1}{2\pi i}\frac{d}{dx}$. The operator D, defined initially on the smooth functions in $H = L^2(S^1)$ (we are thinking now of S^1 as $\mathbb{R}/\mathbb{Z}$), has a unique extension to a self-adjoint operator on H, and the triple (A, H, D) incorporating this extension is a spectral triple in the sense of Definition 3.1.

3.4. Remark. If the algebra A has a unit, which acts as the identity operator on the Hilbert space H, then item (i) is equivalent to the assertion that $(D \pm i)^{-1}$ be compact operators, which is equivalent to the assertion that there exist an orthonormal basis for H consisting of eigenvectors v_j for D, with eigenvalues λ_j converging to ∞ in absolute value.

In a way which is similar to our treatment of Fredholm modules, we shall call a spectral triple *even* if the Hilbert space H is equipped with a self-adjoint grading operator ε, decomposing H as a direct sum $H = H_0 \oplus H_1$, such that ε maps the domain of D into itself, anitcommutes with D, and commutes with every $a \in A$. Spectral triples without a grading operator will be referred to as *odd*.

Let (A, H, D) be a spectral triple, and assume that D is invertible. In the polar decomposition $D = |D|F$ of D the operator F is self-adjoint and satisifies $F^2 = 1$.

3.5. Lemma. *If (A, H, D) is a spectral triple (A, H, F) is a Fredholm module in the sense of Definition 2.2.* □

3.6. Example. The Fredholm module described in Example 2.1 is obtained in this way from the spectral triple of Example 3.3, after we make a small modification to D to make it invertible — for example by replacing $\frac{1}{2\pi i}\frac{d}{dx}$ with $\frac{1}{2\pi i}\frac{d}{dx} + \frac{1}{2}$.

3.2. The Residue Trace. We are going to develop for spectral triples a refinement of the notion of finite summability that we introduced for Fredholm modules. For this purpose we need to quickly review the following facts about compact operators and their singular values (see [**25**] for more details).

3.7. Definition. If K is a compact operator on a Hilbert space then the *singular value sequence* $\{\mu_j\}$ of K is defined by the formulas

$$\mu_j = \inf_{\dim(V)=j-1} \sup_{v \perp V} \frac{\|Kv\|}{\|v\|} \qquad j = 1, 2, \ldots.$$

The infimum is over all linear subspaces of H of dimension $j-1$. Thus μ_1 is just the norm of K, μ_2 is the smallest possible norm obtained by restricting K to a codimension 1 subspace, and so on.

The trace ideal is easily described in terms of the sequence of singular values:

3.8. Lemma. *A compact operator K belongs to the trace ideal if and only if $\sum_j \mu_j < \infty$. Moreover if K is a positive, trace-class operator then* $\operatorname{Trace}(K) = \sum_j \mu_j$. □

Now, *any* self-adjoint trace-class operator can be written as a difference of positive, trace-class operators, $K = K^{(1)} - K^{(2)}$, and we therefore have a corresponding formula for the trace

$$\operatorname{Trace}(K) = \operatorname{Trace}(K^{(1)}) - \operatorname{Trace}(K^{(2)}) = \sum_j \mu_j^{(1)} - \sum_j \mu_j^{(2)}.$$

And since every trace-class operator is a linear combination of two self-adjoint, trace-class operators, we can go on to obtain a formula for the trace of a general trace-class operator.

We are going to define a new sort of trace by means of formulas like the one above.

3.9. Definition. Denote by $\mathcal{L}^{1,\infty}(H)$ the set of all compact operators K on H for which

$$\sum_{j=1}^N \mu_j = \mathcal{O}(\log N).$$

Thus every trace-class operator belongs to $\mathcal{L}^{1,\infty}(H)$ but operators in $\mathcal{L}^{1,\infty}(H)$ need not be trace class, since the sum $\sum_j \mu_j$ is permitted to diverge logarithmically.

3.10. Remark. The set $\mathcal{L}^{1,\infty}(H)$ is a two-sided ideal in $B(H)$.

Suppose now that we are given a linear subspace of $\mathcal{L}^{1,\infty}(H)$ consisting of operators for which the sequence of numbers

$$\frac{1}{\log N}\sum_{j=1}^N \mu_j$$

is not merely bounded but in fact convergent. It may be shown using fairly standard singular value inequalities that the functional Tr_ω which assigns to each positive operator in the subspace the limit of the sequence is *additive*:

$$\operatorname{Tr}_\omega(K^{(1)}) + \operatorname{Tr}_\omega(K^{(2)}) = \operatorname{Tr}_\omega(K^{(1)} + K^{(2)}).$$

As a result, if we assume that the linear subspace we are given is spanned by its positive elements, the prescription Tr_ω extends by linearity from positive operators to all operators and yields a linear functional.

A theorem of Dixmier [**14**] (see also [**7**, Section IV.2]) improves this construction by replacing limits with generalized limits, thereby obviating the need to assume that the sequence of partial sums $\frac{1}{\log N}\sum_{j=1}^N \mu_j$ is convergent:

3.11. Theorem. *There is a linear functional* $\operatorname{Tr}_\omega \colon \mathcal{L}^{1,\infty}(H) \to \mathbb{C}$ *with the following properties:*

(a) *If* $K \geq 0$ *then* $\operatorname{Tr}_\omega(K)$ *depends only on the singular value sequence* $\{\mu_j\}$*, and*
(b) *If* $K \geq 0$ *then* $\liminf_N \frac{1}{\log N}\sum_{j=1}^N \mu_j \leq \operatorname{Tr}_\omega(K) \leq \limsup_N \frac{1}{\log N}\sum_{j=1}^N \mu_j$. □

It follows from (a) that $\operatorname{Tr}_\omega(K) = \operatorname{Tr}_\omega(U^*KU)$ for every positive $K \in \mathcal{L}^{1,\infty}(H)$ and every unitary operator U on H. Since the positive operators in $\mathcal{L}^{1,\infty}(H)$ span $\mathcal{L}^{1,\infty}(H)$, it follows that $\operatorname{Tr}_\omega(T) = \operatorname{Tr}_\omega(U^*TU)$, for every $T \in \mathcal{L}^{1,\infty}(H)$ and every unitary operator U. Replacing T by U^*T we get $\operatorname{Tr}_\omega(UT) = \operatorname{Tr}_\omega(TU)$, for every $T \in \mathcal{L}^{1,\infty}(H)$ and every unitary U. Since the unitary operators span all of $B(H)$, we finally conclude that

$$\operatorname{Tr}_\omega(ST) = \operatorname{Tr}_\omega(TS),$$

for every $T \in \mathcal{L}^{1,\infty}(H)$ and every bounded operator S.

3.12. Remark. The *Dixmier trace* Trace_ω is not unique — depends on a choice of suitable generalized limit for the sequence of partial sums $\frac{1}{\log N}\sum_{j=1}^N \mu_j$. But of course it is unique on (positive) operators for which this sequence is convergent, which turns out to be the case in many geometric examples.

3.3. The Hochschild Character Theorem. If (A, H, F) is a finitely summable Fredholm module then Connes' cyclic Chern character ch_n^F is defined for all large enough n of the correct parity (even or odd, according as the Fredholm module is even or odd). It is a cyclic n-cocycle, and in particular, disregarding its cyclicity, it is a Hochschild n-cocycle. We are going to present a formula for the Hochschild cohomology class of this cocycle, in certain cases.

3.13. Lemma. *Let (A, H, D) be a spectral triple, and let n be a positive integer. Assume that D is invertible and that*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H),$$

for every $a \in A$. Then the associated Fredholm module (A, H, F) has the property that the operators

$$[F, a^0][F, a^1] \cdots [F, a^n]$$

are trace-class, for every $a^0, \dots, a^n \in A$. In particular, the Fredholm module (A, H, F) is finitely summable and if n has the correct parity, then the Chern character ch_n^F is defined. □

3.14. Definition. A spectral triple (A, H, D) is *regular* if there exists an algebra B of bounded operators on H such that

(a) $A \subseteq B$ and $[D, A] \subseteq B$, and
(b) if $b \in B$ then b maps the domain of $|D|$ (which is equal to the domain of D) into itself, and moreover $|D|B - B|D| \in B$.

3.15. Example. The spectral triple $(C^\infty(S^1), L^2(S^1), D)$ of Example 3.3 is regular.

3.16. Remark. We shall look at the notion of regularity in more detail in Section 4, when we discuss elliptic estimates.

We can now state Connes' Hochschild class formula:

3.17. Theorem. *Let (A, H, D) be a regular spectral triple. Assume that D is invertible and that for some positive integer n of the same parity as the triple, and every $a \in A$,*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H).$$

The Chern character ch_n^F of Definition 2.22 is cohomologous, as a Hochschild cocycle, to the cocycle

$$\Phi(a^0, \dots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!}\,\mathrm{Trace}_\omega(\varepsilon a^0[D, a^1][D, a^2]\cdots[D, a^n]|D|^{-n}).$$

Here ε is 1 in the odd case, and the grading operator on H in the even case.

3.18. Remark. Actually this is a slight strengthening of what is actually provable. For the correct statement, see Appendix C.

3.4. A Simple Example. We shall prove Theorem 3.17 in Appendix C, as a by-product of our proof of the local index theorem (at the moment, it is probably not even obvious that the functional Φ given in the theorem is a Hochschild cocycle). Right now what we want to do is to compute a simple example of the Hochschild cocycle Φ.

We shall consider the spectral triple $(C^\infty(S^1), L^2(S^1), D)$, where D is the unique self-adjoint extension of the operator $\frac{1}{2\pi i}\frac{d}{dx} + \frac{1}{2}$ (recall that the term $\frac{1}{2}$ was added to guarantee that D is invertible).

The operator D is diagonalizable, with eigenfunctions $e_n(x) = \exp(2\pi i n x)$ and eigenvalues $n + \frac{1}{2}$, where $n \in \mathbb{Z}$. We see then that $|D|$ given by the formula

$$|D| e_n = |n + \tfrac{1}{2}| e_n \qquad (n \in \mathbb{Z}).$$

As a result, $|D|^{-1} \in \mathcal{L}^{1,\infty}(H)$, and a brief computation shows that

$$\operatorname{Trace}_\omega(|D|^{-1}) = 2.$$

3.19. Lemma. *If $f \in C(S^1)$ then* $\operatorname{Trace}_\omega(f \cdot |D|^{-1}) = 2\int_{S^1} f(x)\, dx$.

Proof. The linear functional $f \mapsto \operatorname{Trace}_\omega f \cdot |D|^{-1}$ is positive, and so by the Riesz representation theorem it is given by integration against a Borel measure on S^1. But the functional, and hence the measure, is rotation invariant. This proves that the measure is a multiple of Lebesgue measure, and the computation $\operatorname{Tr}_\omega(|D|^{-1}) = 2$ fixes the multiple. □

With this computation in hand, we can now determine the cocycle Φ which appears in Theorem 3.17:

$$\Phi(f_0, f_1) = \operatorname{Trace}_\omega(f_0[D, f_1]|D|^{-1}) = \frac{1}{\pi i}\int_{S^1} f_0(x) f_1'(x)\, dx.$$

Now if $a^0, a^1 \in C^\infty(S^1)$, and if Ψ is any 1-cocycle which is in fact a Hochschild coboundary, then it is easily computed that $\Psi(a^0, a^1) = 0$. As a result, of Theorem 3.17 it therefore follows that

$$\Gamma(\tfrac{3}{2}) \cdot \tfrac{1}{2}\operatorname{Trace}(F[F, a^0][F, a^1]) \stackrel{def}{=} \operatorname{ch}_1^F(a^0, a^1) = \Gamma(\tfrac{3}{2})\Phi(a^0, a^1)$$

If we combine this with the Fredholm index formula in Theorem 2.10 we arrive at a proof of the well-known index formula

$$\operatorname{Index}(PuP) = -\tfrac{1}{4}\operatorname{Trace}(F[F, u^{-1}][F, u]) = -\tfrac{1}{2}\Phi(u^{-1}, u) = -\frac{1}{2\pi i}\int_{S^1} u^{-1} du$$

associated to an invertible element $u \in C^\infty(S^1)$, which we already mentioned in Example 2.1.

3.20. Remark. In this very simple example we have determined not only the Hochschild cocycle Φ but also the cyclic cocycle ch_1^F. This is an artifact of the low-dimensionality of the example: the natural map from the first cyclic cohomology group into the first Hochschild group happens always to be injective. In higher dimensional examples a determination of Φ will in general stop quite a bit short of a determination of ch_n^F.

3.5. Weyl's Theorem. The simple computation which we carried out above has a general counterpart which originates with a famous theorem of Weyl. We shall state the theorem in the context of Dirac-type operators, for which we refer the reader to Roe's introductory text [**24**] (this book also contains a proof of Weyl's theorem).

3.21. THEOREM. *Let M be a closed Riemannian manifold of dimension n, and let D be a Dirac-type operator on M, acting on the sections of some complex Hermitian vector bundle S over M. The operator D has a unique self-adjoint extension, and $|D|^{-n} \in \mathcal{L}^{1,\infty}(H)$. Moreover*

$$Trace_\omega(|D|^{-n}) = \frac{\dim(S)}{(2\sqrt{\pi})^n} \frac{\mathrm{Vol}(M)}{\Gamma(\frac{n}{2}+1)}.$$

3.22. REMARK. If D is not invertible then we define $|D|^{-n}$ by for example $|D|^{-1} = |D+P|^{-n}$, where P is the orthogonal projection onto the kernel of D. (Incidentally, we might note that altering an operator in $\mathcal{L}^{1,\infty}(H)$ by any finite rank operator — or indeed any trace-class operator — has no effect on the Dixmier trace.)

The theorem may be extended, as follows:

3.23. THEOREM. *Let M be a closed Riemannian manifold of dimension n, and let D be a Dirac-type operator on M, acting on the sections of some complex Hermitian vector bundle S over M. The operator D has a unique self-adjoint extension, and $|D|^{-n} \in \mathcal{L}^{1,\infty}(H)$. If F is any endomorphism of S then*

$$Trace_\omega(|D|^{-n}) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2}+1)} \int_M \mathrm{trace}(F(x))\, dx.$$

Thanks to the theorem, the Hochschild character Φ of Theorem 3.17 may be computed in the case where $A = C^\infty(M)$, $H = L^2(S)$, and D is a Dirac-type operator acting on sections of S (it may be shown that this consitutes an example of a regular spectral triple; compare Section 4). The commutators $[D,a]$ are endomorphisms of S, and so

$$\Phi(a^0,\dots,a^n) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2}+1)} \int_M \mathrm{trace}(\varepsilon a^0[D,a^1]\cdots[D,a^n])\, dx.$$

3.24. REMARK. In many cases the pointwise trace which appears here can be further computed. For example if D is the Dirac operator associated to a Spin^c-structure on M then we obtain the simple formula

$$\Phi(a^0,\dots,a^n) = \frac{1}{(2\sqrt{\pi})^n \Gamma(\frac{n}{2}+1)} \int_M a^0 da^1 \cdots da^n.$$

In summary, we see that $\Phi(a^0,\dots,a^n)$ is an integral over M of an explicit expression involving the a^j and their derivatives. Unfortunately, in higher dimensions, this very precise information about Φ is not enough to deduce an index theorem, since it is impossible to recover the pairing between cyclic coycles and idempotents or invertibles from the Hochschild cohomology class of the cyclic cocycle. For the purposes of index theory we need to obtain a similar formula for the cyclic cocycle ch_n^F itself, or for a cocycle which is cohomologous to it in cyclic or periodic theory. This is what the Connes-Moscovici formula achieves.

The formula involves in a crucial way a *residue trace* which in certain circumstances extends to a certain class of operators, including some unbounded operators, 2 times the Dixmier trace on $\mathcal{L}^{1,\infty}(H)$. We shall discuss this in detail in the next section, but we shall conclude here with a somewhat vague formulation of the local index formula, to give the reader some idea of the direction in which we are heading. The statement will be refined in the coming sections.

3.25. THEOREM. *Let (A, H, D) be a suitable even spectral triple*[5] *and let (A, H, F) be the associated Fredholm module. The Chern character ch_n^F is cohomologous, as a (b, B)-cocycle to the cocycle $\phi = (\phi_0, \phi_2, \dots)$ given by the formulas*

$$\phi_p(a^0, \dots, a^p) = \sum_{k \geq 0} c_{pk} \operatorname{Res} \operatorname{Tr} \left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2} - |k|} \right).$$

The sum is over all multi-indices $(k_1, \dots, k_p)$ with non-negative integer entries, and the constants c_{pk} are given by the formula

$$c_{pk} = \frac{(-1)^k}{k!} \frac{\Gamma(k_1 + \cdots + k_p + \frac{p}{2})}{(k_1 + 1)(k_1 + k_2 + 2) \cdots (k_1 + \cdots + k_p + p)}.$$

The operators $X^{(k)}$ are defined inductively by $X^{(0)} = X$ and $X^{(k)} = [D^2, X^{(k-1)}]$.

3.26. REMARK. Note that when $p = n$ and $k = 0$ we obtain the term

$$\begin{aligned} &\frac{\Gamma(\frac{n}{2})}{n!} \operatorname{Res} \operatorname{Tr} \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right) \\ &\qquad = \frac{2\Gamma(\frac{n}{2})}{n!} \operatorname{Trace}_\omega \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right) \\ &\qquad\qquad = \frac{\Gamma(\frac{n}{2} + 1)}{n \cdot n!} \operatorname{Trace}_\omega \left(\varepsilon a^0 [D, a^1] \cdots [D, a^p] |D|^{-n} \right). \end{aligned}$$

Thus we recover precisely the Hochschild cocycle of Theorem 3.17. The relation between Theorem 3.17 and the local index formula will be further discussed in Appendix C.

4. Differential Operators and Zeta Functions

Apart from cyclic theory, the local index theorem requires a certain amount of Hilbert space operator theory. We shall develop the necessary topics in this section, beginning with a very rapid review of the basic theory of linear elliptic operators on manifolds.

4.1. Elliptic Operators on Manifolds. Let M be a smooth, closed manifold, let S be a smooth vector bundle over M. Let us equip M with a smooth measure and S with an inner product, so that we can form the Hilbert space $L^2(M, S)$.

Let $\mathcal{D}$ be the algebra of linear differential operators on M acting on smooth sections of S. This is an associative algebra of operators and it is filtered by the usual notion of order of a differential operator: an operator X has order q or less if any local coordinate system it can be written in the form

$$X = \sum_{|\alpha| \leq q} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

[5] There is an analogous theorem in the odd case.

where α is a multi-index $(\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

If s is a non-negative integer then the space of order s or less operators is a finitely generated module over the ring $C^\infty(M)$. If $X_1, \ldots, X_N$ is a generating set then the Sobolev space $W^s(M, S)$ is defined to be the completion of $C^\infty(M, S)$ induced from the norm

$$\|\phi\|^2_{W^s(M,S)} = \sum_j \|X_j \phi\|^2_{L^2(M,S)}.$$

Different choices of generating set result in equivalent norms and the same space $W^s(M, S)$. Every differential operator of order q extends to a bounded linear operator from $W^s(M, S)$ to $W^{s-q}(M, S)$, for all $s \geq q$. The *Sobolev Embedding Theorem* implies that that $\cap_{s \geq 0} W^s(M, S) = C^\infty(M, S)$.

Now let Δ be a linear *elliptic* operator of order r. The reader unfamiliar with the definition of ellipticity can take the following *basic estimate* as the definition: if Δ is elliptic of order r, then there is some $\varepsilon > 0$ such that

$$\|\Delta\phi\|_{W^s(M,S)} + \|\phi\|_{L^2(M,S)} \geq \varepsilon \|\phi\|_{W^{s+r}(M,S)},$$

for every $\phi \in C^\infty(M, S)$.

Suppose now that Δ is also *positive*, which is to say that $\langle \Delta\phi, \phi \rangle_{L^2(M,S)} \geq 0$, for all $\phi \in C^\infty(M, S)$. It may be shown then that Δ is essentially self-adjoint on the domain $C^\infty(M, S)$, and for $s \geq 0$ we can define the linear spaces

$$H^s = \operatorname{dom}(\Delta^{\frac{s}{r}}) \subseteq H,$$

which are Hilbert spaces in the norm

$$\|\phi\|^2_{H^s} = \|\phi\|^2_{L^2} + \|\Delta^{\frac{r}{s}}\|^2_{L^2}.$$

It follows from the basic estimate that $H^s = W^s(M, S)$.

Let us say that an operator Δ of order r operator is of *scalar type* if in every local coordinate system Δ can be written in the form

$$\Delta = \sum_{|\alpha| \leq r} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha},$$

where the $a_\alpha(x)$, for $|\alpha| = r$, are scalar multiples of the identity operator (acting on the fiber S_x of S). Good examples are the Laplace operators $\Delta = \nabla^* \nabla$ associated to affine connections on S, which are positive, elliptic of order 2, and of scalar type. Other examples are the squares of Dirac-type operators on Riemannian manifolds. If Δ is of scalar type then

$$\operatorname{order}([\Delta, X]) \leq \operatorname{order}(X) + \operatorname{order}(\Delta) - 1$$

(whereas the individual products $X\Delta$ and ΔX have one greater order, in general).

The following theorem, which is quite well known, will be fundamental to what follows in these notes. For a proof which is somewhat in the spirit of these notes see [**19**].

4.1. Theorem. *Let Δ be elliptic of order r, positive and of scalar type, and assume for simplicity that Δ is invertible as a Hilbert space operator. Let X be any differential operator. If $\operatorname{Re}(z)$ is sufficiently large then the operator $X\Delta^{-z}$ is of trace class. Moreover the function $\zeta(z) = \operatorname{Trace}(X\Delta^{-z})$ extends to a meromorphic function on $\mathbb{C}$ with only simple poles.*

4.2. Remarks. The assumption that Δ is of scalar type is not necessary, but it simpifies the proof. and covers the cases of interest. The meaning of the complex power Δ^{-z} will be clarified in the coming paragraphs.

4.2. Abstract Differential Operators. In this section we shall give abstract counterparts of the ideas presented in the previous section.

Let H be a complex Hilbert space. We shall assume as given an unbounded, positive, self-adjoint operator Δ on H. The operator Δ and its powers Δ^k are provided with definite domains $\operatorname{dom}(\Delta^k) \subseteq H$, which are dense subspaces of H. We shall denote by H^∞ the intersections of the domains of all the Δ^k:

$$H^\infty = \cap_{k=1}^\infty \operatorname{dom}(\Delta^k).$$

We shall assume as given an algebra $\mathcal{D}(\Delta)$ of linear operators on the vector space H^∞. We shall assume the following things about $\mathcal{D}(\Delta)$:[6]

(i) If $X \in \mathcal{D}(\Delta)$ then $[\Delta, X] \in \mathcal{D}(\Delta)$ (we shall not insist that Δ belongs to $\mathcal{D}(\Delta)$).

(ii) The algebra $\mathcal{D}(\Delta)$ is filtered,

$$\mathcal{D}(\Delta) = \cup_{q=0}^\infty \mathcal{D}_q(\Delta) \quad \text{(an increasing union).}$$

We shall write $\operatorname{order}(X) \leq q$ to denote the relation $X \in \mathcal{D}_q(\Delta)$. Sometimes we shall use the term "differential order" to refer to this filtration. This is supposed to call to mind the standard example, in which $\operatorname{order}(X)$ is the order of X as a differential operator.

(iii) There is an integer $r > 0$ (the "order of Δ") such that

$$\operatorname{order}([\Delta, X]) \leq \operatorname{order}(X) + r - 1$$

for every $X \in \mathcal{D}(\Delta)$.

To state the final assumption, we need to introduce the linear spaces

$$H^s = \operatorname{dom}(\Delta^{\frac{s}{r}}) \subseteq H$$

for $s \geq 0$. These are Hilbert spaces in their own right, in the norms

$$\|v\|_s^2 = \|v\|^2 + \|\Delta^{\frac{s}{r}} v\|^2.$$

The following key condition connects the algebraic hypotheses we have placed on $\mathcal{D}(\Delta)$ to operator theory:

(iv) If $X \in \mathcal{D}(\Delta)$ and if $\operatorname{order}(X) \leq q$ then there is a constant $\varepsilon > 0$ such that

$$\|v\|_q + \|v\| \geq \varepsilon \|Xv\|, \quad \forall v \in H^\infty.$$

4.3. Example. The standard example is of course that in which Δ is a Laplace-type operator $\Delta = \nabla^*\nabla$, or Δ is the square of a Dirac-type operator, on a closed manifold M and $\mathcal{D}(\Delta)$ is the algebra of differential operators on M. We can obtain a slightly more complicated example by dropping the assumption that M is compact, and defining $\mathcal{D}(\Delta)$ to be the algebra of compactly supported differential operators on M (Δ is still a Lapacian or the square of a Dirac operator). Item (i) above was formulated with the non-compact case in mind.

4.4. Remark. In the standard example the "order" r of Δ is $r = 2$. But other orders are possible. For example Connes and Moscovici consider an important example in which $r = 4$.

[6]Various minor modifications of these axioms are certainly possible.

4.5. Remark. For the purposes of these notes we could get by with something a little weaker than condition (iv), namely this:

(iv′) If $X \in \mathcal{D}(\Delta)$ and if $\operatorname{order}(X) \le kr$ then there is a constant $\varepsilon > 0$ such that
$$\|\Delta^k v\| + \|v\| \ge \varepsilon \|Xv\|, \quad \forall v \in H^\infty.$$

The advantage of this condition is that it involves only integral powers of the operator Δ (in contrast the $\| \, \|_s$ involve fractional powers of Δ). Condition (iv′) is therefore in principle easier to verify. However in the main examples, for instance the one developed by Connes and Moscovici in [**12**], the stronger condition holds.

4.6. Definition. We shall refer to an algebra $\mathcal{D}(\Delta)$ (together with the distinguished operator Δ) satisfying the axioms (i)-(iv) above as an *algebra of generalized differential operators*.

4.7. Lemma. *If $X \in \mathcal{D}(\Delta)$, and if X has order q or less, then for every $s \ge 0$ the operator X extends to a bounded linear operator from H^{s+q} to H^s.*

Proof. If s is an integer multiple of the order r of Δ then the lemma follows immediately from the elliptic estimate above. The general case (which we shall not actually need) follows by interpolation. □

4.3. Zeta Functions. Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators, as in the previous sections. We are going to define certain zeta-type functions associated with $\mathcal{D}(\Delta)$.

To simplify matters we shall now assume that the operator Δ is invertible. *This assumption will remain in force until Section 6*, where we shall first consider more general operators Δ.

The complex powers Δ^{-z} may be defined using the functional calculus. They are, among other things, well-defined operators on the vector space H^∞.

4.8. Definition. The algebra $\mathcal{D}(\Delta)$ has *finite analytic dimension* if there is some $d \ge 0$ with the property that if $X \in \mathcal{D}(\Delta)$ has order q or less, then, for every $z \in \mathbb{C}$ with real part greater than $\frac{q+d}{r}$, the operator $X\Delta^{-z}$ extends by continuity to a trace-class operator on H (here r is the order of Δ, as described in Section 4.2).

4.9. Remark. The condition on $\operatorname{Re}(z)$ is meant to imply that the order of $X\Delta^{-z}$ is less than $-d$. (We have not yet assigned a notion of order to operators such as $X\Delta^{-z}$, but we shall do so in Definition 4.15.)

4.10. Definition. The smallest value $d \ge 0$ with the property described in Definition 4.8 will be called the *analytic dimension* of the algebra $\mathcal{D}(\Delta)$.

Assume that $\mathcal{D}(\Delta)$ has finite analytic dimension d. If $X \in \mathcal{D}(\Delta)$ and if $\operatorname{order}(X) \le q$ then the complex function $\operatorname{Trace}(X\Delta^{-z})$ is holomorphic in the right half-plane $\operatorname{Re}(z) > \frac{q+d}{r}$.

4.11. Definition. An algebra $\mathcal{D}(\Delta)$ of generalized differential operators which has finite analytic dimension has the *analytic continuation property* if for every $X \in \mathcal{D}(\Delta)$ the analytic function $\operatorname{Trace}(X\Delta^{-z})$, defined initially on a half-plane in $\mathbb{C}$, extends to a meromorphic function on the full complex plane.

Actually, for what follows it would be sufficient to assume that $\operatorname{Trace}(X\Delta^{-s})$ has an analytic continuation to $\mathbb{C}$ with only isolated singularities, which could perhaps be essential singularities.[7] The analytic continuation property is obviously an abstraction of Theorem 4.1 concerning elliptic operators on manifolds.

We are ready to present what is, in effect, the main definition of these notes, in which we describe the "elementary quantities" which were mentioned in the introduction. The reasoning which leads to this definition will be explained in Appendix B.

In order to accommodate the cyclic cohomology constructions to be carried out in Section 5 we shall now assume that the Hilbert space H is $\mathbb{Z}/2$-graded, that Δ has even grading-degree, and that the grading operator $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ multiplies $\mathcal{D}(\Delta)$ into itself. (The case $\varepsilon = I$, where the grading is trivial, is one important possibility.)

4.12. Definition. Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators which has finite analytic dimension. Define, for $\mathrm{Re}(z) \gg 0$ and $X^0, \dots, X^p \in \mathcal{D}(\Delta)$, the quantity

$$\begin{aligned} (4.1) \quad & \langle X^0, X^1, \dots, X^p \rangle_z = \\ & (-1)^p \frac{\Gamma(z)}{2\pi i} \operatorname{Trace}\left(\int \lambda^{-z} \varepsilon X^0 (\lambda - \Delta)^{-1} X^1 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1}\, d\lambda \right) \end{aligned}$$

(the factors in the integral alternate between the X^j and copies of $(\lambda - \Delta)^{-1}$). The contour integral is evaluated down a vertical line in $\mathbb{C}$ which separates 0 and $\mathrm{Spectrum}(\Delta)$.

4.13. Remark. The operator $(\lambda - \Delta)^{-1}$ is bounded on all of the Hilbert spaces H^s, and moreover its norm on each of these spaces is bounded by $|\operatorname{Im}(\lambda)|^{-1}$. As a result, if

$$\mathrm{order}(X^0) + \cdots + \mathrm{order}(X^p) \le q$$

and if the integrand in equation (4.1) is viewed as a bounded operator from H^{s+q} to H^s, then the integral converges absolutely in the operator norm whenever $\mathrm{Re}(z) + p > 0$. In particular, if $\mathrm{Re}(z) > 0$ then the integral (4.1) converges to a well defined operator on H^∞.

The following result establishes the traceability of the integral (4.1), when $\mathrm{Re}(z) \gg 0$.

4.14. Proposition. *Let $\mathcal{D}(\Delta)$ be an algebra of generalized differential operators and let $X^0, \dots, X^p \in \mathcal{D}(\Delta)$. Assume that*

$$\mathrm{order}(X^0) + \cdots + \mathrm{order}(X^p) \le q.$$

If $\mathcal{D}(\Delta)$ has finite analytic dimension d, and if $\mathrm{Re}(z) + p > \frac{1}{r}(q + d)$, then the integral in Equation (4.1) extends by continuity to a trace-class operator on H, and the quantity $\langle X^0, \dots, X^p \rangle_z$ defined by Equation (4.1) is a holomorphic function of z in this half-plane. If in addition the algebra $\mathcal{D}(\Delta)$ has the analytic continuation property then the quantity $\langle X^0, \dots, X^p \rangle_z$ extends to a meromorphic function on $\mathbb{C}$.

For the purpose of proving the proposition it is useful to develop a little more terminology, as follows.

[7] The exception to this is Appendix A, which is independent of the rest of the notes, where we shall assume at one point that the singularities are all simple poles.

4.15. Definition. Let $m \in \mathbb{R}$. We shall say that an operator $T\colon H^\infty \to H^\infty$ has *analytic order m or less* if, for every s,[8] T extends to a bounded operator from H^{m+s} to H^s.

4.16. Example. The resolvents $(\lambda - \Delta)^{-1}$ have analytic order $-r$, or less.

Let us note the following simple consequence of our definitions:

4.17. Lemma. *Let $\mathcal{D}(\Delta)$ have finite analytic dimension d. If T has analytic order less than $-d-q$, and if $X \in \mathcal{D}(\Delta)$ has order q, then XT is a trace-class operator.* □

4.18. Definition. Let T and T_α ($\alpha \in A$) be operators on H^∞. We shall write

$$T \approx \sum_{\alpha \in A} T_\alpha$$

if, for every $m \in \mathbb{R}$, there is a finite set $F \subseteq A$ with the property that if $F' \subseteq A$ is a finite subset containing F then T and $\sum_{\alpha \in F'} T_\alpha$ differ by an operator of analytic order m or less.

One should think of m as being large and negative. Thus $T \approx \sum_{\alpha \in A} T_\alpha$ if every sufficiently large finite partial sum agrees with T up to operators of large negative order.

4.19. Definition. If $Y \in \mathcal{D}(\Delta)$ then denote by $Y^{(k)}$ the k-fold commutator of Y with Δ. Thus $Y^{(0)} = Y$ and $Y^{(k)} = [\Delta, Y^{(k-1)}]$ for $k \geq 1$.

4.20. Lemma. *Let $Y \in \mathcal{D}(\Delta)$ and let h be a positive integer. For every positive integer k there is an asymptotic expansion*

$$\begin{aligned}[(\lambda - \Delta)^{-h}, Y] \approx hY^{(1)}(\lambda - \Delta)^{-(h+1)} + \frac{h(h+1)}{2} Y^{(2)}(\lambda - \Delta)^{-(h+2)} + \dots \\ + \frac{h(h+1)\cdots(h+k)}{k!} Y^{(k)}(\lambda - \Delta)^{-(h+k)} + \cdots ,\end{aligned}$$

4.21. Remark. If $\operatorname{order}(Y) \leq q$ then, according to the axioms in Section 4.2, $\operatorname{order}(Y^{(p)}) \leq q + p(r-1)$. Therefore, thanks to the elliptic estimate of Section 4.2, the operator $Y^{(p)}(\lambda - \Delta)^{-(h+p)}$ has analytic order $q - hr - p$ or less. Hence the terms in the asymptotic expansion of the lemma are of decreasing analytic order.

Proof of Lemma 4.20. Let us write $L = \lambda - \Delta$ and observe that the k-fold iterated commutator of Y with L is $(-1)^k$ times $Y^{(k)}$, the k-fold iterated commutator of Y with Δ. Let us also write $z = -h$.

To prove the lemma we shall use Cauchy's formula,

$$\binom{z}{p} L^{z-p} = \frac{1}{2\pi i} \int w^z (w - L)^{-p-1}\, dw.$$

The integral (which is carried out along the same contour as the one in Definition 4.12) is norm-convergent in the operator norm on any $\mathcal{B}(H^s)$. Applying this

[8]Strictly speaking we should say "for every $s \geq 0$ such that $m + s \geq 0$," since we have not defined H^s for negative s.

formula in the case $p = 0$ we get

$$\begin{aligned}
[L^z, Y] &= \frac{1}{2\pi i}\int w^z\big[(w-L)^{-1}, Y\big]\,dw \\
&= -\frac{1}{2\pi i}\int w^z(w-L)^{-1}Y^{(1)}(w-L)^{-1}\,dw \\
&= -Y^{(1)}\frac{1}{2\pi i}\int w^z(w-L)^{-2}\,dw \\
&\quad -\frac{1}{2\pi i}\int w^z\big[(w-L)^{-1}, Y^{(1)}\big](w-L)^{-1}\,dw \\
&= -\binom{z}{1}Y^{(1)}L^{z-1} + \frac{1}{2\pi i}\int w^z(w-L)^{-1}Y^{(2)}(w-L)^{-2}\,dw.
\end{aligned}$$

The integrals all converge in the operator norm of $\mathcal{B}(H^{s+q}, H^s)$ for any q large enough (and in fact any $q \geq \operatorname{order}(Y)$ would do). By carrying out a sequence of similar manipulations on the remainder integral we arrive at

$$\begin{aligned}
[L^z, Y] = -&\binom{z}{1}Y^{(1)}L^{-z-1} + \binom{z}{2}Y^{(2)}L^{-z-2} - \dots \\
&+ (-1)^p\binom{z}{p}Y^{(p)}L^{-z-p} + \frac{(-1)^p}{2\pi i}\int w^z(w-L)^{-1}Y^{(p)}(w-L)^{-p}\,dw.
\end{aligned}$$

Simple estimates on the remainder integral now prove the lemma. □

We are now almost ready to prove Proposition 4.14. In the proof we shall use asymptotic expansions, as in Definition 4.18. But we shall be considering operators which, like $(\lambda - \Delta)$, depend on a parameter λ. In this situation we shall amend Definition 4.18 by writing $T \approx \sum_\alpha T_\alpha$ if, for every $m \ll 0$, every sufficiently large finite partial sum agrees with T up to an operator of analytic order m or less, whose norm as an operator from H^{s+m} to H^s is $O(|\operatorname{Im}(\lambda)|^m)$. The reason for doing so is that we shall then be able to integrate with respect to λ, and obtain an asymptotic expansion for the integrated operator.

Proof of Proposition 4.14. The idea of the proof is to use Lemma 4.20 to move all the terms $(\lambda - \Delta)^{-1}$ which appear in the basic quantity $X^0(\lambda - \Delta)^{-1}\cdots X^p(\lambda - \Delta)^{-1}$ to the right. If the operators X^j actually commuted with Δ then we would of course get

$$X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1} = X^0\cdots X^p(\lambda-\Delta)^{-(p+1)},$$

and after integrating and applying Cauchy's integral formula we could conclude without difficulty that

$$\langle X^0, \dots, X^p\rangle_z = \frac{\Gamma(z+p)}{p!}\operatorname{Trace}(\varepsilon X^0\cdots X^p\Delta^{-z-p})$$

(compare with the manipulations below). The proposition would then follow immediately from this formula. The general case is only a little more difficult: we shall see that the above formula gives the leading term in a sort of asymptotic expansion for $\langle X^0, \dots, X^p\rangle_z$.

It will be helpful to define quantities

$$c(k_1, \dots, k_j) = \frac{(k_1 + \cdots + k_j + j)!}{k_1!\cdots k_j!(k_1+1)\cdots(k_1+\cdots+k_j+j)},$$

which depend on non-negative integers $k_1, \dots, k_j$. These have the property that $c(k_1) = 1$, for all k_1, and

$$c(k_1, \dots, k_j) = c(k_1, \dots, k_{j-1}) \frac{(k_1 + \cdots + k_{j-1} + j) \cdots (k_1 + \cdots + k_j + j - 1)}{k_j!}$$

(the numerator in the fraction is the product of the k_j successive integers from $(k_1 + \cdots + k_{j-1} + j)$ to $(k_1 + \cdots + k_j + j - 1)$). Using this notation and Lemma 4.20 we obtain an asymptotic expansion

$$(\lambda - \Delta)^{-1} X^1 \approx \sum_{k_1 \geq 0} c(k_1) X^{1^{(k_1)}} (\lambda - \Delta)^{-(k_1+1)},$$

and then

$$\begin{aligned}(\lambda - \Delta)^{-1} X^1 (\lambda - \Delta)^{-1} X^2 &\approx \sum_{k_1 \geq 0} c(k_1) X^{1^{(k_1)}} (\lambda - \Delta)^{-(k_1+2)} X^2 \\ &\approx \sum_{k_1, k_2 \geq 0} c(k_1, k_2) X^{1^{(k_1)}} X^{2^{(k_2)}} (\lambda - \Delta)^{-(k_1+k_2+2)},\end{aligned}$$

and finally

$$(\lambda - \Delta)^{-1} X^1 \cdots (\lambda - \Delta)^{-1} X^p \approx \sum_{k \geq 0} c(k) X^{1^{(k_1)}} \cdots X^{p^{(k_p)}} (\lambda - \Delta)^{-(|k|+p)},$$

where we have written $k = (k_1, \dots, k_p)$ and $|k| = k_1 + \cdots + k_p$. It follows that

$$\begin{aligned}&\frac{(-1)^p \Gamma(z)}{2\pi i} \int \lambda^{-z} (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \, d\lambda \\ &\qquad \approx \sum_{k \geq 0} c(k) X^{1^{(k_1)}} \cdots X^{p^{(k_p)}} \frac{(-1)^p \Gamma(z)}{2\pi i} \int \lambda^{-z} (\lambda - \Delta)^{-(|k|+p+1)} \, d\lambda \\ &\qquad = \sum_{k \geq 0} c(k) X^{1^{(k_1)}} \cdots X^{p^{(k_p)}} (-1)^p \Gamma(z) \binom{-z}{|k| + p} \Delta^{-z-|k|-p}.\end{aligned}$$

The terms of this asymptotic expansion have analytic order $q - k - r(\mathrm{Re}(z) + p)$ or less, and therefore if $\mathrm{Re}(z) + p > \frac{1}{r}(q + d)$ then the terms all have analytic order less than $-d$. This proves the first part of the proposition: after multiplying by εX^0, if $\mathrm{Re}(z) + p > \frac{1}{r}(q + d)$ then all the terms in the asymptotic expansion are trace-class, and the integral extends to a trace-class operator on H. To continue, it follows from the functional equation for $\Gamma(z)$ that

$$(-1)^p \Gamma(z) \binom{-z}{|k| + p} = (-1)^{|k|} \Gamma(z + p + |k|) \frac{1}{(|k| + p)!}.$$

So multiplying by εX^0 and taking traces we get

$$\begin{aligned}\langle X^0, \dots, X^p \rangle_z \approx \sum_{k \geq 0} &(-1)^{|k|} \Gamma(z + p + |k|) \frac{1}{(|k| + p)!} c(k) \\ &\times \operatorname{Trace}\left(\varepsilon X^0 X^{1^{(k_1)}} \cdots X^{p^{(k_p)}} \Delta^{-z-|k|-p} \right),\end{aligned}$$

where the symbol $\approx$ means that, given any right half-plane in $\mathbb{C}$, any sufficiently large finite partial sum of the right hand side agrees with the left hand side modulo a function of z which is holomorphic in that half-plane. The second part of the proposition follows immediately from this asymptotic expansion and Definition 4.11. □

4.22. Remark. In the coming sections we shall make use of a modest generalization of the first part of Proposition 4.14, in which the operators $X^0, \dots, X^p$ are chosen from the algebra generated by $\mathcal{D}(\Delta)$ and D (a square root of the operator Δ that we shall discuss next), with at least one X^j actually in $\mathcal{D}(\Delta)$ itself. The conclusion of the proposition and the proof are the same.

At the end of Section 7 we shall also need a version of Lemma 4.20 involving powers Δ^{-h} for non-integral h. Once again the formulation of the lemma, and the proof, are the otherwise unchanged.

4.4. Square Root of the Laplacian. We shall now assume that a self-adjoint operator D is specified, for which $D^2 = \Delta$. If the Hilbert space H is nontrivially $\mathbb{Z}/2$-graded we shall also assume that the operator D has grading degree 1. We shall also assume that an algebra $A \subseteq \mathcal{D}(\Delta)$ is specified, consisting of operators of differential order zero (the operators in A are therefore bounded operators on H).

4.23. Example. In the standard example, D will be a Dirac-type operator and A will be the algebra of C^∞-functions on M.

Continuing the axioms listed in Section 4.2, we shall assume that

(v) If $a \in A \subseteq \mathcal{D}(\Delta)$ then $[D, a] \in \mathcal{D}(\Delta)$.

We shall also assume that

(vi) If $a \in A$ then $\operatorname{order}\big([D, a]\big) \leq \operatorname{order}(D) - 1$, where we set $\operatorname{order}(D) = \frac{r}{2}$.

4.5. Spectral Triples. In Section 5 we shall use the square root D to construct cyclic cocycles for the algebra A from the quantities $\langle X^0, \dots, X^p \rangle_z$. But first we shall conclude our discussion of analytic preliminaries by briefly discussing the relation between our algebras $\mathcal{D}(\Delta)$ and the notion of spectral triple.

4.24. Definition. A *spectral triple* is a triple (A, H, D), composed of a complex Hilbert space H, an algebra A of bounded operators on H, and a self-adjoint operator D on H with the following two properties:

(i) If $a \in A$ then the operator $a \cdot (1 + D^2)^{-1}$ is compact.
(ii) If $a \in A$ then $a \cdot \operatorname{dom}(D) \subseteq \operatorname{dom}(D)$ and the commutator $[D, a]$ extends to a bounded operator on H

Various examples are listed in [**12**]; in the standard example A is the algebra of smooth functions on a complete Riemannian manifold M, D is a Dirac-type operator on M, and H is the Hilbert space of L^2-sections of the vector bundle on which D acts.

Let (A, H, D) be a spectral triple. Let $\Delta = D^2$, and as in Section 4.2 let us define

$$H^\infty = \cap_{k=1}^\infty \operatorname{dom}(\Delta^k) = \cap_{k=1}^\infty \operatorname{dom}(D^k).$$

Let us assume that A maps the space H^∞ into itself (this does not follow automatically). Having done so, let us define $\mathcal{D}(A, D)$ to be the smallest algebra of linear operators on H^∞ which contains A and $[D, A]$ and which is closed under the operation $X \mapsto [\Delta, X]$. Note that $\mathcal{D}(A, D)$ does not necesssarily contain D.

Equip the algebra $\mathcal{D}(A, D)$ with the smallest filtration so that

(i) If $a \in A$ then $\operatorname{order}(a) = 0$ and $\operatorname{order}\big([D, a]\big) = 0$.
(ii) If $X \in \mathcal{D}(A, D)$ then $\operatorname{order}\big([\Delta, X]\big) \leq \operatorname{order}(X) + 1$.

The term "smallest" means here that we write $\operatorname{order}(X) \leq q$ if and only if the order of X is q or less in every filtration satisfying the above conditions (there is at least one such filtration). Having filtered $\mathcal{D}(A, D)$ in this way we obtain an example of the sort of algebra of generalized differential operators which was considered in Section 4.2.

Denote by δ the unbounded derivation of $\mathcal{B}(H)$ given by commutator with $|D|$. Thus the domain of δ is the set of all bounded operators T which maps the domain of $|D|$ into itself, and for which the commutator extends to a bounded operator on H.

4.25. Definition. A spectral triple is *regular* if A and $[D, A]$ belong to $\cap_{n=1}^{\infty} \delta^n$.

We want to prove the following result.

4.26. Theorem. *Let (A, H, D) be a spectral triple with the property that every $a \in A$ maps H^∞ into itself. It is regular if and only if the algebra $\mathcal{D}(A, D)$ satisfies the basic estimate (iv) of Section 4.2.*

The proof is based on the following computation. Denote by B the algebra of operators on H^∞ generated by all the spaces $\delta^n[A]$ and $\delta^n\big[[D, A]\big]$, for all $n \geq 0$. According to the definition of regularity every operator in B extends to a bounded operator on H.

4.27. Lemma. *Assume that (A, H, D) is a regular spectral triple. Every operator in $\mathcal{D}$ of order k may be written as a finite sum of operators $b|D|^\ell$, where b belongs to the algebra B and where $\ell \leq k$.*

Proof. The spaces $\mathcal{D}_k$ of operators of order k or less in $\mathcal{D}(A, D)$ may be defined inductively as follows:
(a) algebra generated by $\mathcal{D}_0 = A + [D, A]$.
(b) $\mathcal{D}_1 = [\Delta, \mathcal{D}_0] + \mathcal{D}_0[\Delta, \mathcal{D}_0]$.
(c) $\mathcal{D}_k = \sum_{j=1}^{k-1} \mathcal{D}_j \cdot \mathcal{D}_{k-j} + [\Delta, \mathcal{D}_{k-1}] + \mathcal{D}_0[\Delta, \mathcal{D}_{k-1}]$.

Define $\mathcal{E}$, a space of operators on H^∞, to be the linear span of the operators of the form $b|D|^k$, where $k \geq 0$. The space $\mathcal{E}$ is an algebra since

$$b_1|D|^{k_1} \cdot b_2|D|^{k_2} = \sum_{j=0}^{k_1} \binom{k_1}{j} b_1 \delta^j(b_2)|D|^{k_1+k_2-j}.$$

Filter the algebra $\mathcal{E}$ by defining $\mathcal{E}_k$ to be the span of all operators $b|D|^\ell$ with $\ell \leq k$. The formula above shows that this does define a filtration of the algebra $\mathcal{E}$. Now the algebra $\mathcal{D}$ of differential operators is contained within $\mathcal{E}$, and the lemma amounts to the assertion that $\mathcal{D}_k \subseteq \mathcal{E}_k$. Clearly $\mathcal{D}_0 \subseteq \mathcal{E}_0$. Using the formula

$$[\Delta, b|D|^{k-1}] = [|D|^2, b|D|^{k-1}] = 2\delta(b)|D|^k + \delta^2(b)|D|^{k-1}$$

and our formula for $\mathcal{D}_k$ the inclusion $\mathcal{D}_k \subseteq \mathcal{E}_k$ is easily proved by induction. □

Proof of Theorem 4.26, Part One. Suppose that (A, H, D) is regular. According to the lemma, to prove the basic estimate for $\mathcal{D}(A, D)$ it suffices to prove that if $k \geq \ell$ and if $X = b|D|^\ell$, where $b \in B$, then there exists $\varepsilon > 0$ such that

$$\|D^k v\| + \|v\| \geq \varepsilon \|Xv\|,$$

for every $v \in H^\infty$. But we have

$$\|Xv\| = \|b|D|^\ell v\| \le \|b\| \cdot \||D|^\ell v\| = \|b\| \cdot \|D^\ell v\|,$$

And since by spectral theory for every $\ell \le k$ we have that

$$\|D^\ell v\|^2 \le \|D^k v\|^2 + \|v\|^2 \le \big(\|D^k v\| + \|v\|\big)^2$$

it follows that

$$\|D^k v\| + \|v\| \ge \frac{1}{\|b\|+1}\|Xv\|,$$

as required. □

To prove the converse, we shall develop a *pseudodifferential calculus*, as follows.

4.28. Definition. Let (A, H, D) be a spectral triple for which $A \cdot H^\infty \subseteq H^\infty$, and for which the basic elliptic estimate holds. Fix an operator $K\colon H^\infty \to H^\infty$ of order $-\infty$ and such that $\Delta_1 = \Delta + K$ is invertible. A *basic pseudodifferential operator of order $k \in \mathbb{Z}$* is a linear operator $T\colon H^\infty \to H^\infty$ with the property that for every $\ell \in Z$ the operator T may be decomposed as

$$T = X\Delta_1^{\frac{m}{2}} + R,$$

where $X \in \mathcal{D}(A, D)$, $m \in \mathbb{Z}$, and $R\colon H^\infty \to H^\infty$, and where

$$\operatorname{order}(X) + m \le k \quad \text{and} \quad \operatorname{order}(R) \le \ell.$$

A *pseudodifferential operator of order $k \in \mathbb{Z}$* is a finite linear combination of basic pseudodifferential operators of order k.

4.29. Remarks. Every pseudodifferential operator is a sum of two basic operators (one with the integer m in Definition 4.28 even, and one with m odd). The class of pseudodifferential operators does not depend on the choice of operator K.

4.30. Lemma. *If T is a pseudodifferential operator and $z \in \mathbb{C}$ then*

$$[\Delta_1^z, T] \approx \sum_{j=1}^{\infty} \binom{z}{j} T^{(j)} \Delta_1^{z-j}.$$

Proof. See the proof of Lemma 4.20. □

4.31. Proposition. *The set of all pseudodifferential operators is a filtered algebra. If T is a pseudodifferential operator then so is $\delta(T)$, and moreover* $\operatorname{order}(\delta(T)) \le \operatorname{order}(T)$.

Proof. The set of pseudodifferential operators is a vector space. The formula

$$X\Delta_1^{\frac{m}{2}} \cdot Y\Delta_1^{\frac{n}{2}} \approx \sum_{j=0}^{\infty} \binom{\frac{m}{2}}{j} XY^{(j)} \Delta_1^{\frac{m+n}{2}-j}$$

shows that it is closed under multiplication. Finally,

$$\begin{aligned}\delta(T) = |D|T - T|D| &\approx \Delta_1^{\frac{1}{2}} T - \Delta_1^{\frac{1}{2}} T \\ &\approx \sum_{j=1}^{\infty} \binom{\frac{1}{2}}{j} T^{(j)} \Delta_1^{\frac{1}{2}-j}.\end{aligned}$$

This computation reduces the second part of the lemma to the assertion that if T is a pseudodifferential operator of order k then $T^{(1)} = [\Delta, T]$ is a pseudodifferential operator of order $k+1$ or less. This in turn follows from the definition of pseudodifferential operator and the fact that if X is a differential operator then the differential operator $[\Delta, X]$ has order at most one more than the order of X. $\square$

Proof of Theorem 4.26, Part Two. Suppose that (A, H, D) is a spectral triple for which $A \cdot H^\infty \subseteq H^\infty$ and for which the basic estimate holds. By the basic estimate, every pseudodifferential operator of order zero extends to a bounded operator on H. Since every operator in A or $[D, A]$ is pseudodifferential of order zero, and since $\delta(T)$ is pseudodifferential of order zero whenever T is, we see that if $b \in A$ or $b \in [D, A]$ then for every n the operator $\delta^n(b)$ extends to a bounded operator on H. Hence the spectral triple (A, H, D) is regular, as required. $\square$

4.32. Definition. A spectral triple (A, H, D) is *finitely summable* if there is a Schatten ideal $\mathcal{L}^p(H)$ (where $1 \le p < \infty$)) such that

$$a \cdot (1 + D^2)^{-\frac{1}{2}} \in \mathcal{L}^p(H)$$

for every $a \in A$.

If the spectral triple (A, H, D) is regular and finitely summable then for every $X \in \mathcal{D}(A, D)$ the zeta function $\operatorname{Trace}(X\Delta^{-\frac{z}{2}})$ is defined in a right half-plane in $\mathbb{C}$, and is holomorphic there. The following concept has been introduced by Connes and Moscovici [**12**, Definition II.1].

4.33. Definition. Let (A, H, D) be a regular and finitely summable spectral triple. It has *discrete dimension spectrum* if[9] there is a discrete subset F of $\mathbb{C}$ with the following property: for every operator X in $\mathcal{D}(A, D)$ if $\operatorname{order}(X) \le q$ then the zeta function $\operatorname{Trace}(X\Delta^{-\frac{z}{2}})$ extends to a meromorphic function on $\mathbb{C}$ with all poles contained in $F + q$. The *dimension spectrum* of (A, H, D) is then the smallest such set F.

4.34. Remark. The definition above extends without change to arbitrary algebras of generalized differential operators, and at one point (in Section 6) we shall use it in this context.

A final item of terminology: in Appendix A we shall make use of the following notion:

4.35. Definition. A regular and finitely summable spectral triple has *simple* dimension spectrum if it has discrete dimension spectrum and if all the zeta-type functions above have only simple poles.

5. The Residue Cocycle

In this section we shall assume as given an algebra $\mathcal{D}(\Delta)$, a square root D of Δ, and an algebra $A \subseteq \mathcal{D}(\Delta)$, as in the previous sections. We shall assume the finite analytic dimension and analytic continuation properties set forth in Definitions 4.8 and 4.11. We shall also assume that the Hilbert space H is nontrivially $\mathbb{Z}/2$-graded and therefore that the operator D has odd grading-degree. This is the "even-dimensional" case. The "odd-dimensional" case, where H has no grading, will be considered separately in Section 7.4.

[9]Connes and Moscovici add a technical condition concerning decay of zeta functions along vertical lines in $\mathbb{C}$; compare Appendix A.

5.1. Improper Cocycle. We are going to define a periodic cyclic cocycle $\Psi = (\Psi_0, \Psi_2, \dots)$ for the algebra A. The cocycle will be *improper*, in the sense that all the Ψ_p will be (typically) nonzero. Moreover the cocycle will assume values in the field of meromorphic functions on $\mathbb{C}$. But in the next section we shall convert it into a proper cocycle with values in $\mathbb{C}$ itself.

We are going to assemble Ψ from the quantities $\langle X^0, \dots, X^p \rangle_z$ defined in Section 4. In doing so we shall follow quite closely the construction of the JLO cocycle in entire cyclic cohomology (see [**20**] and [**17**]), which is assembled from the quantities

$$\langle X^0, \dots, X^p \rangle^{\mathrm{JLO}} = \operatorname{Trace}\left(\int_{\Sigma^p} \varepsilon X^0 e^{-t_0 \Delta} \dots X^p e^{-t_p \Delta} \, dt \right) \tag{5.1}$$

(the integral is over the standard p-simplex). In Appendix A we shall compare our cocycle to the JLO cocycle. For now, let us note that the quantities in Equation (5.1) are scalars, while the quantities $\langle X^0, \dots, X^p \rangle_z$ are functions of the parameter z. But this difference is superficial, and the computations which follow in this section are more or less direct copies of computations already carried out for the JLO cocycle in [**20**] and [**17**].

We begin by establishing some "functional equations" for the quantities $\langle \cdots \rangle_z$. In order to keep the formulas reasonably compact, if $X \in \mathcal{D}(\Delta)$ then we shall write $(-1)^X$ to denote either $+1$ or -1, according as the $\mathbb{Z}/2$-grading degree of X is even or odd.

5.1. Lemma. *The meromorphic functions $\langle X^0, \dots, X^p \rangle_z$ satisfy the following functional equations:*

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_{z+1} = \sum_{j=0}^{p} \langle X^0, \dots, X^{j-1}, 1, X^j, \dots, X^p \rangle_z \tag{5.2}$$

$$\langle X^0, \dots, X^{p-1}, X^p \rangle_z = (-1)^{X^p} \langle X^p, X^0, \dots, X^{p-1} \rangle_z \tag{5.3}$$

Proof. The first identity follows from the fact that

$$\begin{aligned} \frac{d}{d\lambda} &\left(\lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \right) \\ &= (-z) \lambda^{-z-1} X^0 (\lambda - \Delta)^{-1} \cdots X^p (\lambda - \Delta)^{-1} \\ &\quad - \sum_{j=0}^{p} \lambda^{-z} X^0 (\lambda - \Delta)^{-1} \cdots X^j (\lambda - \Delta)^{-2} X^{j+1} \cdots X^p (\lambda - \Delta)^{-1} \end{aligned}$$

and the fact that the integral of the derivative is zero. As for the second identity, if $p \gg 0$ then the integrand in Equation (4.1) is a trace-class operator, and Equation (5.3) is an immediate consequence of the trace-property. In general we can repeatedly apply Equation (5.2) to reduce to the case where $p \gg 0$. $\square$

5.2. Lemma.

$$\begin{aligned} &\langle X^0, \dots, [D^2, X^j], \dots, X^p \rangle_z = \\ &\qquad \langle X^0, \dots, X^{j-1} X^j, \dots X^p \rangle_z - \langle X^0, \dots, X^j X^{j+1}, \dots X^p \rangle_z \end{aligned} \tag{5.4}$$

PROOF. This follows from the identity

$$X^{j-1}(\lambda-\Delta)^{-1}[D^2,X^j](\lambda-\Delta)^{-1}X^{j+1} \\ = X^{j-1}(\lambda-\Delta)^{-1}X^jX^{j+1} - X^{j-1}X^j(\lambda-\Delta)^{-1}X^{j+1}.$$

Note that when $j=p$ equation (5.4) should read as

$$\langle X^0,\dots,X^{p-1},[D^2,X^p]\rangle_z = \langle X^0,\dots,X^{p-1}X^p\rangle_z - (-1)^{X^p}\langle X^pX^0,\dots,X^{p-1}\rangle_z.$$

□

5.3. LEMMA.

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^p\rangle_z = 0 \tag{5.5}$$

PROOF. The identity is equivalent to the formula

$$\operatorname{Trace}\left(\varepsilon\Big[D,\int\lambda^{-z}X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda\Big]\right)=0,$$

which holds since the supertrace of any (graded) commutator is zero. □

With these preliminaries out of the way we can obtain very quickly the (improper) (b,B)-cocycle which is the main object of study in these notes.

5.4. DEFINITION. If p is a non-negative and even integer then define a $(p+1)$-multi-linear functional on A with values in the meromorphic functions on $\mathbb{C}$ by the formula

$$\Psi_p(a^0,\dots,a^p) = \langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}$$

5.5. THEOREM. *The even (b,B)-cochain $\Psi=(\Psi_0,\Psi_2,\Psi_4\cdots)$ is an (improper) (b,B)-cocycle.*

PROOF. First of all, it follows from the definition of B and Lemma 5.1 that

$$\begin{aligned} B\Psi_{p+2}(a^0,\dots,a^{p+1}) &= \sum_{j=0}^{p+1}(-1)^j\langle 1,[D,a^j],\dots,[D,a^{j-1}]\rangle_{s-\frac{p+2}{2}} \\ &= \sum_{j=0}^{p+1}\langle [D,a^0],\dots,[D,a^{j-1}],1,[D,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p+2}{2}} \\ &= \langle [D,a^0],[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}. \end{aligned}$$

Next, it follows from the definition of b and the Leibniz rule $[D,a^ja^{j+1}]=a^j[D,a^{j+1}]+[D,a^j]a^{j+1}$ that

$$\begin{aligned} b\Psi_p(a^0,\dots,a^{p+1}) &= \big(\langle a^0a^1,[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^0,a^1[D,a^2],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big) \\ &\quad - \big(\langle a^0,[D,a^1]a^2,[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^0,[D,a^1],a^2[D,a^3],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big) \\ &\quad + \cdots \\ &\quad + \big(\langle a^0,[D,a^1],\dots,[D,a^p]a^{p+1}\rangle_{s-\frac{p}{2}} \\ &\qquad - \langle a^{p+1}a^0,[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}\big). \end{aligned}$$

Applying Lemma 5.2 we get

$$b\Psi_p(a^0,\dots,a^{p+1}) = \sum_{j=1}^{p+1}(-1)^{j-1}\langle a^0,[D,a^1],\dots,[D^2,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}}$$

Setting $X^0 = a^0$ and $X^j = [D,a^j]$ for $j \geq 1$, and applying Lemma 5.3 we get

$$\begin{aligned} &B\Psi_{p+2}(a^0,\dots,a^{p+1}) + b\Psi_p(a^0,\dots,a^{p+1}) \\ &\qquad = \sum_{j=0}^{p+1}(-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^{p+1}\rangle_{s-\frac{p}{2}} = 0. \end{aligned}$$

□

5.2. Residue Cocycle. By taking residues at $s = 0$ we map the space of meromorphic functions on $\mathbb{C}$ to the scalar field $\mathbb{C}$, and we obtain from any (b,B)-cocycle with coefficients in the space of meromorphic functions a (b,B)-cocycle with coefficients in $\mathbb{C}$. This operation transforms the improper cocycle Ψ that we constructed in the last section into a *proper* cocycle $\operatorname{Res}_{s=0}\Psi$. Indeed, it follows from Proposition 4.14 that if p is greater than the analytic dimension d of $\mathcal{D}(\Delta)$ then the function

$$\Psi_p(a^0,\dots,a^p)_s = \langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}$$

is holomorphic at $s = 0$.

The following proposition identifies the proper (b,B)-cocycle $\operatorname{Res}_{s=0}\Psi$ with the residue cocycle studied by Connes and Moscovici.

5.6. Theorem. *For all $p \geq 0$ and all $a^0,\dots,a^p \in A$,*

$$\begin{aligned} &\operatorname{Res}_{s=0}\Psi_p(a^0,\dots,a^p) \\ &\qquad = \sum_{k\geq 0} c_{p,k}\operatorname{Res}_{s=0}\operatorname{Tr}\left(\varepsilon a^0[D,a^1]^{(k_1)}\cdots[D,a^p]^{(k_p)}\Delta^{-\frac{p}{2}-|k|-s}\right). \end{aligned}$$

The sum is over all multi-indices $(k_1,\dots,k_p)$ with non-negative integer entries, and the constants c_{pk} are given by the formula

$$c_{pk} = \frac{(-1)^k}{k!}\frac{\Gamma(|k|+\frac{p}{2})}{(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_p+p)}$$

5.7. Remarks. Before proving the theorem we need to make one or two comments about the above formula.

First, the constant $c_{00} = \Gamma(0)$ is not well defined since 0 is a pole of the Γ-function. To cope with this problem we replace the term $c_{00}\operatorname{Res}_{s=0}\big(\operatorname{Tr}(\varepsilon a^0\Delta^{-s})\big)$ with $\operatorname{Res}_{s=0}\big(\Gamma(s)\operatorname{Tr}(\varepsilon a^0\Delta^{-s})\big)$.

Second, it follows from Proposition 4.14 that if $|k| + p > d$ then the (p,k)-contribution to the above sum of residues is actually zero. Hence for every p the sum is in fact finite (and as we already noted above, the sum is 0 when $p > d$).

Proof of Theorem 5.6. We showed in the proof of Proposition 4.14 that

$$\langle X^0, \dots, X^p\rangle_z \approx \sum_{k\geq 0} (-1)^{|k|}\Gamma(z+p+|k|)\frac{c(k)}{(|k|+p)!} \times \operatorname{Trace}\left(\varepsilon X^0 {X^1}^{(k_1)} \cdots {X^p}^{(k_p)} \Delta^{-z-|k|-p}\right).$$

After we note that

$$c_{pk} = (-1)^{|k|}\Gamma\big(|k|+\frac{p}{2}\big)\frac{c(k)}{(p+|k|)!}$$

the proof of the theorem follows immediately from the asymptotic expansion upon setting $z = s - \frac{p}{2}$ and taking residues at $s = 0$. □

6. The Local Index Formula

The objective of this section is to compute the pairing between the periodic cyclic cocycle $\operatorname{Res}_{s=0}\Psi$ and idempotents in the algebra A (compare Section 2.5). We shall prove the following result.

6.1. Theorem. *Let $\operatorname{Res}_{s=0}\Psi$ be the index cocycle associated to an algebra $\mathcal{D}(\Delta)$ of generalized differential operators with finite analytic dimension and the analytic continuation property, together with a square-root D of Δ and a subalgebra $A \subseteq \mathcal{D}(\Delta)$. If e is an idempotent element of A then*

$$\langle \operatorname{Res}_{s=0}\Psi, e\rangle = \operatorname{Index}\big(eDe\colon eH_0 \to eH_1\big).$$

Theorem 6.1 will later be superseded by a more precise result at the level of cyclic cohomology, and we shall we shall only sketch one or to parts of the proof which will be dealt with in more detail later. Furthermore, to slightly simplify the analysis we shall assume that $\mathcal{D}(\Delta)$ has discrete dimension spectrum, in the sense of Definition 4.33.

6.1. Invertibility Hypothesis Removed. Up to now we have been assuming that the self-adjoint operator Δ is invertible (in the sense of Hilbert space operator theory, meaning that Δ is a bijection from its domain to the Hilbert space H). We shall now remove this hypothesis.

To do so we shall begin with an operator D which is not necessarily invertible (with $D^2 = \Delta$). We shall assume that the axioms (i)-(vi) in Sections 4.2 and 4.4 hold. Fix a bounded self-adjoint operator K with the following properties:

(i) K commutes with D.
(ii) K has analytic order $-\infty$ (in other words, $K \cdot H \subseteq H^\infty$).
(iii) The operator $\Delta + K^2$ is invertible.

Having done so, let us construct the operator

$$D_K = \begin{pmatrix} D & K \\ K & -D \end{pmatrix}$$

acting on the Hilbert space $H \oplus H^{\mathrm{opp}}$, where H^{opp} is the $\mathbb{Z}/2$-graded Hilbert space H but with the grading reversed. It is invertible.

6.2. Example. If D is a Fredholm operator then we can choose for K the projection onto the kernel of D.

Let $\Delta_K = (D_K)^2$ and denote by $\mathcal{D}(\Delta_K)$ the smallest algebra of operators on $H \oplus H^{\mathrm{opp}}$ which contains the 2×2 matrices over $\mathcal{D}(\Delta)$ and which is closed under multiplication by operators of analytic order $-\infty$.

The axioms (i)-(iv) of Section 4.2 are satisfied for the new algebra. Moreover if we embed A into $\mathcal{D}(\Delta_K)$ as matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ then the axioms (v) and (vi) in Section 4.4 are satisfied too.

6.3. Lemma. *Assume that the operators K_1 and K_2 both have the properties* (i)-(iii) *listed above. Then $\mathcal{D}(\Delta_{K_1}) = \mathcal{D}(\Delta_{K_1})$. Moreover the algebra has finite analytic dimension d and has the analytic continuation property with respect to Δ_{K_1} if and only if it has the same with respect to Δ_{K_2}. If these properties do hold then the quantities $\langle X^0, \ldots, X^p \rangle_z$ associated to Δ_{K_1} and Δ_{K_2} differ by a function which is analytic in the half-plane* $\mathrm{Re}(z) > -p$.

Proof. It is clear that $\mathcal{D}(\Delta_{K_1}) = \mathcal{D}(\Delta_{K_2})$. To investigate the analytic continuation property it suffices to consider the case where K_1 is a fixed function of Δ, in which case K_1 and K_2 commute. Let us write

$$X\Delta^{-z} = \frac{1}{2\pi i} \int \lambda^{-z} X (\lambda - \Delta)^{-1}\, d\lambda$$

for $\mathrm{Re}(z) > 0$. Observe now that

$$(\lambda - \Delta_{K_1})^{-1} - (\lambda - \Delta_{K_2})^{-1} \approx M(\lambda - \Delta_{K_1})^{-2} - M(\lambda - \Delta_{K_1})^{-3} + \cdots,$$

where $M = \Delta_{K_1} - \Delta_{K_2}$ (this is an asymptotic expansion in the sense described prior to the proof of Proposition 4.14). Integrating and taking traces we see that

$$\mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big) \approx \sum_{k \geq 1} (-1)^{k-1} \binom{-z}{k} \mathrm{Trace}\big(XM\Delta_{K_1}^{-z-k}\big), \tag{6.1}$$

which shows that the difference $\mathrm{Trace}\big(X\Delta_{K_1}^{-z}\big) - \mathrm{Trace}\big(X\Delta_{K_2}^{-z}\big)$ has an analytic continuation to an entire function. Therefore Δ_{K_1} has the analytic continuation property if and only if Δ_{K_2} does (and moreover the analytic dimensions are equal).

The remaining part of the lemma follows from the asymptotic formula

$$\langle X^0, \ldots, X^p \rangle_z \approx \sum_{k \geq 0} (-1)^{|k|} \Gamma(z + p + |k|) \frac{1}{(|k| + p)!} c(k) \times \mathrm{Trace}\left(\varepsilon X^0 {X^1}^{(k_1)} \cdots {X^p}^{(k_p)} \Delta^{-z-|k|-p} \right)$$

that we proved earlier. □

6.4. Definition. The *residue cocycle* associated to the possibly non-invertible operator D is the residue cocycle $\mathrm{Res}_{s=0}\, \Psi$ associated to the invertible operator D_K, as above.

Lemma 6.3 shows that if $p > 0$ then the residue cocycle given by Definition 6.4 is independent of the choice of the operator K. In fact this is true when $p = 0$ too. Indeed Equation (6.1) shows that not only is the difference $\mathrm{Trace}(\varepsilon a^0 \Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0 \Delta_{K_2}^{-s})$ analytic at $s = 0$, but it vanishes there too. Therefore

$$\begin{aligned} &\mathrm{Res}_{s=0}\, \Psi_0^{K_1}(a^0) - \mathrm{Res}_{s=0}\, \Psi_0^{K_2}(a^0) \\ &\qquad = \mathrm{Res}_{s=0}\, \Gamma(s) \big(\mathrm{Trace}(\varepsilon a^0 \Delta_{K_1}^{-s}) - \mathrm{Trace}(\varepsilon a^0 \Delta_{K_2}^{-s}) \big) = 0. \end{aligned}$$

6.5. Example. If D happens to be invertible already then we obtain the same residue cocycle as before.

6.6. Example. In the case where D is Fredholm, the residue cocycle is given by the same formula that we saw in Theorem 5.6:

$$\operatorname{Res}_{s=0} \Psi_p(a^0, \dots, a^p) = \sum_{k\geq 0} c_{p,k} \operatorname{Res}_{s=0} \operatorname{Tr}\left(\varepsilon a^0 [D, a^1]^{(k_1)} \cdots [D, a^p]^{(k_p)} \Delta^{-\frac{p}{2}-|k|-s}\right).$$

The complex powers Δ^{-z} are defined to be zero on the kernel of D (which is also the kernel of Δ). When $p = 0$ the residue cocycle is

$$\operatorname{Res}_{s=0}\bigl(\Gamma(s) \operatorname{Trace}(\varepsilon a^o \Delta^{-s})\bigr) + \operatorname{Trace}(\varepsilon a^0 P),$$

where the complex power Δ^{-s} is defined as above and P is the orthogonal projection onto the kernel of D.

6.2. Proof of the Index Theorem. Let us fix an idempotent $e \in A$ and define a family of operators by the formula

$$D_t = D + t[e, [D, e]], \qquad t \in [0, 1].$$

Note that $D_0 = D$ while $D_1 = eDe + e^{\perp} D e^{\perp}$, so that in particular D_1 commutes with e. Denote by Ψ^t the improper cocycle associated to D_t (via the mechanism just described in the last section which involves the incorporation of some order $-\infty$ operator K_t, which we shall assume depends smoothly on t).

6.7. Lemma. *Define an improper (b, B)-cochain Θ^t by the formula*

$$\Theta^t_p(a^0, \dots, a^p) = \sum_{j=0}^{p} (-1)^{j-1} \langle a^0, \dots, [D_{K_t}, a^j], \dot{D}_{K_t}, [D_{K_t}, a^{j+1}], \dots, [D_{K_t}, a^p]\rangle_{s-\frac{p+1}{2}},$$

where $\dot{D} = \frac{d}{dt} D_{K_t} \in \mathcal{D}(\Delta_K)$. Then

$$B\Theta^t_{p+1} + b\Theta^t_{p-1} + \frac{d}{dt}\Psi^t_p = 0.$$

□

The lemma, which is nothing more than an elaborate computation, can be proved by following the steps taken in Section 7.1 below (compare Remark 7.9).

Proof of Theorem 6.1. It follows from the asymptotic expansion method used to prove Lemma 6.3 that each Ψ^t and each Θ^t is meromorphic. Since we are assuming that $\mathcal{D}(\Delta)$ has discrete dimension spectrum the poles of all these functions are located within the same discrete set in $\mathbb{C}$. As a result, the integral $\int_0^1 \Theta^t \, dt$ is clearly meromorphic too. Since

$$B \int_0^1 \Theta^t_{p+1} \, dt + b \int_0^1 \Theta^t_{p-1} \, dt = \Psi^0 - \Psi^1,$$

it follows by taking residues that $\operatorname{Res}_{s=0} \Psi^0$ and $\operatorname{Res}_{s=1} \Psi^1$ are cohomologous. As a result, we can compute the pairing $\langle \operatorname{Res}_{s=0} \Psi, e\rangle$ using Ψ^1 in place of Ψ^0. If we choose the operator K_1 to commute with not only D_1 but also e, then D_{K_1}

commutes with e and the explicit formula for the pairing $\langle \mathrm{Res}_{s=0}\Psi, e\rangle$ given in Theorem 2.27 simplifies, as follows:

$$\begin{aligned}\langle \mathrm{Res}_{s=0}\Psi^1, e\rangle &= \mathrm{Res}_{s=0}\Psi^1_0(e) + \sum_{k\geq 1}(-1)^k\frac{(2k)!}{k!}\mathrm{Res}_{s=0}\Psi^1_{2k}(e-\frac{1}{2}, e, \dots, e)\\ &= \mathrm{Res}_{s=0}\Psi^1_0(e)\end{aligned}$$

(all the higher terms vanish since they involve commutators $[D_{K_1}, e]$). We conclude that

$$\begin{aligned}\langle \mathrm{Res}_{s=0}\Psi^1, e\rangle &= \mathrm{Res}_{s=0}\Psi^1_0(e)\\ &= \mathrm{Res}_{s=0}\Big(\Gamma(s)\,\mathrm{Trace}(\varepsilon e\,(\Delta_{K_1})^{-s})\Big)\\ &= \mathrm{Index}\big(eDe\colon eH_0 \to eH_1\big),\end{aligned}$$

as required (the last step is the index computation made by Atiyah and Bott that we mentioned in the introduction).

□

6.8. Remark. The proof of the corresponding odd index formula (involving the odd pairing in Theorem 2.27) is not quite so simple, but could presumably be accomplished following the argument developed by Getzler in [**16**] for the JLO cocycle.

7. The Local Index Theorem in Cyclic Cohomology

Our goal in this section is to identify the cohomology class of the residue cocycle $\mathrm{Res}_{s=0}\Psi$ with the cohomology class of the Chern character cocycle ch^F_n associated to the operator $F = D|D|^{-1}$ (see Section 2.1). Here n is any even integer greater than or equal to the analytic dimension d. It follows from the definition of analytic dimension and some simple manipulations that

$$[F, a^0]\cdots[F, a^n] \in \mathcal{L}^1(H),$$

for such n, so that the Chern character cocycle is well-defined.

We shall reach the goal in two steps. First we shall identify the cohomology class of $\mathrm{Res}_{s=0}\Psi$ with the class of a certain specific cyclic cocycle, which involves no residues. Secondly we shall show that this cyclic cocycle is cohomologous to the Chern character ch^F_n.

To begin, we shall return to our assumption that D is invertible, and then deal with the general case at the end of the section.

7.1. Reduction to a Cyclic Cocycle.

The following result summarizes step one.

7.1. Theorem. *Fix an even integer n strictly greater than $d-1$. The multilinear functional*

$$(a^0, \dots, a^n) \mapsto \frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}\langle [D,a^0], \dots, [D,a^j], D, [D,a^{j+1}], \dots, [D,a^n]\rangle_{-\frac{n}{2}}.$$

is a cyclic n-cocycle which, when considered as a (b, B)-cocycle, is cohomologous to the residue cocycle $\mathrm{Res}_{s=0}\Psi$.

7.2. Remark. It follows from Proposition 4.14 that the quantities

$$\langle [D,a^0],\dots,[D,a^j],D,[D,a^{j+1}],\dots,[D,a^n]\rangle_z$$

which appear in the theorem are holomorphic in the half-plane $\operatorname{Re}(z) > -\frac{n}{2} + \frac{1}{r}(d-(n+1))$. Therefore it makes sense to evaluate them at $z=-\frac{n}{2}$, as we have done. Appearances might suggest otherwise, because the term $\Gamma(z)$ which appears in the definition of $\langle \dots \rangle_z$ has poles at the non-positive integers (and in particular at $z=-\frac{n}{2}$ if n is even). However these poles are canceled by zeroes of the contour integral in the given half-plane.

Theorem 7.1 and its proof have a simple conceptual explanation, which we shall give in a little while (after Lemma 7.8). However a certain amount of elementary, if laborious, computation is also involved in the proof, and we shall get to work on this first. For this purpose it is useful to introduce the following notation.

7.3. Definition. If $X^0,\dots,X^p$ are operators in the algebra generated by $\mathcal{D}(\Delta)$, 1 and D, and if at least one belongs to $\mathcal{D}(\Delta)$ then define

$$\langle\!\langle X^0,\dots,X^p\rangle\!\rangle_z = \sum_{k=0}^{p} (-1)^{X^0\cdots X^k}\langle X^0,\dots,X^k,D,X^{k+1},\dots,X^p\rangle_z,$$

which is a meromorphic function of $z\in\mathbb{C}$.

The new notation allows us to write a compact formula for the cyclic cocycle appearing in Theorem 7.1:

$$(a^0,\dots,a^n)\mapsto \tfrac{1}{2}\langle\!\langle [D,a^0],\dots,[D,a^n]\rangle\!\rangle_{-\frac{n}{2}}.$$

We shall now list some properties of the quantities $\langle\!\langle \cdots \rangle\!\rangle_z$ which are analogous to the properties of the quantities $\langle \cdots \rangle_z$ that we verified in Section 5. The following lemma may be proved using the formulas in Lemmas 5.1 and 5.2.

7.4. Lemma. *The quantity $\langle\!\langle X^0,\dots,X^p\rangle\!\rangle_z$ satisfies the following identities:*

$$\langle\!\langle X^0,\dots,X^p\rangle\!\rangle_z = \langle\!\langle X^p,X^0,\dots,X^{p-1}\rangle\!\rangle_z \tag{7.1}$$

$$\sum_{j=0}^{p}\langle\!\langle X^0,\dots,X^j,1,X^{j+1},\dots,X^p\rangle\!\rangle_{z+1} = \langle\!\langle X^0,\dots,X^p\rangle\!\rangle_z \tag{7.2}$$

In addition,

$$\begin{aligned}(7.3)\quad &\langle\!\langle X^0,\dots,X^{j-1}X^j,\dots,X^p\rangle\!\rangle_z - \langle\!\langle X^0,\dots,X^jX^{j+1},\dots,X^p\rangle\!\rangle_z\\ &= \langle\!\langle X^0,\dots,[D^2,X^j],\dots,X^p\rangle\!\rangle_z - (-1)^{X^0\cdots X^{j-1}}\langle X^0,\dots,[D,X^j],\dots,X^p\rangle_z.\end{aligned}$$

(In both instances within this last formula the commutators are graded commutators.) □

7.5. Remark. When $j=p$ equation (7.3) should be read as

$$\begin{aligned}&\langle\!\langle X^0,\dots,X^{p-1}X^p\rangle\!\rangle_z - \langle\!\langle X^pX^0,\dots,X^{p-1}\rangle\!\rangle_z\\ &\qquad = \langle\!\langle X^0,\dots,X^{p-1},[D^2,X^p]\rangle\!\rangle_z - (-1)^{X^0\cdots X^{p-1}}\langle X^0,\dots,X^{p-1},[D,X^p]\rangle_z.\end{aligned}$$

We shall also need a version of Lemma 5.3, as follows.

7.6. Lemma.

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\!\langle X^0,\ldots,[D,X^j],\ldots,X^p\rangle\!\rangle_z = 2\sum_{k=0}^{p}\langle X^0,\ldots,X^{k-1},D^2,X^k,\ldots,X^p\rangle_z. \tag{7.4}$$

□

Proof. This follows from Lemma 5.3. Note that $[D,D]=2D^2$, which helps explain the factor of 2 in the formula. □

The formula in Lemma 7.6 can be simplified by means of the following computation:

7.7. Lemma.

$$\sum_{j=0}^{p}\langle X^0,\ldots,X^j,D^2,X^{j+1},\ldots,X^p\rangle_z = (z+p)\langle X^0,\ldots,X^p\rangle_z$$

Proof. If we substitute into the integral which defines $\langle X^0,\ldots,D^2,\ldots,X^p\rangle_z$ the formula

$$D^2 = \lambda-(\lambda-\Delta)$$

we obtain the (supertrace of the) terms

$$(-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z+1}X^0(\lambda-\Delta)^{-1}\cdots 1(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda - (-1)^{p+1}\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z}X^0(\lambda-\Delta)^{-1}\cdots X^p(\lambda-\Delta)^{-1}\,d\lambda$$

Using the functional equation $\Gamma(z)=(z-1)\Gamma(z-1)$ we therefore obtain the quantity

$$(z-1)\langle X^0,\ldots,X^j,1,X^{j+1},\ldots,X^p\rangle_{z-1}+\langle X^0,\ldots,X^p\rangle_z$$

(the change in the sign preceding the second bracket comes from the fact that the bracket contains one fewer term, and the fact that $(-1)^{p+1}=-(-1)^p$). Adding up the terms for each j, and using Lemma 5.1 we therefore obtain

$$\begin{aligned}\sum_{j=0}^{p}\langle X^0,\ldots,X^j,D^2,X^{j+1},\ldots,X^p\rangle_z &= (z-1)\langle X^0,\ldots,X^p\rangle_z+(p+1)\langle X^0,\ldots,X^p\rangle_z\\ &= (z+p)\langle X^0,\ldots,X^p\rangle_z\end{aligned}$$

as required. □

Putting together the last two lemmas we obtain the formula

$$\sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\!\langle X^0,\ldots,[D,X^j],\ldots,X^p\rangle\!\rangle_z = 2(z+p)\langle X^0,\ldots,X^p\rangle_z. \tag{7.5}$$

With this in hand we can proceed to the following computation:

7.8. Lemma. *Define multi-linear functionals Θ_p on A, with values in the space of meromorphic functions on $\mathbb{C}$, by the formulas*

$$\Theta_p(a^0,\dots,a^p) = \langle\langle a^0,[D,a^1],\dots,[D,a^p]\rangle\rangle_{s-\frac{p+1}{2}}.$$

Then

$$B\Theta_{p+1}(a^0,\dots,a^p) = \langle\langle [D,a^0],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}.$$

and in addition

$$b\Theta_{p-1}(a^0,\dots,a^p) + B\Theta_{p+1}(a^0,\dots,a^p) = 2s\Psi_p(a^0,\dots,a^p)$$

for all $s\in\mathbb{C}$ and all $a^0,\dots,a^p\in A$.

Proof. The formula for $B\Theta_{p+1}(a^0,\dots,a^p)$ is a consequence of Lemma 7.4. The computation of $b\Theta_{p-1}(a^0,\dots,a^p)$ is a little more cumbersome, although still elementary. It proceeds as follows. First we use the Leibniz rule to write

$$\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) &= \sum_{j=0}^{p-1}(-1)^j\langle\langle a^0,\dots,[D,a^ja^{j+1}],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad + (-1)^p\langle\langle a^pa^0,[D,a^1],\dots,[D,a^{p-1}]\rangle\rangle_{s-\frac{p}{2}}\\
&= \langle\langle a^0a^1,[D,a^2],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad + \sum_{j=1}^{p-1}(-1)^j\langle\langle a^0,\dots,a^j[D,a^{j+1}],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad + \sum_{j=1}^{p-1}(-1)^j\langle\langle a^0,\dots,[D,a^j]a^{j+1},\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad + (-1)^p\langle\langle a^pa^0,[D,a^1],\dots,[D,a^{p-1}]\rangle\rangle_{s-\frac{p}{2}}
\end{aligned}$$

Next we rearrange the terms to obtain the formula

$$\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) &= \Big(\langle\langle a^0a^1,[D,a^2],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle\langle a^0,a^1[D,a^2],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\Big)\\
&\quad + \sum_{j=1}^{p-2}(-1)^j\Big(\langle\langle a^0,\dots,[D,a^j]a^{j+1},\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle\langle a^0,\dots,a^{j+1}[D,a^{j+2}],\dots,[D,a^p]\rangle\rangle_{s-\frac{p}{2}}\Big)\\
&\quad + (-1)^{p-1}\Big(\langle\langle a^0,[D,a^1],\dots,[D,a^{p-1}]a^p\rangle\rangle_{s-\frac{p}{2}}\\
&\qquad - \langle\langle a^pa^0,[D,a^1],\dots,[D,a^{p-1}]\rangle\rangle_{s-\frac{p}{2}}\Big).
\end{aligned}$$

We can now apply Lemma 7.6:

$$
\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) = \Big(&\langle\!\langle a^0,[D^2,a^1],[D,a^2],\dots,[D,a^p]\rangle\!\rangle_{s-\frac{p}{2}} \\
&+ \langle a^0,[D,a^1],[D,a^2],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}\Big) \\
&+\sum_{j=1}^{p-2}(-1)^j\Big(\langle\!\langle a^0,[D,a^1],\dots,[D^2,a^{j+1}],\dots,[D,a^p]\rangle\!\rangle_{s-\frac{p}{2}} \\
&\qquad + (-1)^j\ \langle a^0,[D,a^1],\dots,[D,a^{j+1}],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}\Big) \\
&+(-1)^{p-1}\Big(\langle\!\langle a^0,[D,a^1],\dots,[D,a^{p-1}],[D^2,a^p]\rangle\!\rangle_{s-\frac{p}{2}} \\
&\qquad + (-1)^{p-1}\langle a^0,[D,a^1],\dots,[D,a^{p-1}],[D,a^p]\rangle_{s-\frac{p}{2}}\Big).
\end{aligned}
$$

Collecting terms we get

$$
\begin{aligned}
b\Theta_{p-1}(a^0,\dots,a^p) = &\sum_{k=1}^{p}(-1)^{k-1}\langle\!\langle a^0,[D,a^1],\dots,[D^2,a^k],\dots,[D,a^p]\rangle\!\rangle_{s-\frac{p}{2}} \\
&+ p\langle a^0,[D,a^1],\dots,[D,a^p]\rangle_{s-\frac{p}{2}}.
\end{aligned}
$$

All that remains now is to add together $b\Theta$ and $B\Theta$, and apply Equation (7.5) to the result. Writing $a^0 = X^0$ and $[D,a^j] = X^j$ for $j = 1,\dots,p$ we get

$$
\begin{aligned}
&b\Theta_{p-1}(a^0,\dots,a^p) + B\Theta_{p+1}(a^0,\dots,a^p) \\
&\quad = \sum_{j=0}^{p}(-1)^{X^0\cdots X^{j-1}}\langle\!\langle X^0,\dots,[D,X^j],\dots,X^p\rangle\!\rangle_{s-\frac{p}{2}} - p\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} \\
&\quad = 2(s-\frac{p}{2}+p)\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} - p\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}} \\
&\quad = 2s\langle X^0,\dots,X^p\rangle_{s-\frac{p}{2}},
\end{aligned}
$$

as the lemma requires. □

7.9. Remark. The statement of Lemma 7.8 can be explained as follows. If we replace D by tD and Δ by $t^2\Delta$ in the definitions of $\langle\cdots\rangle_z$ and Ψ_p, so as to obtain a new improper (b,B)-cocycle $\Psi^t = (\Psi^t_0,\Psi^t_2,\dots)$, then it is easy to check from the definitions that

$$\Psi^t_p(a^0,\dots,a^p) = t^{-2s}\Psi_p(a^0,\dots,a^p).$$

Now, we expect that as t varies the cohomology class of the cocycle Ψ^t should not change. And indeed, by borrowing known formulas from the theory of the JLO cocycle (see for example [**17**], or [**18**, Section 10.2], or Section 6 below) we can construct a (b,B)-cochain Θ such that

$$B\Theta + b\Theta + \frac{d}{dt}\Psi^t = 0.$$

This is the same Θ as that which appears in the lemma.

The proof of Theorem 7.1 is now very straightforward:

Proof of Theorem 7.1. According to Lemma 7.8 the (b,B)-cochain

$$\big(\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_1\big),\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_3\big),\dots,\mathrm{Res}_{s=0}\big(\frac{1}{2s}\Theta_{n-1}\big),0,0,\dots\big)$$

cobounds the difference of $\mathrm{Res}_{s=0}\,\Psi$ and the cyclic n-cocycle $\mathrm{Res}_{s=0}\big(\frac{1}{2s}B\Theta_{n+1}\big)$. Since

$$\mathrm{Res}_{s=0}\big(\frac{1}{2s}B\Theta_{n+1}\big)(a^0,\dots,a^n) = \frac{1}{2}\langle\!\langle [D,a^0],\dots,[D,a^n]\rangle\!\rangle_{-\frac{n}{2}}$$

the theorem is proved. □

7.2. Computation with the Cyclic Cocycle. We turn now to the second step. We are going to alter D by means of the following homotopy:

$$D_t = D|D|^{-t} \qquad (0 \le t \le 1)$$

(the same strategy is employed by Connes and Moscovici in [**10**]). We shall similarly replace Δ with $\Delta_t = D_t^2$, and we shall use Δ_t in place of Δ in the definitions of $\langle\cdots\rangle_z$ and of $\langle\!\langle\cdots\rangle\!\rangle_z$.

To simplify the notation we shall drop the subscript t in the following computation and denote by $\dot D = -D_t \cdot \log|D|$ the derivative of the operator D_t with respect to t.

7.10. Lemma. *Define a multi-linear functional on A, with values in the analytic functions on the half-plane* $\mathrm{Re}(z) + n > \frac{d-1}{2}$*, by the formula*

$$\Phi_n^t(a^0,\dots,a^n) = \langle\!\langle a^0\dot D, [D,a^1],\dots,[D,a^n]\rangle\!\rangle_z.$$

Then $B\Phi_n^t$ is a cyclic $(n-1)$-cochain and

$$\begin{aligned} &bB\Phi_n^t(a^0,\dots,a^n)\\ &\qquad = \frac{d}{dt}\langle\!\langle [D,a^0],\dots,[D,a^n]\rangle\!\rangle_z + (2z+n)\sum_{j=0}^{n}\langle \dot D, [D,a^j],\dots,[D^{j-1}]\rangle_z. \end{aligned}$$

7.11. Remark. Observe that the operator $\log|D|$ has analytic order δ or less, for every $\delta > 0$. As a result, the proof of Proposition 4.14 shows that the quantity is a holomorphic function of z in the half-plane $\mathrm{Re}(z) + n > \frac{d-1}{2}$. But we shall not be concerned with any possible meromorphic continuation to $\mathbb{C}$.

Proof. Let us take advantage of the fact that $bB + Bb = 0$ and compute $Bb\Phi^t$ instead (fewer minus signs and wrap-around terms are involved).

A straightforward application of the definitions in Section 2 shows that the quantity $Bb\Phi_n^t(a^0,\dots,a^n)$ is the sum, from $j=0$ to $j=n$, of the following terms:

$$\begin{aligned} &-\langle\!\langle \dot D, [D,a^j a^{j+1}], [D,a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z\\ &\quad+\langle\!\langle \dot D, [D,a^j], [D,a^{j+1}a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z\\ &\quad\quad-\cdots\\ &\quad\quad\quad+(-1)^n\langle\!\langle \dot D, [D,a^j], [D,a^{j+1}],\dots,[D,a^{j-2}a^{j-1}]\rangle\!\rangle_z. \end{aligned}$$

If we add the term $\langle\!\langle \dot D a^j, [D,a^{j+1}],\dots,[D,a^{j-1}\rangle\!\rangle_z$ to the beginning of this expression, and also the terms

$$\begin{aligned} &-\langle\!\langle a^{j-1}\dot D, [D,a^j a^{j+1}], [D,a^{j+2}],\dots,[D,a^{j-2}]\rangle\!\rangle_z\\ &\quad-\langle\!\langle [\dot D,a^j], [D,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z \end{aligned}$$

at the end, then, after summing over all j, we have added zero in total. But we can now invoke Leibniz's rule to expand $[D,a^k a^{k+1}]$ and apply part (iii) of Lemma 7.4

to obtain the quantity

$$\begin{aligned}
&\Big(\langle\!\langle \dot D, [D^2,a^j],[D,a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z \\
&\quad - \langle\!\langle \dot D, [D,a^j],[D^2,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z \\
&\qquad + \dots \\
&\qquad\quad + (-1)^n \langle\!\langle \dot D, [D,a^j],[D,a^{j+1}],\dots,[D^2,a^{j-1}]\rangle\!\rangle_z\Big) \\
&+ (n+1)\langle \dot D, [D,a^j],[D,a^{j+1}],\dots,[D,a^{j-1}]\rangle_z \\
&\qquad\qquad - \langle\!\langle [\dot D,a^j],[D,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z
\end{aligned}$$

(summed over j, as before). Applying Equation 7.5 we arrive at the following formula

$$\begin{aligned}
Bb\Phi^t_n(a^0,\dots,a^n) &= \textstyle\sum_{j=0}^n \langle\!\langle [D,\dot D],[D,a^j],[D,a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z \\
&\quad - (2z+(n+1))\textstyle\sum_{j=0}^n \langle \dot D,[D,a^j],\dots,[D,a^{j-1}]\rangle_z \\
&\quad - \textstyle\sum_{j=0}^n \langle\!\langle [\dot D,a^j],[D,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z .
\end{aligned}$$

To complete the proof we write

$$\langle\!\langle [D,a^0],\dots,[D,a^p]\rangle\!\rangle_z = \textstyle\sum_{j=0}^n \langle D,[D,a^j],\dots,[D,a^{j-1}]\rangle_z$$

and differentiate with respect to t, bearing in mind the definition of the quantities $\langle \dots\rangle_z$ and the fact that $\frac{d}{dt}(\lambda-\Delta)^{-1} = (\lambda-\Delta)^{-1}[D,\dot D](\lambda-\Delta)^{-1}$. We obtain

$$\begin{aligned}
\tfrac{d}{dt}\langle\!\langle [D,a^0],\dots,[D,a^p]\rangle\!\rangle_z &= -\textstyle\sum_{j=0}^n \langle\!\langle [D,\dot D],[D,a^j],[D,a^{j+2}],\dots,[D,a^{j-1}]\rangle\!\rangle_z \\
&\quad + \textstyle\sum_{j=0}^n \langle \dot D,[D,a^j],\dots,[D,a^{j-1}]\rangle_z \\
&\quad + \textstyle\sum_{j=0}^n \langle\!\langle [\dot D,a^j],[D,a^{j+1}],\dots,[D,a^{j-1}]\rangle\!\rangle_z .
\end{aligned}$$

This proves the lemma. □

We can now complete the second step, and with it the proof of the Connes-Moscovici Residue Index Theorem:

7.12. Theorem (Connes and Moscovici). *The residue cocycle* $\mathrm{Res}_{s=0}\Psi$ *is cohomologous, as a* (b,B)*-cocycle, to the Chern character cocycle of Connes.*

Proof. Thanks to Theorem 7.1 it suffices to show that the cyclic cocycle

$$\frac{1}{2}\langle\!\langle [D,a^0],\dots,[D,a^n]\rangle\!\rangle_{-\frac{n}{2}} \tag{7.6}$$

is cohomologous to the Chern character. To do this we use the homotopy D_t above. Thanks to Lemma 7.10 the coboundary of the cyclic cochain

$$\int_0^1 B\Psi^t_n(a^0,\dots,a^{n-1})\,dt$$

is the difference of the cocycles (7.6) associated to $D_0 = D$ and $D_1 = F$. For D_1 we have $D_1^2 = \Delta_1 = I$ and so

$$\frac{1}{2}\langle\langle [D_1, a^0], \dots, [D_1, a^n]\rangle\rangle_z$$
$$= \frac{1}{2}\sum_{j=1}^{n}(-1)^{j+1}\frac{(-1)^{n+1}\Gamma(z)}{2\pi i}\times$$
$$\operatorname{Trace}\left(\int \lambda^{-z}\varepsilon[F, a^0]\cdots[F, a^j]F\cdots[F, a^n](\lambda - I)^{-(n+2)}\, d\lambda\right)$$

Since F anticommutes with each operator $[F, a^j]$ this simplifies to

$$\frac{1}{2}\sum_{j=1}^{n}\frac{(-1)^{n+1}\Gamma(z)}{2\pi i}\operatorname{Trace}\left(\int \lambda^{-z}\varepsilon F[F, a^0]\cdots[F, a^n](\lambda - I)^{-(n+2)}\, d\lambda\right).$$

The terms in the sum are now all the same, and after applying Cauchy's formula we get

$$\frac{n+1}{2}(-1)^{n+1}\Gamma(z)\cdot\operatorname{Trace}\left(\varepsilon F[F, a^0]\cdots[F, a^n]\right)\cdot\binom{-z}{n+1}.$$

Using the functional equation for the Γ-function this reduces to

$$\frac{\Gamma(z+n+1)}{2\cdot n!}\operatorname{Trace}\left(\varepsilon F[F, a^0]\cdots[F, a^n]\right)$$

and evaluating at $z = -\frac{n}{2}$ we obtain the Chern character of Connes. □

7.3. Invertibility Hypothesis Removed. In the case where D is non-invertible we employ the device introduced in Section 6.1, and associate to D the residue cocycle for the operator D_K.

Now Connes' Chern character cocycle is defined for a not necessarily invertible operator D by forming first D_K, then $F_K = D_K|D_K|^{-1}$, then $\mathrm{ch}_n^{F_K}$. See [**4**, Part I]. The following result therefore follows immediately from our calculations in the invertible case.

7.13. Theorem. *For any operator D, invertible or not, the class in periodic cyclic cohomology of the residue cocycle* $\operatorname{Res}_{s=0}\Psi$ *is equal to the class of the Chern character cocycle of Connes.* □

7.4. The Odd-Dimensional Case. We shall briefly indicate the changes which must be made to deal with the "odd" degree case, consisting of a self-adjoint operator D on a trivially graded Hilbert space H.

The basic definition of the quantity $\langle\cdots\rangle_z$ is unchanged, except of course that now we set $\varepsilon = I$, and so we could omit ε from Equation (4.1). The formula

$$\Psi^p(a^0, \dots, a^p) = \langle a^0, [D, a^1], \dots, [D, a^p]\rangle_{s-\frac{p}{2}}$$

now defines an *odd*, improper cocycle, with values in the meromorphic functions on $\mathbb{C}$. The proof of this is almost the same as the proof of Theorem 5.5. We obtain the formula

$$
\begin{aligned}
(7.7)\quad B\Psi_{p+2}(a^0,\dots,a^{p+1}) &+ b\Psi_p(a^0,\dots,a^{p+1}) \\
&= \langle [D,a^0],[D,a^1],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}} \\
&\quad + \sum_{j=1}^{p+1}(-1)^{j-1}\langle a^0,[D,a^1],\dots,[D^2,a^j],\dots,[D,a^{p+1}]\rangle_{s-\frac{p}{2}},
\end{aligned}
$$

as in that proof, but instead of appealing to Lemma 5.3 we now note that

$$[D^2,a^j] = D[D,a^j] + [D,a^j]D$$

and

$$\langle \cdots,[D,a^{j-1}],D[D,a^j],\dots\rangle_z = \langle \cdots,[D,a^{j-1}]D,[D,a^j],\dots\rangle_z .$$

Using these relations the right hand side of Equation (7.7) telescopes to 0. The computation of $\mathrm{Res}_{s=0}\,\Psi$ is unchanged from the proof of Theorem 5.6, except for the omission of ε.

With similar modifications to the proofs of Lemmas 7.8 and 7.10 we obtain without difficulty the odd version of Theorem 7.12.

Appendix A. Comparison with the JLO Cocycle

In this appendix we shall use the residue theorem and the Mellin transform of complex analysis to compare the residue cocycle with the JLO cocycle.

The JLO cocycle, discovered by Jaffe, Lesniewski and Osterwalder [**20**], was developed in the context of spectral triples, as in Section 4.5, and accordingly we shall begin with such a spectral triple (A,H,D). Since we are going to compare the JLO cocycle with the residue cocycle we shall assume that (A,H,D) has the additional properties considered in Section 4.5 (although the theory of the JLO cocycle itself can be developed in greater generality). Thus we shall assume that our spectral triple is regular, is finitely summable, and has discrete dimension spectrum. We shall also make an additional assumption later on in this section.

We shall consider only the even, $\mathbb{Z}/2$-graded case here, but the odd case can be developed in exactly the same way.

A.1. Definition. If $X^0,\dots,X^p$ are bounded operators on H, and if $t>0$, let us define

$$\langle X^0,\dots,X^p\rangle_t^{\mathrm{JLO}} = t^{\frac{p}{2}}\,\mathrm{Trace}\left(\int_{\Sigma^p}\varepsilon X^0 e^{-u_0 t\Delta}\dots X^p e^{-u_p t\Delta}\,du\right)$$

The integral is over the standard p-simplex

$$\Sigma^p = \left\{\ (u_0,\dots,u_p)\,\middle|\, u_j\ge 0\ \&\ u_0+\dots+u_p=1\right\}.$$

The *JLO cocycle* is the improper (b,B)-cocycle

$$(a^0,\dots,a^p)\mapsto \langle a^0,[D,a^1],\dots,[D,a^p]\rangle_t^{\mathrm{JLO}},$$

which should be thought of here as a cocycle with coefficients in the space of functions of $t>0$.

Strictly speaking the "traditional" JLO cocycle is given by the above formula for the particular value $t=1$. Our formula for $\langle a^0,[D,a^1],\dots,[D,a^p]\rangle_t^{\mathrm{JLO}}$ corresponds to the traditional cocycle associated to the operator $t^{\frac{1}{2}}D$. It will be quite convenient to think of the JLO cocycle as a function of $t>0$.

Of course, it is a basic result that the JLO cocycle really *is* a cocycle. See [**20**] or [**17**].

The proper context for the JLO cocycle is Connes' entire cyclic cohomology [**5, 7**] for Banach algebras. We shall not describe this theory here, except to say that there is a natural map

$$HCP^*(A) \to HCP^*_{\text{entire}}(A),$$

and that the arguments which follow show that the image of the residue cocycle in entire cyclic cohomology is the JLO cocycle.[10]

The following formula (which is essentially due to Connes [**6**, Equation (17)]) exhibits the connection between the JLO cocycle and the cocycle that we constructed in Section 5.

A.2. Lemma. *If $p > 0$ and if $X^0, \ldots, X^p$ are generalized differential operators in $\mathcal{D}(A, D)$, then*

$$\langle X^0, \ldots, X^p \rangle_t^{\text{JLO}} = t^{-\frac{p}{2}} \frac{(-1)^p}{2\pi i} \operatorname{Trace}\left(\int e^{-t\lambda} \varepsilon X^0 (\lambda - \Delta)^{-1} X^1 \cdots X^p (\lambda - \Delta)^{-1} \, d\lambda \right).$$

As in Section 4.3 the contour integral should be evaluated along a (downward pointing) vertical line in the complex plane which separates 0 from the spectrum of Δ. The hypotheses guarantee the absolute convergence of the integral, in the norm-topology. The formula in the lemma is also correct for $p = 0$, but in this case the integral has to be suitably interpreted since it does not converge in the ordinary sense.

Proof of the Lemma. For simplicity let us assume that the operators X^j are bounded (this is the only case of the lemma that we shall use below).

By Cauchy's Theorem, we may replace the contour of integration along which the contour integral is computed by the imaginary axis in $\mathbb{C}$ (traversed upward). Having done so we obtain the formula

$$\text{(A.1)} \quad \frac{(-1)^p}{2\pi i} \int e^{-t\lambda} X^0 (\lambda - \Delta)^{-1} X^1 \cdots X^p (\lambda - \Delta)^{-1} \, d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itv} X^0 (iv + \Delta)^{-1} \cdots X^p (iv + \Delta)^{-1} \, dv.$$

Note that this has the appearance of an inverse Fourier transform. As for the JLO cocycle, if we define functions g^j from $\mathbb{R}$ into the bounded operators on H by

$$u \mapsto \begin{cases} X^j e^{-u\Delta} & \text{if } u \geq 0 \\ 0 & \text{if } u < 0. \end{cases}$$

then we obtain the formula

$$\text{(A.2)} \quad t^{\frac{p}{2}} \int_{\Sigma^p} X^0 e^{-u_0 t\Delta} \ldots X^p e^{-u_p t\Delta} \, du = t^{-\frac{p}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g^0(t - u_1) g^1(u_1 - u_2) \cdots g^{n-1}(u_{n-1} - u_n) g^n(u_n) \, du_1 \ldots du_n,$$

[10] This result can be improved somewhat. Entire cyclic cohomology is defined for locally convex algebras, and one can identify the JLO cocycle and the residue cocycle in the entire cyclic cohomology of various completions of A.

which has the form of a convolution product, evaluated at t.

Suppose now that $f^0, \ldots, f^n$ are Schwartz-class functions from $\mathbb{R}$ into a Banach algebra B. Define their Fourier transforms in the obvious way, by the formulas

$$\widehat{f^j}(v) = \int_{-\infty}^{\infty} e^{-iuv} f(u)\, du.$$

The just as in ordinary Fourier theory one has the formula

$$(f^0 \star \cdots \star f^n)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itv} \widehat{f^0}(v) \cdots \widehat{f^n}(v)\, du. \tag{A.3}$$

Returning to the case at hand, let $B = B(H)$ and let $f^j_\delta(u)$ be the convolution product of a C^∞, compactly supported bump function $\delta^{-1}\phi(\delta^{-1}x)$ with the function g^j. Applying the formula (A.3) to the functions f^j_δ and then taking the limit as $\delta \to 0$ we obtain the equality of (A.1) and (A.2), which proves the lemma. □

A.3. Lemma. *If $p \geq 0$ and $a^0, \ldots, a^p \in A$, then there is some $\alpha > 0$ such that*

$$\left| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle^{\mathrm{JLO}}_t \right| = O(e^{-\alpha t})$$

as $t \to \infty$. In addition if $k > \frac{d-p}{2}$ then

$$\left| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle^{\mathrm{JLO}}_t \right| = O\big(t^{-k}\big)$$

as $t \to 0$, □

Proof. See [**18**, Equation (10.47)] for the first relation, and [**18**, Equation (10.43)] for the second. □

The following proposition now shows that the improper cocycle which we considered in Section 5 is the Mellin transform of the JLO cocycle.

For the rest of this section let us fix a real number $k > \frac{d-p}{2}$.

A.4. Proposition. *If $p \geq 0$ and $a^0, \ldots, a^p \in A$, and if $\mathrm{Re}(s) > k$ then*

$$\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_{s-\frac{p}{2}} = \int_0^\infty \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle^{\mathrm{JLO}}_t t^s \, \frac{dt}{t}.$$

Proof. By Lemma A.3 the integral is absolutely convergent as long as $\mathrm{Re}(s) > k$. The identity follows from Lemma A.2 and the formula

$$\Gamma(z)\lambda^{-z} = \int_0^\infty e^{-t\lambda} t^z \, \frac{dt}{t}$$

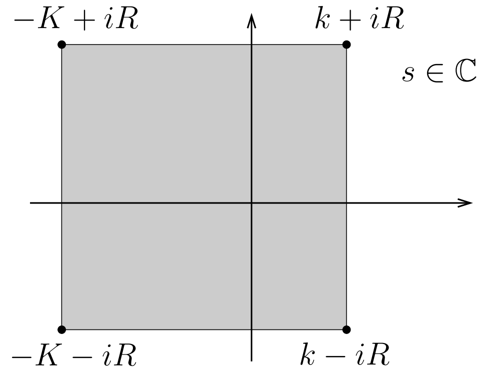
which is valid for all $\lambda > 0$ and for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. (In the case $p = 0$ Lemma A.2 does not apply, but then the proposition is a direct consequence of the displayed formula). □

Having established this basic relation, we are now going to apply the inversion formula for the Mellin transform to obtain an asymptotic formula for the JLO cocycle. In order to do so we shall need to make an additional analytic assumption, as follows: *the function $\langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z$ has only finitely many poles in each vertical strip $\alpha < \mathrm{Re}(z) < \beta$, and in each such strip and every N one has*

$$\Big| \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle_z \Big| = O(|z|^{-N})$$

as $|z| \to \infty$. Note that a similar assumption is made by Connes and Moscovici in [**12**].

Consider the rectangular contour in the complex plane which is indicated in the figure.



Here R is a large positive number (we shall take the limit as $R \to \infty$). The real numbers $k > \frac{d-p}{2}$ and K should be chosen so that there are no poles on the vertical lines $\mathrm{Re}(s) = k$ and $\mathrm{Re}(s) = -K$.

Let us integrate the function $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}$ around this contour.

The function $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}}$ is holomorphic in the region $\mathrm{Re}(s) > \frac{d_1-p}{2}$, and converges rapidly to zero along each vertical line there. Therefore, by the inversion formula for the Mellin transform,

$$\text{(A.4)} \qquad \lim_{R\to\infty} \frac{1}{2\pi i} \int_{k-iR}^{k+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, ds = \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_t^{\mathrm{JLO}}$$

(the contour integral is along the vertical line from $k - iR$ to $k + iR$).

Turning our attention to the left vertical side of the contour, we note that

$$\int_{-K-iR}^{-K+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, dz = t^K \int_{-R}^{-R} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{-K-\frac{p}{2}+ir} t^{-ir}\, dr$$

By hypothesis the quantity $\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{-K-\frac{p}{2}+ir}$ is an integrable function of $r \in \mathbb{R}$. Taking the limit as $R \to \infty$, the integral on the right-hand side (not including the term t^K) is the Fourier transform, evaluated at $\log(t)$, of an integrable function of r. It is therefore a bounded function of t. Hence

$$\text{(A.5)} \qquad \lim_{R\to\infty} \int_{-K-iR}^{-K+iR} \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s}\, ds = O(t^K)$$

as $t \to 0$.

Since the horizontal components of the contour contribute zero to the contour integral, in the limit as $R \to \infty$, it follows from the Residue Theorem that

$$\text{(A.6)} \qquad \langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_t^{\mathrm{JLO}} = \sum_{-K<\mathrm{Re}(w)<k} \mathrm{Res}_{s=w} \left(\langle a^0, [D,a^1], \ldots, [D,a^p]\rangle_{s-\frac{p}{2}} t^{-s} \right) + O(t^K)$$

as $t \to 0$.

A.5. THEOREM. *Assume that the spectral triple (A, H, D) is regular and has simple dimension spectrum. Then for every $p \geq 0$ and all $a^0, \dots a^p \in A$ the quantity $\langle a^0, [D, a^1], \dots, [D, a^p]\rangle_t^{\mathrm{JLO}}$ has an asymptotic expansion in powers of t as decreases to zero. The residue cocycle $\mathrm{Res}_{s=0}\,\Psi(a^0, \dots, a^p)$ is the coefficient of the constant term in this asymptotic expansion.* □

Appendix B. Complex Powers in a Differential Algebra

In this appendix we shall try to sketch out a more conceptual view of the improper cocycle which was constructed in Section 5. This involves Quillen's cochain picture of cyclic cohomology [**23**], and in fact it was Quillen's account of the JLO cocycle from this perspective which first led to the formula for the quantity $\langle X^0, \dots, X^p\rangle_z$ given in Definition 4.12.

We shall not attempt to carefully reconstruct the results of Sections 5 and 7 from the cochain perspective, and in fact for the sake of brevity we shall disregard analytic niceties altogether. Our purpose is only to set the main definitions of these notes against a background which may (or may not) make them seem more natural.

With this limited aim in mind we shall assume, as we did in the body of the notes, that the operator Δ is invertible. We shall also consider only the even case, in which the Hilbert space H on which Δ acts is $\mathbb{Z}/2$-graded.

As we did when we looked at cyclic cohomology in Section 2, let us fix an algebra A. But let us now also fix a second algebra L. For $n \geq 0$ denote by $\mathrm{Hom}^n(A, L)$ the vector space of n-linear maps from A to L. By a 0-linear map from A to L we shall mean a linear map from $\mathbb{C}$ to L, or in other words just an element of L. Let $\mathrm{Hom}^{**}(A, L)$ be the direct product

$$\mathrm{Hom}^{**}(A, L) = \prod_{n=0}^{\infty} \mathrm{Hom}^n(A, L).$$

Thus an element ϕ of $\mathrm{Hom}^{**}(A, L)$ is a sequence of multi-linear maps from A to L. We shall denote by $\phi(a^1, \dots, a^n)$ the value of the n-th component of ϕ on the n-tuple $(a^1, \dots, a^n)$.

The vector space $\mathrm{Hom}^{**}(A, L)$ is $\mathbb{Z}/2$-graded in the following way: an element ϕ is even (resp. odd) if $\phi(a^1, \dots, a^n) = 0$ for all odd n (resp. for all even n). We shall denote by $\deg_M(\phi) \in \{0, 1\}$ the grading-degree of ϕ. (The letter "M" stands for "multi-linear;" a second grading-degree will be introduced below.)

B.1. LEMMA. *If $\phi, \psi \in \mathrm{Hom}^{**}(A, L)$, then define*

$$\phi \vee \psi(a^1, \dots, a^n) = \sum_{p+q=n} \phi(a^1, \dots, a^p)\psi(a^{p+1}, \dots, a^n)$$

and

$$d_M\phi(a^1, \dots, a^{n+1}) = \sum_{i=1}^{n} (-1)^{i+1}\phi(a^1, \dots, a^i a^{i+1}, \dots, a^{n+1}).$$

The vector space $\mathrm{Hom}^{**}(A, L)$, so equipped with a multiplication and differential, is a $\mathbb{Z}/2$-graded differential algebra. □

Let us now suppose that the algebra L is $\mathbb{Z}/2$-graded. If $\phi \in \mathrm{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 0$ if $\phi(a^1, \dots, a^n)$ belongs to the degree-zero part of L for every n and every n-tuple $(a^1, \dots, a^n)$. Similarly, if $\phi \in \mathrm{Hom}^{**}(A, L)$ then let us write $\deg_L(\phi) = 1$ if $\phi(a^1, \dots, a^n)$ belongs to the degree-one part of L for every n and every n-tuple $(a^1, \dots, a^n)$. This is a new $\mathbb{Z}/2$-grading on the vector space $\mathrm{Hom}^{**}(A, L)$. The formula

$$\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$$

defines a third $\mathbb{Z}/2$-grading—the one we are really interested in. Using this last $\mathbb{Z}/2$-grading, we have the following result:

B.2. Lemma. *If $\phi, \psi \in \mathrm{Hom}^{**}(A, L)$, then define*

$$\phi \diamond \psi = (-1)^{\deg_M(\phi)\deg_L(\psi)} \phi \vee \psi$$

and

$$d\phi = (-1)^{\deg_L(\phi)} d'\phi$$

These new operations once again provide $\mathrm{Hom}^{**}(A, L)$ with the structure of a $\mathbb{Z}/2$-graded differential algebra (for the total $\mathbb{Z}/2$-grading $\deg(\phi) = \deg_M(\phi) + \deg_L(\phi)$). □

We shall now specialize to the following situation: A will be, as in Section 5, an algebra of differential order zero and grading degree zero operators contained within an algebra $\mathcal{D}(\Delta)$ of generalized differential operators, and L will be the algebra of all operators on the $\mathbb{Z}/2$-graded vector space $H^\infty \subseteq H$.

Denote by ρ the inclusion of A into L. This is of course a 1-linear map from A to L, and we can therefore think of ρ as an element of $\mathrm{Hom}^{**}(A, L)$ (all of whose n-linear components are zero, except for $n = 1$).

Denote by D a square root of Δ, as in Section 4.4. Think of D as a 0-linear map from A to L, and therefore as an element of $\mathrm{Hom}^{**}(A, L)$ too. Combining D and ρ let us define the "superconnection form"

$$\theta = D - \rho \in \mathrm{Hom}^{**}(A, L)$$

This has odd $\mathbb{Z}/2$-grading degree (that is, $\deg(\theta) = 1$). Let K be its "curvature:"

$$K = d\theta + \theta^2,$$

which has even $\mathbb{Z}/2$-grading degree. Using the formulas in Lemma B.2 the element K may be calculated, as follows:

B.3. Lemma. *One has*

$$K = \Delta - E \in \mathrm{Hom}^{**}(A, L),$$

where $E\colon A \to L$ is the 1-linear map defined by the formula

$$E(a) = [D, \rho(a)].$$ □

In all of the above we are following Quillen, who then proceeds to make the following definition, which is motivated by the well-known Banach algebra formula

$$e^{b-a} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 a} b e^{-t_1 a} \cdots b e^{-t_n a}\, dt.$$

B.4. Definition. Denote by $e^{-K} \in \mathrm{Hom}^{**}(A, L)$ the element

$$e^{-K} = \sum_{n=0}^{\infty} \int_{\Sigma^n} e^{-t_0 \Delta} E e^{-t_1 \Delta} \dots E e^{-t_n \Delta}\, dt.$$

The n-th term in the sum is an n-linear map from A to L, and the series should be regarded as defining an element of $\mathrm{Hom}^{**}(A, L)$ whose n-linear component is this term. As Quillen observes, in [**23**, Section 8] the exponential e^{-K} defined in this way has the following two crucial properties:

B.5. Lemma (Bianchi Identity). $d(e^{-K}) + [e^{-K}, \theta] = 0$. □

B.6. Lemma (Differential Equation). *Suppose that* δ *is a derivation of* $\mathrm{Hom}^{**}(A, L)$ *into a bimodule. Then*

$$\delta(e^{-K}) = -\delta(K)e^{-K},$$

modulo (limits of) commutators. □

Both lemmas follow from the "Duhamel formula"

$$\delta(e^{-K}) = \int_0^1 e^{-tK}\delta(K)e^{-(1-t)K}\,dt,$$

which is familiar from semigroup theory and which may be verified for the notion of exponential now being considered. (Once more, we remind the reader that we are disregarding analytic details.)

Suppose we now introduce the "supertrace" $\mathrm{Trace}_\varepsilon(X) = \mathrm{Trace}(\varepsilon X)$ (which is of course defined only on a subalgebra of L). Quillen reinterprets the Bianchi Identity and the Differential Equation above as coboundary computations in a complex which computes periodic cyclic cohomology (using improper cocycles, in our terminology here). As a result he is able to recover the following basic fact about the JLO cocycle — namely that it really is a cocycle:

B.7. Theorem (Quillen). *The formula*

$$\Phi_{2n}(a^0, \ldots, a^{2n}) = \int_{\Sigma^n} \mathrm{Trace}\left(\varepsilon a^0 e^{-t_0\Delta}[D, a^1]e^{-t_1\Delta}[D, a^2]\ldots[D, a^n]e^{-t_n\Delta}\right)\,dt$$

defines a (b, B)*-cocycle.* □

The details of the argument are not important here. What *is* important is that using the Bianchi Identity and a Differential Equation one can construct cocycles for cyclic cohomology from elements of the algebra $\mathrm{Hom}^{**}(A, L)$. With this in mind, let us consider other functions of the curvature operator K, beginning with resolvents.

B.8. Lemma. *If* $\lambda \notin \mathrm{Spectrum}(\Delta)$ *then the element* $(\lambda - K) \in \mathrm{Hom}^{**}(A, L)$ *is invertible.*

Proof. Since $(\lambda - K) = (\lambda - \Delta) + E$ we can write

$$\begin{aligned}(\lambda - K)^{-1} &= (\lambda - \Delta)^{-1} - (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} \\ &\quad + (\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1}E(\lambda - \Delta)^{-1} - \cdots\end{aligned}$$

This is a series whose nth term is an n-linear map from A to L, and so the sum has an obvious meaning in $\mathrm{Hom}^{**}(A, L)$. One can then check that the sum defines $(\lambda - K)^{-1}$, as required. □

With resolvents in hand, we can construct other functions of K using formulas modeled on the holomorphic functional calculus.

B.9. Definition. For any complex z with positive real part define $K^{-z} \in \operatorname{Hom}^{**}(A, L)$ by the formula

$$K^{-z} = \frac{1}{2\pi i} \int \lambda^{-z}(\lambda - K)^{-1} \, d\lambda,$$

in which the integral is a contour integral along a downward vertical line in $\mathbb{C}$ separating 0 from Spectrum(Δ).

The assumption that $\operatorname{Re}(z) > 0$ guarantees convergence of the integral (in each component within $\operatorname{Hom}^{**}(A, L)$ the integral converges in the pointwise norm topology of n-linear maps from A to the algebra of bounded operators on H; the limit is also an operator from H^∞ to H^∞, as required). The complex powers K^{-z} so defined satisfy the following key identities:

B.10. Lemma (Bianchi Identity). $d(K^{-z}) + [K^{-z}, \theta] = 0.$ □

B.11. Lemma (Differential Equation). *If δ is a derivation of* $\operatorname{Hom}^{**}(A, L)$ *into a bimodule, then*

$$\delta(K^{-z}) = -z\delta(K)K^{-z-1},$$

modulo (limits of) commutators. □

These follow from the derivation formula

$$\delta(K^{-z}) = \frac{1}{2\pi i} \int \lambda^{-z}(\lambda - K)^{-1}\delta(K)(\lambda - K)^{-1} \, d\lambda.$$

In order to simplify the Differential Equation it is convenient to introduce the Gamma function, using which we can write

$$\delta\Big(\Gamma(z)K^{-z}\Big) = -\delta(K)\Gamma(z+1)K^{-(z+1)}$$

(modulo limits of commutators, as before). Except for the appearance of $z+1$ in place of z in the right hand side of the equation, this is exactly the same as the differential equation for e^{-K}. Meanwhile even after introducing the Gamma function we still have available the Bianchi identity:

$$d\Big(\Gamma(z)K^{-z}\Big) + \Big[\Gamma(z)K^{-z}, \theta\Big] = 0.$$

The degree n component of $\Gamma(z)K^{-z}$ is the multi-linear function

$$(a^1, \ldots, a^n) \mapsto \frac{(-1)^n}{2\pi i}\Gamma(z) \int \lambda^{-z}(\lambda - \Delta)^{-1}[D, a^1] \ldots [D, a^n](\lambda - \Delta)^{-1} \, d\lambda,$$

Quillen's approach to JLO therefore suggests (and in fact upon closer inspection proves) the following result:

B.12. Theorem. *If we define*

$$\Psi_p^s(a^0, \ldots, a^p) = \frac{(-1)^p\Gamma(s - \frac{p}{2})}{2\pi i} \operatorname{Trace}\Big(\int \lambda^{\frac{p}{2}-s}\varepsilon a^0(\lambda - \Delta)^{-1}[D, a^1] \ldots [D, a^p](\lambda - \Delta)^{-1} \, d\lambda \Big),$$

then $b\Psi_p^s + B\Psi_{p+2}^s = 0.$ □

This is of course precisely the conclusion that we reached in Section 5.

Appendix C. Proof of the Hochschild Character Theorem

In this final appendix we shall prove Connes' Hochschild character theorem by appealing to some of the computations that we made in Section 7.

C.1. Definition. A *Hochschild n-cycle* over an algebra A is an element of the $(n+1)$-fold tensor product $A \otimes \cdots \otimes A$ which is mapped to zero by the differential

$$b(a^0 \otimes \cdots a^n) = \sum_{j=0}^{n-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots a^n \\ + (-1)^n a^n a^0 \otimes a^1 \otimes \cdots \otimes a^{n-1}.$$

C.2. Remark. Obviously, two Hochschild n-cochains which differ by a Hochschild coboundary will agree when evaluated on any Hochschild cycle. The converse is not quite true.

C.3. Theorem. *Let (A, H, D) be a regular spectral triple. Assume that D is invertible and that for some positive integer n of the same parity as the triple, and every $a \in A$,*

$$a \cdot |D|^{-n} \in \mathcal{L}^{1,\infty}(H).$$

The Chern character ch_n^F of Definition 2.22 and the cochain

$$\Phi(a^0, \ldots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!} \operatorname{Trace}_\omega(\varepsilon a^0 [D, a^1][D, a^2] \cdots [D, a^n] |D|^{-n})$$

are equal when evaluated on any Hochschild cycle $\sum_i a_i^0 \otimes \cdots \otimes a_i^n$. Here ε is 1 in the odd case, and the grading operator on H in the even case.

Proof. We showed in Lemma 7.8 that

$$b\Theta_{n-1}(a^0, \ldots, a^n) + B\Theta_{n+1}(a^0, \ldots, a^n) = 2s\Psi_n(a^0, \ldots, a^n),$$

at least for all s whose real part is large enough that all the terms are defined (since we are no longer assuming any sort of analytic continuation property this is an issue now). It follows that $B\Theta_{n+1}$ and $2s\Psi_n$ agree on any Hochschild cycle. Now, it is not hard to compute that $2s\Psi_n$ is defined when $\mathrm{Re}(s) > 0$, and

$$\lim_{s \to 0} 2s\Psi_n(a^0, \ldots, a^n) = \frac{\Gamma(\frac{n}{2}+1)}{n \cdot n!} \operatorname{Trace}_\omega(\varepsilon a^0 [D, a^1][D, a^2] \cdots [D, a^n] |D|^{-n}).$$

On the other hand $B\Theta_{n+1}(a^0, \ldots, a^n)$ is defined when $\mathrm{Re}(s) \geq 0$ (compare Remark 7.2). Since the computations in Section 7.2 show that $B\Theta_{n+1}$ is cohomologous, even as a cyclic cocycle, to the Chern character ch_n^F, the theorem is proved. □

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Department of Mathematics, Penn State University, University Park, PA 16802
E-mail address: higson@psu.edu
URL: http://www.math.psu.edu/higson