On the K-Theory Proof of the Index Theorem

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1. Introduction

This paper is an exposition of the K-theory proof of the Atiyah-Singer Index Theorem. I have tried to separate, as much as possible, the analytic parts of the proof from the topological calculations. For the topology I have taken advantage of the Chern isomorphism to work mostly within the world of ordinary cohomology. The analytic part of the proof is done within the framework of asymptotic morphisms [6] [7]. Depending on the reader’s outlook this may or may not be simpler than the usual approach through pseudodifferential operators.

The approach we take is due, more or less, to Kasparov [12]. It differs a little from the argument in [2] and has the useful feature that embeddings into Euclidean space are not required. This will be used in the article [4] which deals with the equivariant index theorem for manifolds equipped with proper actions of discrete groups.

See [8] for another K-theoretic proof of the index theorem, based on ideas of P. Baum.

2. Elliptic Operators

Let $M$ be a smooth closed manifold, let $E$ and $F$ be smooth complex vector bundles over $M$, and let

$$D : C^\infty(M, E) \to C^\infty(M, F)$$

be a linear elliptic operator on $M$, mapping sections of $E$ to sections of $F$. For simplicity assume that $D$ is a differential—as opposed to pseudodifferential—operator, and that it has order one. So choosing local coordinates on $M$, along

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with local frames for $E$ and $F$, the operator $D$ is of the form

$$D = \sum a_i \partial/\partial x_i + b,$$

where $a_i: E \to F$, $b: E \to F$ are smooth matrix valued functions.

The symbol of $D$ is the function $\sigma$ which associates to each cotangent vector $\xi \in T^*M_p$ a linear transformation $E_p \to F_p$, according to the formula

$$\sigma(p, \xi) = \sqrt{-1} \sum \xi_i a_i(p) \quad (\xi = \sum \xi_i dx_i).$$

It does not depend on the choice of local coordinates. The definition of ellipticity asserts that if $\xi \neq 0$ then the linear transformation $\sigma(p, \xi): E_p \to F_p$ is invertible.

The rudiments of the theory of elliptic operators imply that the kernel and cokernel of $D$ are finite dimensional complex vector spaces, and our objective is to calculate the quantity

$$\text{Index}(D) = \dim_{\mathbb{C}}(\text{kernel } D) - \dim_{\mathbb{C}}(\text{cokernel } D) \in \mathbb{Z}$$

in terms of the symbol of $D$ and the algebraic topology of $M$. See [15].

3. $K$-Theory

We review a few facts about the $K$-theory of $C^*$-algebras. See [5] and [7] for details. In fact we shall scarcely go beyond the $K$-theory of commutative $C^*$-algebras, which amounts to the same thing as topological $K$-theory [1], but for one or two constructions it is convenient to adopt the $C^*$-algebra point of view.

Let $A$ be a $C^*$-algebra. Recall that if $A$ has a unit then $K(A)$ is the abelian group generated by homotopy classes of projections in matrix algebras over $A$, subject to the relation that addition of disjoint projections correspond to addition in $K(A)$.

A homomorphism $A \to B$ between $C^*$-algebras with unit determines a homomorphism of abelian groups $K(A) \to K(B)$, making $K(A)$ into a covariant functor.

If $A$ does not have a unit then we define $K(A)$ by adjoining a unit to $A$, so as to obtain a $C^*$-algebra $A^+$, and setting

$$K(A) = \text{kernel}\{K(A^+) \to K(A^+/A)\}.$$ 

Since any homomorphism of $C^*$-algebras $A \to B$ extends to a homomorphism $A^+ \to B^+$ we obtain a covariant functor on the category of all $C^*$-algebras and all $C^*$-algebra homomorphisms.
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Definition. Let $A$ and $B$ be $C^*$-algebras. An asymptotic morphism from $A$ to $B$ is a family of functions $T^\omega: A \to B$ ($\omega \in [1, \infty)$) such that

1. $T^\omega(a)$ is jointly continuous in $a$ and $\omega$;
2. $\limsup_{\omega \to \infty} \|T^\omega(a)\| < \infty$ for every $a \in A$; and
3. we have

\[
\lim_{\omega \to \infty} \|T^\omega(a) + \lambda T^\omega(a') - T^\omega(a + \lambda a')\| = 0,
\]
\[
\lim_{\omega \to \infty} \|T^\omega(a^*) - T^\omega(a)^*\| = 0,
\]
\[
\lim_{\omega \to \infty} \|T^\omega(a)T^\omega(a') - T^\omega(aa')\| = 0,
\]

and the convergence is uniform on compact subsets of $A$.

This differs a little from the definition in [6,7], but not in any essential way. We remark that condition (2) is in fact a consequence of conditions (1) and (3).

An asymptotic morphism $T^\omega: A \to B$ determines a homomorphism of $K$-theory groups

\[ T: K(A) \to K(B), \]
as follows. Suppose first that $A$ has a unit. Let $p$ be a projection in $A$, or in a matrix algebra over $A$ (in which case we note that $T^\omega$ applied entrywise gives an asymptotic morphism from matrices over $A$ to matrices over $B$). Consider the continuous family $T^\omega(p)$ of elements in $B$ (or in a matrix algebra over $B$). It is uniformly bounded, and

\[ \|T^\omega(p) - T^\omega(p)^2\| \to 0, \]
as $\omega \to \infty$, so that $T^\omega(p)$ is “asymptotically” a projection. It follows easily from the functional calculus that there is a continuous family of projections $q^\omega$ in $B$ such that

\[ \|T^\omega(p) - q^\omega\| \to 0 \]
as $\omega \to \infty$. We define

\[ T[p] = [q^1]. \]

If $A$ does not have a unit then note that $T^\omega$ extends to an asymptotic morphism $T^\omega: A^+ \to B^+$ (mapping one adjoined unit to the other). We obtain a map $K(A^+) \to K(B^+)$ which restricts to a map from $K(A)$ into $K(B)$, as required.

Let $X$ be a compact Hausdorff space. As usual, denote by $C(X)$ the continuous, complex valued functions on $X$. The group $K(C(X))$ has the structure of a commutative ring, for if $p \in M_n(C(X))$ and $q \in M_{n'}(C(X))$ are projections then we may form

\[ p \otimes q(x) = p(x) \otimes q(x) \in M_{nn'}(C(X)) \]
(here we view matrices of functions on $X$ as matrix valued functions on $X$). The multiplicative unit of $C(X)$ determines a unit

$$1 = [1] \in K(C(X)).$$

Denote by $A(X)$ the $C^*$-algebra of continuous functions from $X$ into a $C^*$-algebra $A$. Then the group $K(A(X))$ is a module over $K(X)$. If $A$ has a unit the module structure is defined by a formula like (3.1). If $A$ has no unit we observe that

$$K(A(X)) \cong \ker\{K(A^+(X)) \to K(A^+/A(X))\},$$

and reduce to the unital case.

An asymptotic morphism $T^\omega: A \to B$ extends in the obvious way to an asymptotic morphism

$$T^\omega_X: A(X) \to B(X),$$

and so we obtain homomorphisms of $K$-theory groups

$$T_X: K(A(X)) \to K(B(X)).$$

**Lemma 3.1.** The maps $T_X$ are $K(C(X))$-module homomorphisms. In addition, if $f: X' \to X$ is any continuous map then the diagram

\[
\begin{array}{ccc}
K(A(X)) & \xrightarrow{T_X} & K(B(X)) \\
\downarrow f^* & & \downarrow f^* \\
K(A(X')) & \xrightarrow{T_X'} & K(B(X'))
\end{array}
\]

commutes. □

Let $K$ denote the $C^*$-algebra of compact operators on a separable Hilbert space. Fix a rank one projection $e$ in $K$, and map $C(X)$ into $K(X)$ by sending a function $f$ to the function $x \mapsto f(x)e$.

**Lemma 3.2.** The induced map

$$K(C(X)) \to K(K(X))$$

(which is a $K(C(X))$-module homomorphism) is an isomorphism. □

Let $Y$ be a locally compact space and let $C_0(Y)$ be the $C^*$-algebra of continuous complex valued functions on $Y$ which vanish at infinity.

For the rest of the paper we shall write $K(Y)$ in place of $K(C_0(Y))$.

Note that $C_0(Y)^+ = C(Y^+)$, where $Y^+$ denotes the one point compactification of $Y$. Thus if $p$ and $q$ are projection valued functions on $Y^+$, which are equal at infinity, then the difference $[p] - [q]$ is an element of $K(Y)$.

Note also that the algebra of continuous functions from $X$ into $C_0(Y)$ is equal to $C_0(X \times Y)$. 
Proposition 3.3. An asymptotic morphism \( T^\omega : C_0(Y) \to K \) determines a family of \( K(X) \)-module maps
\[
T_X : K(X \times Y) \to K(X),
\]
which are natural in \( X \) as in Lemma 3.1. □

4. The Symbol Class

We shall define two sorts of \( K \)-theory classes, the first associated to an elliptic operator on a manifold, and the second associated to the manifold itself.

Let \( M \) be a smooth, closed manifold and let
\[
D : C^\infty(M, E) \to C^\infty(M, F)
\]
be an elliptic operator with symbol
\[
\sigma : \pi^*E \to \pi^*F
\]
(\( \pi \) is the projection from the cotangent bundle \( T^*M \) to \( M \)). Endow the \( E \) and \( F \) with metrics and form the self-adjoint endomorphism
\[
\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix} : \pi^*E \oplus \pi^*F \to \pi^*E \oplus \pi^*F.
\]

Lemma 4.1. The resolvent operators
\[
(\sigma \pm i)^{-1} : \pi^*E \oplus \pi^*F \to \pi^*E \oplus \pi^*F
\]
are endomorphisms which vanish at infinity (in the operator norm induced from the metrics on \( E \) and \( F \)).

Proof. Ellipticity implies that \( \sigma \) is bounded below on the complement of any neighbourhood of the zero section in \( T^*M \). Using the homogeneity \( \sigma(x, t\xi) = t\sigma(x, \xi) \) we see that for any \( C > 0 \) there is a compact subset of \( T^*M \) outside of which \( \sigma \) is bounded below by \( C \). The lemma follows from this. □

Now form the Cayley transform
\[
\sigma = (\sigma + i)(\sigma - i)^{-1} = 1 + 2i(\sigma - i)^{-1}.
\]

Embed \( E \) and \( F \) into trivial bundles \( \mathbb{C}^{N_1} \) and \( \mathbb{C}^{N_2} \) over \( M \), and extend the automorphism \( \epsilon \) of \( \pi^*E \oplus \pi^*F \) to the trivial bundle \( \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2} \) over \( T^*M \) by setting it equal to the identity on the complement of \( \pi^*E \oplus \pi^*F \). By Lemma 4.1 \( \epsilon \) extends continuously to \( (T^*M)^+ \) upon setting \( \epsilon(\infty) = I \).

Let
\[
\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
viewed as an automorphism of the trivial bundle $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$ over $(T^*M)^+$. Then of course $\epsilon^2 = 1$, but in addition
\[(u\epsilon)^2 = 1.\]
This is a simple consequence of the fact that $\epsilon$ anticommutes with the endomorphism $\sigma$.

Now to each involution $w$ (meaning $w^2 = 1$) there is associated a projection $p(w)$ (meaning $p(w)^2 = p(w)$) according to the formula
\[p(w) = \frac{1}{2}(w + 1).\]
In the case at hand we obtain two projection valued functions $p(\epsilon)$ and $p(\epsilon u)$ on $(T^*M)^+$ which are equal at infinity. Each defines an element of $K((T^*M)^+)$ and their difference defines an element
\[\sigma_D = [p(\epsilon)] - [p(\epsilon u)] \in K(T^*M).\]
This is the symbol class of the elliptic operator $D$. (Our construction of it, using the Cayley transform, is taken from [16].)

Now let $V$ be a Euclidean vector bundle over a compact space $X$. We shall define a class
\[\lambda_V \in K(V \oplus V),\]
and from it, using the tangent bundle, we shall obtain a class
\[\lambda_M \in K(M \times T^*M).\]
Form the complexified exterior algebra bundle $\bigwedge^* \mathbb{C} V$ and for $w \in V_x \otimes \mathbb{C}$ define
\[c(w): \bigwedge^*_\mathbb{C} V_x \to \bigwedge^*_\mathbb{C} V_x \]
\[c(w)\eta = d_w\eta - \delta_w\eta,\]
where $d_w$ denotes the operator of exterior multiplication by $w$, and $\delta_w$ denotes its adjoint. Define an endomorphism
\[c: \pi^*(\bigwedge^*_\mathbb{C} V) \to \pi^*(\bigwedge^*_\mathbb{C} V)\]
of the vector bundle $\bigwedge^*_\mathbb{C} V$ pulled back to the space $V \oplus V$ by the formula
\[c(v, v') = c(v) + c(\sqrt{-1}v').\]
It is self adjoint and
\[c(v, v')^2 = \|v\|^2 + \|v'\|^2,\]
so that the resolvents $(c \pm i)^{-1}$ vanish at infinity in $V \oplus V$. In addition, $c$ anticommutes with the “grading operator” $\epsilon$ which multiplies a form by $\pm 1$ according as the form is even or odd. Because of this we can follow the same
procedure as above to define a $K$-theory class in $K(V \oplus V)$: we form the Cayley transform $v$ of $c$, and then define
\[ \lambda_V = [p(e)] - [p(e v)] \in K(V \oplus V). \]

If $V$ is a vector space (= vector bundle over a point) then $\lambda_V$ is the “Bott element” familiar from the Periodicity Theorem.

Endow the smooth, closed manifold $M$ with a Riemannian metric and define a map
\[
\phi: TM \to M, \\
\phi(v) = \exp \left( \frac{\delta}{1 + \|v\|^2} v \right).
\]
Here we use the exponential map from differential geometry, and $\delta > 0$ is chosen to be small enough so that the associated map
\[ v \mapsto (\pi(v), \phi(v)) \]
is a diffeomorphism from $TM$ onto an open subset of $M \times M$ (see for example [14]). Define a diffeomorphism from $TM \oplus T^*M$ onto an open subset of $M \times T^*M$ as follows. For $m \in M$ the fibre of $TM \oplus T^*M$ over $m$ may be identified with the cotangent bundle of $TM_m$. The map $\phi$ is a diffeomorphism from $TM_m$ to an open subset of $W_m \subset M$, and so the transpose of the derivative of $\phi^{-1}$ is a diffeomorphism
\[ \tilde{\phi}: T^*(TM_m) \to T^*W_m \subset T^*M. \]
We define
\[
(4.2) \\
TM \oplus T^*M \to M \times T^*M \\
(v, \xi) \mapsto (\pi(v), \tilde{\phi}(v, \xi)).
\]
Identifying $T^*M$ and $TM$ using the metric, we define $\lambda_M \in K(M \times T^*M)$ to be the image of $\lambda_{TM} \in K(TM \oplus TM)$ under the map on $K$-theory groups induced from (4.2).

5. The Analytic Index

The $K$-theory proof of the Index Theorem is based on the following result of Atiyah and Singer.

**Theorem 5.1.** (Atiyah and Singer [4]) There are maps
\[ \text{Ind}_X: K(X \times T^*M) \to K(X) \]
for each compact space $X$ such that:

(1) $\text{Ind}_X$ is a $K(X)$-module homomorphism;
(2) Ind is a natural transformation, in the sense that for every continuous
map \( f: X' \to X \) the diagram
\[
\begin{array}{ccc}
K(X \times T^*M) & \xrightarrow{\text{Ind}_X} & K(X) \\
\downarrow f^* & & \downarrow f^* \\
K(X' \times T^*M) & \xrightarrow{\text{Ind}_{X'}} & K(X')
\end{array}
\]
commutes;
(3) if \( D \) is an elliptic operator on \( M \) then
\[
\text{Ind}_{pt}(\sigma_D) = \text{Index}(D)
\]
in \( K(\text{pt}) \cong \mathbb{Z} \); and
(4) \( \text{Ind}_M(\lambda_M) = 1 \in K(M) \).

We shall prove this by constructing in Section 8 an appropriate asymptotic
morphism from \( C_0(T^*M) \) into \( K(L^2(M)) \) and applying the remarks made in
Section 3. The verification of parts (3) and (4) will be done in Section 9.

6. Chern Character and Cohomology

Let \( Y \) be a locally compact space. Denote by \( H^*(Y) \) the direct sum of the
cohomology groups of \( Y \) with real coefficients and compact supports. Denote by
\( H^{ev}(Y) \) the direct sum of the even cohomology groups with real coefficients and
compact supports.

For our purposes \( Y \) will always be a reasonable space, in fact a smooth man-
ifold, so it is not necessary to specify a choice of cohomology theory.

Let \( X \) be a compact space. The cup product in cohomology makes \( H^*(X) \)
into a graded commutative ring, and \( H^{ev}(X) \) is a subring. A continuous map
\( f: Y \to X \) provides \( H^*(Y) \) with the structure of an \( H^*(X) \)-module. (If we are
working with de Rham theory and if \( f \) is smooth then the module structure is
given by pulling back forms from \( X \) to \( Y \) and taking wedge product.) We shall
use the cup product symbol \( a \smile b \) for the module action. It will be convenient
to work with both left and right modules.

There is a Chern character homomorphism
\[
\text{ch}: K(Y) \to H^{ev}(Y)
\]
(see [11]). It is a natural transformation which is multiplicative with respect to
the ring and module structures on \( K \)-theory and cohomology described above
and in Section 3.

As a consequence of the Bott Periodicity Theorem we have:
Chern Isomorphism Theorem. The map
\[ \text{ch} \otimes \text{id}_R : K(Y) \otimes \mathbb{R} \to H^{ev}(Y) \]
is an isomorphism. □

7. Poincaré Duality and the Index Theorem

In this section we shall use Theorem 5.1 and the Chern isomorphism to obtain the Atiyah-Singer Index Theorem.

Given a smooth closed manifold \( M \), orient the manifold \( T^* M \) as follows. Choose local coordinates \( x_1, \ldots, x_n \) on \( M \). Define functions \( y_1, \ldots, y_n \) on \( T^* M \) by
\[ y_i(\xi) = \langle \xi, \partial/\partial x_i \rangle \]
(the angle brackets denote the pairing between cotangent and tangent vectors). Then we deem \( x_1, y_1, x_2, y_2, \ldots, x_n, y_n \) to be an oriented system of local coordinates on \( T^* M \).

The orientation gives a linear functional
\[ (7.1) \quad p^*: H^*(T^* M) \to \mathbb{R} \]
in de Rham theory, take the degree \( 2n \) component of an element in \( H^*(T^* M) \), represent it as a compactly supported \( 2n \)-form and integrate it over \( T^* M \).

The projection \( \pi: T^* M \to M \) gives \( H^*(T^* M) \) the structure of an \( H^*(M) \)-module. For bookkeeping purposes take it to be a right module.

Poincaré Duality Theorem. The pairing
\[ b \otimes a \mapsto p^*(b \circ a) \]
from \( H^p(T^* M) \otimes H^{2n-p}(M) \) into \( \mathbb{R} \) induces an isomorphism from \( H^p(M) \) to the dual space of \( H^{2n-p}(T^* M) \). □

This simple version of Poincaré Duality is easily proved using a Mayer-Vietoris argument, as is the following result.

Kunneth Formula. View \( H^*(X \times T^* M) \) as a left \( H^*(X) \) module via the projection \( p \) of \( X \times T^* M \) onto \( X \). Denote by \( q: X \times T^* M \to T^* M \) the other projection. Then the map \( x \otimes y \mapsto x \circ q^*(y) \) is an isomorphism from \( H^*(X) \otimes H^*(T^* M) \) to \( H^*(X \times T^* M) \). □

In view of the Kunneth Formula, the recipe
\[ p^*(x \circ q^*y) = x \cdot p^*y, \]
where \( x \in H^*(X) \) and \( y \in H^*(T^* M) \), extends (7.1) above, giving maps
\[ p^*: H^*(X \times T^* M) \to H^*(X). \]
They are \( H^*(X) \)-module homomorphisms, functorial in \( X \).
These preliminaries dispensed with, we turn to an analysis of the maps

\[ \text{Ind}_X : K(X \times T^*M) \to K(X) \]

of Theorem 5.1. By the Chern isomorphism Theorem, there are homomorphisms

\[ I_X^{ev} : H^{ev}(X \times T^*M) \to H^{ev}(X) \]

such that the diagrams

\[
\begin{array}{ccc}
K(X \times T^*M) & \xrightarrow{\text{Ind}_X} & K(X) \\
\downarrow \text{ch} & & \downarrow \text{ch} \\
H^{ev}(X \times T^*M) & \xrightarrow{I_X^{ev}} & H^{ev}(X)
\end{array}
\]

commute. They are \( H^{ev}(X) \)-module homomorphisms, functorial with respect to maps \( X' \to X \).

Replacing \( X \) with \( X \times S^1 \), it is easily checked that the \( I_X^{ev} \) extend to maps

\[ I_X : H^*(X \times T^*M) \to H^*(X) \]

which are functorial \( H^*(X) \)-module homomorphisms. We shall work with these below.

**Lemma 7.1.** View \( H^*(X \times T^*M) \) as a right \( H^*(M) \) module via the projection map

\[ X \times T^*M \to T^*M \to M. \]

There is a cohomology class \( a_M \in H^*(M) \) such that

\[ I_X(x) = p_*(x \cup a_M), \]

for every \( x \in H^*(X \times T^*M) \). Thus if \( D \) is an elliptic operator on \( M \) then

\[ \text{Index}(D) = p_*(\text{ch}(\sigma_D) \cup a_M). \]

**Proof.** Poincaré duality asserts that \( I_{pt} \) is given by multiplication with some element \( a_M \) of \( H^*(M) \), followed by evaluation against the fundamental class. The formula for \( I_X \) follows from this in view of the Kunneth formula and the fact that \( I_X \) is natural and an \( H^*(X) \)-module homomorphism. \( \square \)

We calculate \( a_M \) as follows. Observe that \( H^*(M \times T^*M) \) is both a left \( H^*(M) \)-module, via the projection of \( M \times T^*M \) onto the first factor, and a right \( H^*(M) \)-module, via the projection of \( M \times T^*M \) onto \( M \) through the second factor.
Lemma 7.2. Let $\lambda_M \in K(M \times T^* M)$ be the class defined in Section 4. Then

$$a \sim \text{ch}(\lambda_M) = \text{ch}(\lambda_M) \sim a$$

for all $a \in H^*(M)$.

Proof. As in Section 4, regard $TM \oplus T^* M$ as an open subset of $M \times T^* M$. The projection of $M \times T^* M$ onto $M$ via the second factor corresponds to the map $TM \oplus T^* M \to M$ given by the formula

$$(v, v') \mapsto \exp \left( \frac{\delta}{(1 + \|v\|^2)^{1/2}} v \right)$$

(compare (4.1)). This is homotopic to the standard projection $(v, v') \mapsto \pi(v)$ (which corresponds to the projection of $M \times T^* M$ onto the first factor) by contracting $\delta$ to zero. Therefore both maps induce the same $H^{ev}(M)$-module action on $H^*(TM \oplus T^* M)$ (note that multiplying $H^{odd}(M)$ against $H^{odd}(TM \oplus T^* M)$ on the right differs by a minus sign from multiplication on the left: this is why we consider only $H^{ev}(M)$). Since $\text{ch}(\lambda_M)$ lies in the image of the map $H^{ev}(TM \oplus T^* M) \to H^{ev}(M \times T^* M)$ given by (4.2) the result follows. \qed

Index Theorem, Preliminary Version. The class $p_*(\text{ch}(\lambda_M))$ is a unit in the ring $H^{ev}(M)$, and for every $x \in H^*(X \times T^* M)$

$$I_X(x) = p_*(x \sim p_*(\text{ch}(\lambda_M))^{-1}).$$

In particular, if $D$ is an elliptic operator on $M$ then

$$\text{Index}(D) = p_*(\text{ch}(\sigma_D) \sim p_*(\text{ch}(\lambda_M))^{-1}).$$

Proof. Let $a_M \in H^*(M)$ be the class obtained in Lemma 7.1. Using Lemma 7.2 and the fact that $p_*$ is a left $H^*(M)$-module homomorphism, we obtain

$$a_M \sim p_*(\text{ch}(\lambda_M)) = p_* (a_M \sim \text{ch}(\lambda_M)) = p_*(\text{ch}(\lambda_M) \sim a_M) = I_M(\text{ch}(\lambda_M)).$$

But according to part (4) of Theorem 5.1 and the definition of $I_M$,

$$I_M(\text{ch}(\lambda_M)) = \text{ch}(\text{Ind}_M(\lambda_M)) = 1. \qed$$

The customary formulation of the index theorem is obtained from the preliminary version above by using some further ideas in algebraic topology. What follows below is a rapid summary of this. For further details see, for example, [3] or [13].
Let $X$ be any compact space, let $V$ be a Euclidean vector bundle over $X$, and let $\lambda_V \in K(V \oplus V)$ be the class defined in Section 4. Using the Thom isomorphism in cohomology,

$$\pi_*: H^*(V \oplus V) \to H^*(X),$$

we form the characteristic class

$$\tau(V) = \pi_*(\text{ch}(\lambda_V)) \in H^*(X),$$

noting that if $M$ is a smooth closed manifold then

$$\tau(TM) = p_*(\text{ch}(\lambda_M)).$$

Using techniques of characteristic class theory one shows that

$$\tau(V) = (-1)^{\dim(V)} \text{Todd}(V \otimes \mathbb{C})^{-1},$$

where $V \otimes \mathbb{C}$ is the complexification of $V$ and $\text{Todd}(V \otimes \mathbb{C})$ denotes its Todd class. Using a more suggestive notation for the functional $p_*: H^{ev}(T^*M) \to \mathbb{R}$ (borrowed from de Rham theory) we get:

**Index Theorem.**

$$\text{Index}(D) = (-1)^{\dim(M)} \int_{T^*M} \text{ch}(\sigma_D) \sim \text{Todd}(TM \otimes \mathbb{C}). \quad \square$$

### 8. The Asymptotic Morphism

In this section we construct the asymptotic morphism

$$T_\omega: C_0(T^*M) \to K(L^2(M))$$

used in the definition of the maps $\text{Ind}_X: K(X \times T^*M) \to K(X)$.

Let $U$ be an open subset of $\mathbb{R}^n$ and let $a(x, \xi)$ be a smooth, compactly supported function on $T^*U$. For $\omega \in [1, \infty)$ define an operator $T_\omega^a: L^2(U) \to L^2(U)$ by the formula

$$T_\omega^a f(x) = \int a(x, \omega^{-1} \xi) e^{ix\xi} \hat{f}(\xi) \, d\xi.$$

Thus

$$T_\omega^a f(x) = \int k_\omega^a(x, y) f(y) \, dy,$$

where

$$k_\omega^a(x, y) = \left( \frac{\omega}{2\pi} \right)^n \int a(x, \xi) e^{i\omega(x-y)\xi} \, d\xi.$$

Each $T_\omega^a$ is a compact operator.

We are interested in the asymptotic behaviour of the operators $T_\omega^a$ as $\omega \to \infty$ (compare [17]).
Lemma 8.1. The operators \( T_\omega^a \) are uniformly bounded.

Proof. For \( f, g \in L^2(U) \) the Cauchy-Schwarz inequality gives
\[
|(f, T_\omega^a g)|^2 = |\int f(x) k_\omega^a(x, y) g(y) \, dxdy|^2 \\
\leq \iint |f(x)|^2 |k_\omega^a(x, y)| \, dydx \cdot \iint |g(y)|^2 |k_\omega^a(x, y)| \, dxdy \\
= \int |f(x)|^2 \left( \int |k_\omega^a(x, y)| \, dy \right) dx \cdot \int |g(y)|^2 \left( \int |k_\omega^a(x, y)| \, dx \right) dy.
\]
It is easily verified that for every \( N \),
\[
|k_\omega^a(x, y)| \leq \text{constant} \cdot \omega^n / (1 + \omega|y - x|)^N.
\]
Using polar coordinates and this estimate for \( N = n + 1 \) we get
\[
\int |f(x)|^2 \left( \int |k_\omega^a(x, y)| \, dy \right) dx \leq \text{constant} \cdot \int |f(x)|^2 \left( \int_0^{\infty} r^{n-1} \omega^n / (1 + \omega r)^{n+1} \, dr \right) dx,
\]
where the term \( r^{n-1} \) comes from the change of variables formula. Substituting \( \rho = \omega r \) we see that the integral is independent of \( \omega \) (and of course finite). Treating the other iterated integral in a similar fashion we obtain
\[
|(f, T_\omega^a g)|^2 \leq \text{constant} \cdot \|f\|_2^2 \|g\|_2^2. \quad \Box
\]

The following lemmas are proved by the same method.

Lemma 8.2. Suppose that \( A^\omega: L^2(U) \to L^2(U) \) are operators of the form
\[
A^\omega f(x) = \int k^\omega(x, y) f(x) \, dy,
\]
where
\[
|k^\omega(x, y)| \leq \text{constant} \cdot \omega^n / (1 + \omega|y - x|)^{n+1}.
\]
For \( L > 0 \) let \( A^\omega_L \) be the operator with kernel
\[
k^\omega_L(x, y) = \begin{cases} 
k^\omega(x, y) & \text{if } |x - y| < L \omega^{-1} \\
0 & \text{if } |x - y| \geq L \omega^{-1}
\end{cases}.
\]
Then \( \|A^\omega - A^\omega_L\| \to 0 \) as \( D \to \infty \), uniformly in \( \omega \). \quad \Box

Lemma 8.3. Let \( A^\omega \) be as above, but suppose that
\[
|k^\omega(x, y)| \leq \text{constant} \cdot \omega^{n-1} / (1 + \omega|y - x|)^{n+1}.
\]
Then \( \|A^\omega\| \to 0 \) as \( \omega \to \infty \). \quad \Box
Proposition 8.4.

(1) If \( b(x, \xi) \) is another smooth, compactly supported function on \( T^* U \) then

\[
T_n a T_n^* b - T_n^* a T_n b \to 0,
\]

in the operator norm, as \( \omega \to \infty \).

(2) Denote by \( a^* \) the complex conjugate of \( a \). Then

\[
T_n^* a^* - (T_n^*)^* \to 0,
\]

in the operator norm, as \( \omega \to \infty \).

Proof. It is easily checked that if the kernels of operators \( A^\omega \) and \( B^\omega \) satisfy the estimate of Lemma 8.2 then so do the kernels of \( A^\omega B^\omega \). Because of this, along with Lemmas 8.3 and 8.4, it suffices to show that for any \( L > 0 \) the kernels of the operators \( T_n^\omega a b - T_n^\omega a T_n^\omega b \) are bounded by a multiple of \( \omega^{n-1} \) on the set \( |x-y| \leq L/\omega \).

We have that

\[
T_n^\omega a T_n^\omega f(x) = \left( \frac{\omega}{2\pi} \right)^n \int \int c_n(x, \xi) e^{i\omega(x-y)\xi} f(y) dy d\xi,
\]

where

\[
c_n(x, \xi) = \left( \frac{\omega}{2\pi} \right)^n \int \int a(x, \eta)b(z, \xi) e^{i\omega(x-z)(\eta-\xi)} dz d\eta
\]

\[
= \left( \frac{\omega}{2\pi} \right)^n \int \int a(x, \xi + \eta)b(x + z, \xi) e^{-i\omega\eta z} dz d\eta.
\]

A simple special case of the stationary phase formula (see Lemma 7.7.3 of [10]) gives us

\[
|c_n(x, \xi) - a(x, \xi)b(x, \xi)| \leq \text{constant} \cdot \omega^{-1}.
\]

Now, the kernel of \( T_n^\omega a b - T_n^\omega a T_n^\omega b \) is

\[
\left( \frac{\omega}{2\pi} \right)^n \int \{a(x, \xi)b(x, \xi) - c_n(x, \xi)\} e^{i\omega(x-y)\xi} d\xi,
\]

and by (8.1), together with the fact that \( c_n(x, \xi) \) is uniformly compactly supported, this is bounded by a multiple of \( \omega^{n-1} \), as required.

Part (2) (which is much easier) is proved in a similar fashion by calculating the kernel of \( T_n^\omega a^* - (T_n^\omega a)^* \) and applying Lemmas 8.2 and 8.3. \(\square\)

Theorem 8.5. There is an asymptotic morphism \( T^\omega : C_0(T^* U) \to K(L^2(U)) \) such that

\[
\|T^\omega(a) - T_n^\omega\| \to 0,
\]

as \( \omega \to \infty \), for all \( a \in C_0^\infty(T^* U) \). \(\square\)

Proof. We start from the fact that a \( * \)-homomorphism from the algebra \( C_0^\infty(T^* U) \) into any \( C^* \)-algebra is automatically continuous in the sup norm, and so extends to \( C_0(T^* U) \) (this is left to the reader).
Form the quotient $\mathcal{K}_\infty/\mathcal{K}_0$ of the algebra of bounded continuous functions from $[1, \infty)$ to $\mathcal{K}(L^2(U))$ by the ideal of functions which vanish at infinity. It is a $C^*$-algebra, and by Lemma 8.1 and Proposition 8.4 the correspondence $a \rightarrow T^\omega_a$ gives a $*$-homomorphism from $C_c^\infty(T^*U)$ into $\mathcal{K}_\infty/\mathcal{K}_0$. Composing the extension to $C_0(T^*U)$ with a continuous (but not necessarily multiplicative, or even linear) section $\mathcal{K}_\infty/\mathcal{K}_0 \rightarrow \mathcal{K}_\infty$ we get the desired asymptotic morphism. □

These considerations are easily generalized from open sets $U$ to arbitrary smooth manifolds $M$ by means of a partition of unity argument and the following calculations.

**Lemma 8.6.** Let $f$ be a smooth, compactly supported function on $U$ and denote by $M_f: L^2(U) \rightarrow L^2(U)$ the operator of pointwise multiplication by $f$. Then

$$M_f T^\omega_a - T^\omega_a M_f \rightarrow 0$$

in the operator norm, as $\omega \rightarrow \infty$.

**Proof.** The kernel of $M_f T^\omega_a - T^\omega_a M_f$ is $(f(x) - f(y))k^\omega(x, y)$. By the Mean Value Theorem, $|f(x) - f(y)| \leq \text{constant} \cdot L/\omega$ when $|x - y| \leq L/\omega$. So the kernel is bounded by a multiple of $\omega^{n-1}$ on the set $|x - y| \leq D/\omega$, and Lemmas 8.2 and 8.3 apply. □

**Lemma 8.7.** Suppose that $U, W$ are open subsets of $\mathbb{R}^n$ and that $\phi: W \rightarrow U$ is a diffeomorphism. Denote by $\tilde{\phi}: T^*W \rightarrow T^*U$ the induced diffeomorphism of cotangent bundles and denote by $U_\phi: L^2(U) \rightarrow L^2(W)$ the induced unitary isomorphism of Hilbert spaces. Then

$$T^\omega_{a_\phi} - U_\phi T^\omega_a U^{-1}_\phi \rightarrow 0$$

in the operator norm as $\omega \rightarrow \infty$.

To explain the notation, we define, as in (4.2),

$$\tilde{\phi}(x, \xi) = (\phi(x), (\phi^{-1}_*)^t \xi),$$

where $\phi_*$ denotes the derivative of $\phi$, mapping tangent vectors at $x$ to tangent vectors at $\phi(x)$, and $(\phi^{-1}_*)^t$ denotes the transpose of its inverse, mapping cotangent vectors at $x$ to cotangent vectors at $\phi(x)$. Also, we define

$$U_\phi f(x) = f(\phi(x)) \cdot J^{1/2}(x),$$

where $J(x)$ denotes the absolute value of the Jacobian of $\phi$ at $x$.

**Proof.** By reducing $U$ and $W$ to smaller sets, if necessary, we may suppose that the derivatives of $\phi$ and its inverse are bounded. Then the operators $U_\phi T^\omega_a U^{-1}_\phi$ are of the sort considered in Lemma 8.2. So it suffices to show that for each $L > 0$ the kernel of $T^\omega_{a\phi} - U_\phi T^\omega_a U^{-1}_\phi$ is $O(\omega^{n-1})$ on the set $|x - y| \leq L/\omega$. 


The kernel of $U_\omega T^\omega_a U^{-1}_\phi$ is
\[
\left(\frac{\omega}{2\pi}\right)^n J(x)^{1/2} J(y)^{1/2} \int a(\phi(x), \xi) e^{i\omega(\phi(x) - \phi(y))\xi} d\xi.
\]
On the other hand the kernel of $T^\omega_{a\phi}$ is
\[
k^\omega_{a\phi}(x, y) = \left(\frac{\omega}{2\pi}\right)^n \int a(\phi(x), (\phi^{-1}_a)^t \xi) e^{i\omega(x-y)\xi} d\xi
= \left(\frac{\omega}{2\pi}\right)^n J(x) \int a(\phi(x), \xi) e^{i\omega(x-y)\phi^*_x \xi} d\xi
= \left(\frac{\omega}{2\pi}\right)^n J(x) \int a(\phi(x), \xi) e^{i\omega\phi^*_x (x-y)\xi} d\xi.
\]

The required estimate follows from the approximation
\[
\phi(x) - \phi(y) = \phi^*_x (x-y) + O(|x-y|^2)
\]
as $|x-y| \to 0$. □

**Theorem 8.8.** Let $M$ be a smooth manifold without boundary, and fix a smooth measure on $M$. There is an asymptotic morphism $T^\omega: C_0(T^*M) \to K(L^2(M))$ such that if $\phi: W \to U$ is a diffeomorphism from an open set in $M$ to $\mathbb{R}^n$ then
\[
\|T^\omega(a \circ \tilde{\phi}) - U_\phi T^\omega_a U^{-1}_\phi\| \to 0
\]
as $\omega \to \infty$, for all $a \in C^\infty_c(T^*U)$. □

**Remark.** In the definition of $U_\phi: L^2(U) \to L^2(W)$ we include the appropriate Radon-Nikodym derivative, so as to make $U_\phi$ a unitary operator.

**9. Completion of the Proof**

Let $M$ be a smooth closed manifold and let $D: C^\infty(M, E) \to C^\infty(M, F)$ be an elliptic operator with symbol $\sigma: \pi^*E \to \pi^*F$.

Put a smooth measure on $M$ and metrics on $E$ and $F$, and consider the formally self-adjoint operator
\[
D = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}: C^\infty(M, E \oplus F) \to C^\infty(E \oplus F).
\]
Its symbol is the endomorphism
\[
\sigma = \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}: \pi^*(E \oplus F) \to \pi^*(E \oplus F)
\]
considered in Section 4.

Basic elliptic operator theory tells us that there is a system of eigenvectors $\{u_n\}$ for the operator $D$ on $C^\infty(M, E \oplus F)$ which constitute an orthonormal basis for $L^2(M, E \oplus F)$. The eigenvalues $\lambda_n$ are real and converge to infinity, in absolute value, as $n \to \infty$. 

Suppose now that $\alpha \in C_0(\mathbb{R})$. We may form the operators

$$\alpha(\omega^{-1}D): L^2(M, E \oplus F) \to L^2(M, E \oplus F)$$

in the sense of spectral theory, so that

$$\alpha(\omega^{-1}D)u_n = \alpha(\omega^{-1}\lambda_n)u_n.$$ 

They are compact (since $|\lambda_n| \to \infty$).

On the other hand we may apply $\alpha$ to the symbol $\sigma$ of $D$. The endomorphism $\alpha(\sigma)$ so obtained vanishes at infinity (compare Lemma 4.1).

Viewing $E$ and $F$ as summands of trivial bundles $\mathbb{C}^N_1$ and $\mathbb{C}^N_2$, we may regard $\alpha(\sigma)$ as an endomorphism of the trivial bundle $\mathbb{C}^N_1 \oplus \mathbb{C}^N_2$ on $T^*M$, by setting it to be zero on the complement of $\pi^*E \oplus \pi^*F$. So thought of it is a matrix valued function on $T^*M$, and we can apply $T_\omega$ to it.

Similarly, we may view $\alpha(\omega^{-1}D)$ as an operator on $L^2(M, \mathbb{C}^N_1 \oplus \mathbb{C}^N_2)$ by setting it to be zero on the complement of the subspace $L^2(M, E \oplus F)$ of $L^2(M, \mathbb{C}^N_1 \oplus \mathbb{C}^N_2)$.

The following key result links the spectral theory of $D$ to the asymptotic morphism of the previous section. We shall only outline a proof.

**Proposition 9.1.** If $D$ is an elliptic operator on $M$ with symbol $\sigma$ then

$$T_\omega(\alpha(\sigma)) - \alpha(\omega^{-1}D) \to 0$$

as $\omega \to \infty$, for every $\alpha \in C_0(\mathbb{R})$.

**Proof (sketch).** One verifies that as $\omega \to \infty$ the operator $M_f \alpha(\omega^{-1}D)$ depends, asymptotically, only on the coefficients of $D$ in a neighbourhood of $\text{supp}(f)$ (see [9], Lemma 2.4). Furthermore it follows from the basic elliptic estimates that $\alpha(\omega^{-1}D)$ varies continuously with the coefficients of $D$. Using these facts and a partition of unity argument we reduce the Lemma to an analogous one for constant coefficient operators, for which $\alpha(\omega^{-1}D)$ may be computed explicitly using the Fourier transform. \(\square\)

**Theorem 9.2.** $\text{Ind}_{pt}(\sigma_D) = \text{Index}(D)$.

**Proof.** For $0 < \omega < \infty$ form the Cayley transform

$$U^\omega = (\omega^{-1}D + i)(\omega^{-1}D - i)^{-1} = I + 2i(\omega^{-1}D - i)^{-1}$$

Extend it to a unitary operator on $L^2(M, \mathbb{C}^N_1 \oplus \mathbb{C}^N_2)$ by setting it equal to the identity on the complement of $L^2(M, E \oplus F)$. If $u$ denotes the Cayley transform of $\sigma$ then it follows from Proposition 9.1 that

$$U^\omega - T_\omega(u) \to 0,$$
as $\omega \to \infty$. Therefore
\[ \text{Ind}_{\text{pt}}(\sigma D) = [p(\epsilon)] - [p(\epsilon U^1)] \in K(K) \]

where $p(\epsilon)$ and $p(\epsilon U^1)$ are the projections associated to the involutions $\epsilon$ and $\epsilon U^1$.

We consider now what happens as $\omega \to 0$. The operator $U^\omega$ converges in norm to minus the identity on the kernel of $D$, and the identity on the complement. So the projection $p(\epsilon U^\omega)$ converges to the projection $p(\epsilon U^0) = P - P_{\ker(D)} + P_{\ker(D^*)}$, where $P_{\ker(D)}$ and $P_{\ker(D^*)}$ are the projections onto the kernels of $D$ and $D^*$.

Therefore
\[ [p(\epsilon)] - [p(\epsilon U^1)] = [P_{\ker(D)}] - [P_{\ker(D^*)}], \]

which proves the theorem. □

It remains to prove part (4) of Theorem 5.1.

Let $V$ be a Euclidean vector space with basis $\{e_1, \ldots, e_n\}$ and corresponding coordinates $x_i(v) = (v, e_i)$. Define operators
\[ B^\omega : S(V, \bigwedge^\ast \mathbb{C}V) \to S(V, \bigwedge^\ast \mathbb{C}V) \]
on Schwartz space by the formula
\[ B^\omega = \sum \frac{1}{\sqrt{-1}\omega} c(\sqrt{-1}e_j) \partial/\partial x_j + x_j c(e_j), \]

where $c(\sqrt{-1}e_j)$ and $c(e_j)$ are as in Section 4. The definition does not depend on the choice of basis.

The spectral theory of $B^\omega$ is easily worked out:

**Lemma 9.3.** There is a system of eigenfunctions $\{u_n\}$ for $B^\omega$ consisting of an orthonormal basis for $L^2(V, \bigwedge^\ast \mathbb{C}V)$. The eigenvalues are real and converge to infinity in absolute value as $n \to \infty$. The kernel of $B^\omega$ is one dimensional and is spanned by the 0-form $e^{-\omega|x|^2}$.

**Proof.** (See [9], Section 5.) Upon squaring $B^\omega$ we obtain
\[ (B^\omega)^2 = -\omega^{-2} \Delta + |x|^2 + \omega^{-1} N, \]

where $\Delta$ is the Laplacian, $|x|^2$ denotes pointwise multiplication by the scalar function $|x|^2$, and $N$ is the operator which multiplies a form of degree $j$ by $2j - n$. So $(B^\omega)^2$ is a direct sum of harmonic oscillators $-a\Delta + b|x|^2 + c$ whose spectral theory is well known from elementary quantum mechanics. □

Needless to say, our interest in $B^\omega$ lies in its relation to the “symbol” $c$ constructed in Section 4.
Lemma 9.4. For every $\alpha \in C_0(\mathbb{R})$, $T_\omega^\omega - \alpha(B^\omega) \to 0$ as $\omega \to \infty$. □

This may be proved either by an approximation argument, as in Proposition 9.1, or by a direct calculation, based on Mehler’s formula for the kernel of the operator $e^{-(B^\omega)^2}$.

Theorem 9.5. $\text{Ind}_M(\lambda_M) = 1$.

Proof. For $m \in M$ let

$$U_m: L^2(TM_m, \wedge^*_{C} TM_m) \to L^2(TM_m, \wedge^*_{C} TM_m)$$

be the Cayley transform of the operator $B = B^1$, and let

$$\varepsilon_m: L^2(TM_m, \wedge^*_{C} TM_m) \to L^2(TM_m, \wedge^*_{C} TM_m)$$

be the grading operator which multiplies a form by $\pm 1$ according as its degree is even or odd. As usual, form the projections $p(\varepsilon_m)$ and $p(\varepsilon U_m)$.

Using the exponential map, identify $TM_m$ with a neighbourhood $W_m$ of $m$ in $M$ (compare (4.1)), and so view $L^2(TM_m, \wedge^*_{C} TM_m)$ as a subspace of $L^2(M, \wedge^*_{C} TM_m)$. By complementing the bundle $TM$ over $M$ we can view $L^2(M, \wedge^*_{C} TM_m)$ as a subspace of the fixed Hilbert space

$$L^2(M, \mathbb{C}^N) = L^2(M) \oplus \ldots L^2(M).$$

In this way, $p(\varepsilon_m)$ and $p(\varepsilon U_m)$ become projection valued functions from $M$ to $M_N(K^+)$.

Using Lemma 9.4 and the invariance under diffeomorphism of $T^\omega$ (Lemma 8.7) we see that $\text{Ind}_M(\lambda_M)$ is represented by the difference of projections

$$[p(\varepsilon)] - [p(\varepsilon U)] \in K(K(M)).$$

In order to calculate this difference we use a homotopy similar to the one in Theorem 9.2, replacing $B$ with $t^{-1}B$ and letting $t \to 0$. Bearing in mind the calculation of the kernel of $B$ we see that the Cayley transform of $\omega B$ converges to the operator $U^0$ which is $-1$ on the 0-form $e^{-\|v\|^2}$ and $+1$ on its orthogonal complement. Therefore

$$\text{Ind}_M(\lambda_M) = [p(\varepsilon)] - [p(\varepsilon U)]$$

$$= [p(\varepsilon)] - [p(\varepsilon U^0)]$$

$$= [p],$$

where $p(m)$ is the projection onto the subspace spanned by the 0-form

$$e^{-|x|^2} \in L^2(TM_m, \wedge^*_{C} TM_m) \subset L^2(M, \mathbb{C}^N).$$

The “rotation”

$$\begin{pmatrix}
\sin^2(\theta)p(m) & \sin(\theta)\cos(\theta)r(m) \\
\sin(\theta)\cos(\theta)r(m) & \cos^2(\theta)e
\end{pmatrix},$$
where $e$ is the projection onto the subspace spanned by a fixed $v \in L^2(M)$ and $r(m)$ is the partial isometry mapping $v$ to $e^{-|x|^2} \in L^2(TM_m, \Lambda^*_{\infty} TM_m)$ (appropriately normalized), shows that $[p] = [e]$ in $K$-theory. Bearing in mind the form of the isomorphism from $K(K(M))$ to $K(M)$ we see that
\[
\text{Ind}_M(\lambda_M) = 1,
\]
as required. □

References


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