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CATEGORIES OF FRACTIONS AND EXCISION IN *KK*-THEORY

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Using elementary ideas from the theory of categories of fractions, we construct bivariant homology/cohomology groups $\mathbf{E}(A,B)$ for C^*-algebras, which satisfy general excision axioms, and are equal to Kasparov's groups $KK(A,B)$ for nuclear (or more generally K-nuclear) C^*-algebras.

Introduction

This article is an investigation into the KK-theory of C^*-algebras introduced by Kasparov [11]; our approach is algebraic in nature and is more or less a continuation of the paper [8], in which KK-theory is characterized by means of some simple axioms.

Kasparov's theory can be described as a sort of calculus for computing K-theory groups of C^*-algebras, which generalizes constructions due to Atiyah and Singer in the index theory of elliptic operators [1,2]. A family of elliptic operators on a smooth, closed manifold M, parametrized by a compact space X, gives rise to an 'index map' $K(M)\to K(X)$ in Atiyah–Hirzebruch K-theory, or in other words a map $K(C(M))\to K(C(X))$ in C^*-algebra K-theory (see [3] or [4] for a survey of K-theory for C^*-algebras). Following and broadly extending ideas of Atiyah [1], Kasparov defines a notion of generalized elliptic operator for a pair of C^*-algebras A and B, which, in the case $A=C(M)$ and $B=C(X)$, includes the notion of a family of elliptic pseudodifferential operators on M, parametrized by X. There are natural notions of isomorphism and homotopy for these generalized elliptic operators, and the resulting set of equivalence classes, which is in a natural way an abelian group, is denoted $KK(A,B)$. The index construction of Atiyah and Singer extends to a pairing

$$K(A)\otimes KK(A,B)\to K(B). \tag{1}$$

The most important aspect of Kasparov's theory is the existence of an associative 'product map'

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$$KK(A,B) \otimes KK(B,C) \to KK(A,C) \tag{2}$$

which is compatible with (1), so that a composition of index maps may be represented as the index map of a 'product' generalized elliptic operator. The formula for the product given in [11] is rather complicated (it is somewhat streamlined in [16]) but it is nevertheless possible to compute it in many circumstances, and in practice each such computation amounts to some sort of index formula. A survey of applications is given in [12].

Following ideas of Cuntz [3], the algebraic structure of Kasparov's theory is worked out in [8]. The basic idea is to regard (2) as the law of composition in an additive category, whose objects are (separable) C^*-algebras and whose morphisms are elements of the groups $KK(A,B)$ (there do exist elements which act as identity morphisms). This category then turns out to have a remarkably simple structure (see [8] or Theorem 3.4 below). It follows, for example, that there is a unique functorial pairing (2), up to normalization, and that associativity is 'automatic'.

Kasparov's point of view in [11] is that the groups $KK(A,B)$ constitute a sort of bivariant homology/cohomology theory for C^*-algebras. The motivation for this is that the group $KK(\mathbf{C},B)$ is naturally isomorphic to $K(B)$, whilst $KK(A,\mathbf{C})$ is the 'K-homology' of A (for example, if $A=C(X)$, and X is a finite complex, then $KK(A,\mathbf{C})$ is the even K-homology group of X). From this point of view, it is natural to study questions of excision in KK-theory. In the present context, this amounts to asking whether or not the sequences

$$KK(C,J) \xrightarrow{j_*} KK(C,A) \xrightarrow{q_*} KK(C,B) \tag{3}$$

and

$$KK(B,D) \xrightarrow{q^*} KK(A,D) \xrightarrow{j^*} KK(J,D) \tag{4}$$

associated with a short exact sequence of C^*-algebras

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0 \tag{5}$$

are exact in the middle. (From (3) and (4), the existence of long exact homology/cohomology sequences associated with (5) follows from Bott periodicity for KK-theory [11, Section 5] and a standard argument adapted from algebraic topology [11, Section 7]. By Bott periodicity the sequences are periodic: they are cyclic six term exact sequences.) Kasparov proves the exactness of (3) under the hypothesis that C is a nuclear C^*-algebra, and the exactness of (4) under the assumption that A is nuclear (see [11, Section 7]). Simpler proofs of somewhat more general results are given in [5]. In [17] the notion of 'K-nuclear' C^*-algebra is introduced and the excision problem is settled for this class.

Whilst from a certain point of view the class of K-nuclear C^*-algebras is rather large (certainly all commutative C^*-algebras are K-nuclear, and so KK-theory does give a homology/cohomology theory for compact metric spaces), the present state of knowledge is rather unsatisfactory. This is so for practical, as well as theoretical,

reasons: see [10, Section 3] for a situation in the K-theory of foliations where stronger excision results are needed. Unfortunately it seems likely to most experts that excision in KK-theory does *not* hold in general. The purpose of this paper is to construct (using the term rather loosely) a theory $\mathbf{E}(A, B)$ which *does* have excision in general. The groups $\mathbf{E}(A, B)$ are obtained from a category $\mathbf{E}$, and so there is automatically a product of the form (2). Furthermore there is a natural transformation $KK(A, B) \to \mathbf{E}(A, B)$, compatible with products, which is an isomorphism if A is K-nuclear. In fact, $\mathbf{E}$ can be regarded as a sort of 'universal' extension of the homology/cohomology theory KK from K-nuclear C^*-algebras to all (separable) C^*-algebras.

The construction of $\mathbf{E}$ is based on the notion of a category of fractions, familiar in category theory. Unfortunately the standard presentation of this, [6], is not quite adequate for our purposes. Thus in Section 1 of the paper we give an elementary account of the necessary algebraic results, which slightly extends the discussion in [6, Chapter 1]. In Section 2 we construct various homotopy categories of C^*-algebras. Much of the theory has been worked out by Rosenberg in [15] and we refer to this paper for a number of computations. (Some of the other basic constructions given at the beginning of Section 2 are taken from the author's M.Sc. thesis [7].) The main part of Section 2—the material following Definition 2.5—is devoted to fitting C^*-algebra categories into the framework of Section 1. Finally, in Section 3 we assemble the results of the previous sections to define $\mathbf{E}$, prove its excision properties (Theorem 3.2), relate it to KK-theory (Theorem 3.5), and characterize $\mathbf{E}$ as a 'universal' homology/cohomology theory (Theorem 3.6).

An interesting and quite different approach to more or less the same problem of excision is given by Skandalis in [17], where subgroups $KK_{\mathrm{nuc}}(A, B) \subset KK(A, B)$ are defined, and shown to satisfy excision. The main distinction between $KK_{\mathrm{nuc}}(A, B)$ and $\mathbf{E}(A, B)$ is that *all* generalized elliptic operators give rise to elements of $\mathbf{E}(A, B)$, and so $\mathbf{E}$ 'contains' the elliptic operator calculus on C^*-algebra K-theory. Now, Skandalis has shown in [17] that certain identities involving products of specific elliptic operators, arising in differential topology, do not hold in KK-theory. A consequence is that KK-theory is inadequate as a tool to prove the corresponding relations among K-groups. However, it is *possible* that within $\mathbf{E}$-theory these elliptic operator identities do hold. Thus from the point of view of applications it would be extremely interesting to develop a concrete realization of $\mathbf{E}$ (or a similar functor). This aspect of $\mathbf{E}$-theory is discussed in a little more detail in [9, Section 7].

1. Categories of fractions

Let $\mathbf{A}$ be a small, additive category (for our purposes it is convenient to include in the definition of additive category the existence of finite products and a zero object). If Σ is a set of morphisms in $\mathbf{A}$ then we shall denote by $\mathbf{A}[\Sigma^{-1}]$ the *additive* category obtained from $\mathbf{A}$ by inverting the morphisms in Σ. To be precise,

$\mathbf{A}[\Sigma^{-1}]$ is an additive category, together with an additive functor $F: \mathbf{A} \to \mathbf{A}[\Sigma^{-1}]$, such that:

(i) the objects of $\mathbf{A}[\Sigma^{-1}]$ are the objects of $\mathbf{A}$;

(ii) for every object A of $\mathbf{A}$, $F(A) = A$;

(iii) for every morphism σ in Σ, $F(\sigma)$ is an isomorphism; and

(iv) if $G: \mathbf{A} \to \mathbf{B}$ is any additive functor such that $G(\sigma)$ is an isomorphism for every σ in Σ then there exists a *unique* additive functor $\hat{G}: \mathbf{A}[\Sigma^{-1}] \to \mathbf{B}$ such that $G = \hat{G} \circ F$.

The existence of $\mathbf{A}[\Sigma^{-1}]$ for an arbitrary $\mathbf{A}$ and Σ is easy to establish (see [13, Chapter 4] for example), whilst of course the uniqueness of $\mathbf{A}[\Sigma^{-1}]$ up to canonical isomorphism is guaranteed by condition (iv).

In order to simplify notation, if ϕ is a morphism in $\mathbf{A}$ then we shall denote the morphism $F(\phi)$ in $\mathbf{A}[\Sigma^{-1}]$ by $\bar{\phi}$.

1.1. Definition (compare [6, Chapter 1]). We shall say that Σ is an *admissible* set of morphisms if it is closed under composition of morphisms, contains all identity morphisms, and if the following four conditions involving morphisms in $\mathbf{A}$ are satisfied.

(L1) Given $\phi: A \to B$ and $\sigma: A \to A'$, with σ in Σ, there exists a commutative square

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\sigma"'] & B \arrow[d, dashed, "\tau"] \\ A' \arrow[r, dashed, "\phi'"'] & B' \end{tikzcd}$$

with τ in Σ.

(L2) Denote by Σ_L the class of morphisms in $\mathbf{A}$ which is generated (under composition) by Σ and all split monomorphisms in $\mathbf{A}$. If a composition $\phi \circ \sigma$ is zero, where σ is an element of Σ, then there exists τ in Σ_L such that $\tau \circ \phi = 0$.

(R1) Given $\sigma: A \to A'$ and $\phi': B' \to A'$, with σ in Σ, there exists a commutative diagram

$$\begin{tikzcd} A' & B' \arrow[l, "\phi'"'] \\ A \arrow[u, "\sigma"] & B \arrow[l, dashed, "\phi"] \arrow[u, dashed, "\tau"'] \end{tikzcd}$$

with τ in Σ.

(R2) Denote by Σ_R the set of morphisms in $\mathbf{A}$ generated by Σ and all split epimorphisms in $\mathbf{A}$. If a composition $\sigma\varphi$ is zero, with σ in Σ, then $\phi\tau = 0$ for some τ in Σ_R.

1.2. Remarks. (1) The above conditions are weakened versions of the conditions given in [6] for Σ to admit a 'calculus of left and right fractions' (unfortunately the C^*-algebra constructions of the next section do not quite fit into the framework of [6], and so the extra generality of Definition 1.1 is necessary). Following [6], we will obtain a concrete description of $\mathbf{A}[\Sigma^{-1}]$ for admissible sets Σ.

(2) The conditions (R1) and (R2) are of course simply the duals of (L1) and (L2). We shall consider in detail the consequences of (L1) and (L2); corresponding results concerning (R1) and (R2) follow by reversing arrows.

(3) Note that if Σ is generated under composition by a set Σ^0 and a set of isomorphisms in $\mathbf{A}$, then in order to verify that Σ is admissible, it suffices to check (L1)–(R2) for morphisms σ in Σ^0, for then (L1)–(R2) hold for all σ in Σ by a simple induction argument.

(4) Assuming that Σ is admissible, the condition (L1) is satisfied not just for all σ in Σ but in fact for all σ in Σ_L. To show this, it suffices, as in the previous remark, to consider the case where σ is a split monomorphism, with a left inverse π. In (L1) we may then take $\phi' = \phi\pi$, and $\tau = 1_B$. We shall refer to this generalized condition simply as (L1) below. Similarly, condition (R1) is satisfied for all σ in Σ_R. Concerning Σ_L and Σ_R, we also note that the images in $\mathbf{A}[\Sigma^{-1}]$ of morphisms in Σ_L and Σ_R are left and right invertible, respectively.

For the rest of this section, we assume that Σ is admissible.

1.3. Definition (compare [6, I.2.3]). Let A and B be objects in $\mathbf{A}$ and denote by $\mathscr{F}_B(A)$ the set of ordered pairs of morphisms (σ, ϕ) in $\mathbf{A}$ of the form

$$A \xrightarrow{\phi} B' \xleftarrow{\sigma} B$$

and with σ in Σ. We shall say that (σ_1, ϕ_1) and (σ_2, ϕ_2) are *related* if there exists a commutative diagram

$$\begin{tikzcd} & B' \arrow[dr, "\alpha_1"] & \\ A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_2"] & C \\ & B_2' \arrow[ur, "\alpha_2"'] & \end{tikzcd} \tag{6}$$

(meaning that $\alpha_1\phi_1 = \alpha_2\phi_2$ and $\alpha_1\sigma_1 = \alpha_2\sigma_2$), where $\alpha_1\sigma_1$ is in Σ_L.

1.4. Lemma. *The relation of Definition* 1.3 *is an equivalence relation.*

Proof. Symmetry and reflexivity are obvious. Suppose that (σ_2, ϕ_2) is related to (σ_3, φ_3) by some diagram

$$\begin{tikzcd}
& B_2' \arrow[dr, "\beta_2"] & \\
A \arrow[ur, "\phi_2"] \arrow[dr, "\phi_3"'] & B \arrow[u, "\sigma_2"'] \arrow[d, "\sigma_3"] & D. \\
& B_3' \arrow[ur, "\beta_3"'] &
\end{tikzcd}$$

By (L1) there is a commuting square

$$\begin{tikzcd}
B \arrow[r, "\alpha_2\sigma_2"] \arrow[d, "\beta_2\sigma_2"'] & C \arrow[d, "\gamma"] \\
D \arrow[r, "\delta"'] & E
\end{tikzcd}$$

with γ in Σ_L, and hence $\gamma\alpha_2\sigma_2=\delta\beta_2\sigma_2\in\Sigma_L$. By (L2), applied to $\phi=\gamma\alpha_2-\delta\beta_2$ and $\sigma=\sigma_2$, there exists $\tau:E\to E_1$ in Σ_L such that $\tau\gamma\alpha_2=\tau\delta\beta_2$. Then the diagram

$$\begin{tikzcd}
& B_1' \arrow[dr, "\tau\gamma\alpha_1"] & \\
A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_3"'] & B \arrow[u, "\sigma_1"'] \arrow[d, "\sigma_3"] & E_1 \\
& B_3' \arrow[ur, "\tau\delta\beta_3"'] &
\end{tikzcd}$$

commutes (note that we need the morphism τ so that $\tau\gamma\alpha_1\phi_1=\tau\delta\beta_3\phi_3$) and so (σ_1,ϕ_1) is related to (σ_3,ϕ_3). $\square$

1.5. Definition. Denote by $[\sigma,\phi]$ the equivalence class of (σ,ϕ), and denote by $F_B(A)$ the quotient of $\mathscr{F}_B(A)$ by this equivalence relation.

1.6. Lemma. (i) *For any finite collection* $\{[\sigma_1,\psi_1],\dots,[\sigma_n,\psi_n]\}$ *of elements of* $F_B(A)$, *there exists some* $\sigma:B\to B'$ *in* Σ *and morphisms* $\phi_1,\dots,\phi_n$ *in* $\mathbf{A}$ *such that* $[\sigma,\phi_i]=[\sigma_i,\psi_i]$ *for* $i=1,\dots,n$.

(ii) *The operation* $[\sigma,\phi_1]+[\sigma,\phi_2]=[\sigma,\phi_1+\phi_2]$ *is well defined and makes* $F_B(A)$ *into an abelian group.*

Proof. (i) If $n=1$ the result is trivial. Suppose then that there exist ϱ and $\theta_1,\dots,\theta_{n-1}$ such that $[\sigma_i,\psi_i]=[\varrho,\theta_i]$ for $i=1,\dots,n-1$. By (L1) there is a commuting diagram

$$\begin{tikzcd}
B \arrow[r, "\varrho"] \arrow[d, "\sigma_n"'] & B' \arrow[d, "\tau"] \\
B'' \arrow[r, "\alpha"'] & B'''
\end{tikzcd}$$

with $\tau\in\Sigma$, and hence $\sigma\in\Sigma$, where $\sigma=\tau\varrho=\alpha\sigma_n$. It is then easily checked that $[\sigma,\alpha\psi_n]=[\sigma_n,\psi_n]$ and $[\varrho,\theta_i]=[\sigma,\tau\theta_i]$ for $i=1,\dots,n-1$.

(ii) We must show that if $[\sigma_1,\phi_1]=[\sigma_2,\phi_2]$ and $[\sigma_1,\psi_1]=[\sigma_2,\psi_2]$ then $[\sigma_1,\phi_1+\psi_1]=[\sigma_2,\phi_2+\psi_2]$ (this, together with part (i) will show that the operation is well defined on $F_B(A)$; it is clear that $F_B(A)$ will then be an abelian group). Given commuting diagrams

$$\begin{tikzcd} & B_1' \arrow[dr, "\alpha_1"] & \\ A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & B \arrow[u, "\sigma_1"] \arrow[d, "\sigma_2"] & C \\ & B_2' \arrow[ur, "\alpha_2"'] & \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} & B_1' \arrow[dr, "\beta_1"] & \\ A \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & B \arrow[u, "\sigma_1"] \arrow[d, "\sigma_2"] & D \\ & B_2' \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

as in Definition 1.1, by (L1) there exists a commuting square

$$\begin{tikzcd} B \arrow[r, "\alpha_1\sigma_1"] \arrow[d, "\beta_1\sigma_1"'] & C \arrow[d, "\gamma"] \\ D \arrow[r, "\delta"'] & E \end{tikzcd}$$

with γ in Σ. By (L2) there exist $\tau_i: E\to E_i$ in Σ_L such that $\tau_i\gamma\alpha_i=\tau_i\delta\beta_i$, for $i=1,2$, and applying (L1) to

$$\begin{tikzcd} E \arrow[r, "\tau_1"] \arrow[d, "\tau_2"'] & E_1 \\ E_2 & \end{tikzcd},$$

we obtain a single $\tau: E\to E'$ in Σ_L such that $\tau\gamma\alpha_i=\tau\delta\beta_i$ for $i=1,2$. Then the commuting diagram

$$\begin{tikzcd} & B_1' \arrow[dr, "\tau\gamma\alpha_1"] & \\ A \arrow[ur, "\phi_1+\psi_1"] \arrow[dr, "\phi_2+\psi_2"'] & B \arrow[u, "\sigma_1"] \arrow[d, "\sigma_2"] & E' \\ & B_2' \arrow[ur, "\tau\gamma\alpha_2"'] & \end{tikzcd}$$

shows that $[\sigma_1,\phi_1+\psi_1]=[\sigma_2,\phi_2+\psi_2]$. $\square$

We are of course thinking of $F_B(A)$ as a model for $\mathbf{A}[\Sigma^{-1}](A,B)$. Observe that

if $[\sigma_1,\phi_1]=[\sigma_2,\phi_2]$ in $F_B(A)$ then $\bar\sigma_1^{-1}\bar\phi_1=\bar\sigma_2^{-1}\bar\phi_2$ in $\mathbf{A}[\Sigma^{-1}](A,B)$. Indeed, given the diagram (6), we have $\bar\alpha_1\bar\phi_1=\bar\alpha_2\bar\phi_2$ and $\bar\alpha_1=\bar\alpha_2\bar\sigma_2\bar\sigma_1^{-1}$, so that $\bar\alpha_2\bar\phi_2=\bar\alpha_2\bar\sigma_2\bar\sigma_1^{-1}\bar\phi_1$. But $\bar\alpha_2\bar\sigma_2$, and hence $\bar\alpha_2$, is a split monomorphism, and so from $\bar\alpha_2\bar\phi_2=\bar\alpha_2\bar\sigma_2\bar\sigma_1^{-1}\bar\phi_1$ we obtain $\bar\phi_2=\bar\sigma_2\bar\sigma_1^{-1}\bar\phi_1$, and hence $\bar\sigma_2^{-1}\bar\phi_2=\bar\sigma_1^{-1}\bar\phi_1$. The following result is the converse of this computation:

1.7. Proposition. *Every element of* $\mathbf{A}[\Sigma^{-1}](A,B)$ *is equal to one of the form* $\bar\sigma^{-1}\bar\phi$ *where* $\sigma\in\Sigma$, *and* $\bar\sigma_1^{-1}\bar\phi_1=\bar\sigma_2^{-1}\bar\phi_2$ *if and only if* $[\sigma_1,\phi_1]=[\sigma_2,\phi_2]$.

Proof. An element of $\mathbf{A}[\Sigma^{-1}](A,B)$ is a $\mathbf{Z}$-linear combination of compositions of the form

$$\bar\varrho_k^{-1}\bar\psi_k\bar\varrho_{k-1}^{-1}\bar\psi_{k-1}\cdots\bar\varrho_1^{-1}\bar\psi_1,$$

where $\varrho_j\in\Sigma$ for $j=1,\dots,k$. Using (L1) repeatedly, any such composition is equal to some $\bar\varrho^{-1}\bar\psi$. Thus any morphism from A to B may be written as a linear combination of morphisms $\bar\sigma_i^{-1}\bar\psi_i$ $(i=1,\dots,n)$. Using Lemma 1.6 and the remark preceding this proposition, these may be written as $\bar\sigma^{-1}\bar\phi_i$ $(i=1,\dots,n)$, and then $\sum_{i=1}^n m_i(\bar\sigma^{-1}\bar\phi_i)=\bar\sigma^{-1}(\overline{\sum m_i\phi_i})$, so the first part of the proposition is proved.

Now we may make $A\to F_B(A)$ into a (contravariant) functor on $\mathbf{A}$ in the obvious way, and it follows easily from the group structure given in Lemma 1.6 that F_B is an additive functor from $\mathbf{A}$ to abelian groups. If $\varrho:A\to A'$ is in Σ then $F(\varrho):F_B(A')\to F_B(A)$ is injective. Indeed, if $F(\varrho)([\sigma,\phi])=0$, so that there exists a commuting diagram

$$\begin{tikzcd} & B' \arrow[dr, "\alpha_1"] & \\ A \arrow[ur, "\phi\varrho"] \arrow[dr, "0"'] & B \arrow[u, "\sigma"'] \arrow[d, equal] & C \\ & B \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

(with $\alpha_1\sigma$ in Σ_L), then by (L2), since $\alpha_1\phi\varrho=0$, there exists $\tau\in\Sigma_L$ such that $\tau\alpha_1\phi=0$. The diagram

$$\begin{tikzcd} & B' \arrow[dr, "\tau\alpha_1"] & \\ A' \arrow[ur, "\phi"] \arrow[dr, "0"'] & B \arrow[u, "\sigma"'] \arrow[d, equal] & D \\ & B \arrow[ur, "\tau\alpha_2"'] & \end{tikzcd}$$

then shows that $[\sigma,\phi]=0$. On the other hand, $F(\varrho)$ is surjective since it is easily seen that if $[\sigma,\phi]\in F_B(A)$ then $F(\varrho)([\tau\sigma,\phi'])=[\sigma,\phi]$, where $\tau\in\Sigma$ and ϕ' are chosen using (L1) so that the square

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\varrho"'] & B' \arrow[d, "\tau"] \\ A' \arrow[r, "\phi'"'] & B'' \end{tikzcd}$$

commutes. Thus F_B passes to a functor on $\mathbf{A}[\Sigma^{-1}]$. The proof is completed by noting that under the homomorphism $F(\bar\sigma^{-1}\bar\phi)=F(\phi)F(\sigma)^{-1}$, the 'unit element' $[1,1]\in F_B(B)$ is sent to $[\sigma,\phi]\in F_B(A)$, since $F(\sigma)([\sigma,1])=[1,1]$ (by the above) and $F(\phi)([\sigma,1])=[\sigma,\phi]$. $\square$

1.8. Proposition. *Every element of* $\mathbf{A}[\Sigma^{-1}](A,B)$ *is equal to one of the form* $\bar\phi\bar\sigma^{-1}$, *where* $\sigma\in\Sigma$, *and* $\bar\phi_1\bar\sigma_1^{-1}=\bar\phi_2\bar\sigma_2^{-1}$ *if and only if there is a commuting diagram*

$$\begin{tikzcd} & A_1' \arrow[dl, "\phi_1"'] \arrow[d, "\sigma_1"] & \\ B & A & C \arrow[ul, "\alpha_1"'] \arrow[dl, "\alpha_2"] \\ & A_2' \arrow[ul, "\phi_2"] \arrow[u, "\sigma_2"'] & \end{tikzcd}$$

with $\sigma_1\alpha_1\in\Sigma_R$.

Proof. Apply Proposition 1.7 to the opposite category of **A**. $\square$

Having so described $\mathbf{A}[\Sigma^{-1}]$, we can proceed to prove the excision result we require.

1.9. Theorem. *Suppose that* $A\xrightarrow{\iota}B\xrightarrow{\pi}C$ *is a sequence of morphisms in* **A** *such that for every object* D *in* **A** *the sequences of abelian groups*

$$\mathbf{A}(C,D)\to\mathbf{A}(B,D)\to\mathbf{A}(A,D)$$

and

$$\mathbf{A}(D,A)\to\mathbf{A}(D,B)\to\mathbf{A}(D,C)$$

are exact in the middle. Then the sequences

$$\mathbf{A}[\Sigma^{-1}](C,D)\to\mathbf{A}[\Sigma^{-1}](B,D)\to\mathbf{A}[\Sigma^{-1}](A,D)$$

and

$$\mathbf{A}[\Sigma^{-1}](D,A)\to\mathbf{A}[\Sigma^{-1}](D,B)\to\mathbf{A}[\Sigma^{-1}](D,C)$$

are also exact in the middle.

Proof. We shall consider just the contravariant sequence; the covariant one is dual. Let $\bar{\sigma}^{-1}\bar{\phi} \in \mathbf{A}[\Sigma^{-1}](B,D)$ and suppose that $\bar{\sigma}^{-1}\bar{\phi}\bar{\iota}=0$. There is a diagram of the form

$$\begin{tikzcd}
 & D' \arrow[dr, "\alpha_1"] & \\
A \arrow[ur, "\varphi_1"] \arrow[dr, "0"'] & D \arrow[u, "\sigma"'] \arrow[d, equal] & E \\
 & D \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

with $\alpha_1\sigma \in \Sigma_L$. We have that $(\alpha_1\phi)\iota = 0$, and so by the hypothesis of the theorem, $\alpha_1\phi = \psi\pi$ for some $\psi : C \to E$ in $\mathbf{A}$. Now, $\bar{\alpha}_1$ has a left inverse $\beta : E \to D'$ in $\mathbf{A}[\Sigma^{-1}]$, and from the equality $\alpha_1\phi = \psi\pi$ we obtain $\bar{\phi} = \beta\bar{\psi}\bar{\pi}$. Therefore $\bar{\sigma}^{-1}\bar{\phi} = (\bar{\sigma}^{-1}\beta\bar{\psi})\bar{\pi}$, and so $\bar{\sigma}^{-1}\bar{\phi}$ has the pre-image $\bar{\sigma}^{-1}\beta\bar{\psi}$ in $\mathbf{A}[\Sigma^{-1}](C,D)$, and the sequence is exact at $\mathbf{A}[\Sigma^{-1}](B,D)$. $\square$

2. Stable homotopy for C^*-algebras

Denote by C^*-**Alg** the category of *separable* C^*-algebras and *-homomorphisms. This is of course not a small category so that the results of the previous section will not immediately apply to the categories associated to C^*-**Alg** that we are about to construct. However, C^*-**Alg** does have a small, equivalent subcategory—take the set of C^*-algebras which are algebras of operators on a fixed separable, infinite-dimensional Hilbert space, for example—and it is sufficient for our purposes to work with it. Since they are of a completely straightforward nature, we shall not bother to comment on the various details associated with this substitution. (It would be equally simple to modify the results of Section 1 so that they apply to categories with a small equivalent subcategory.)

Denote by $\mathbf{H}$ the category of separable C^*-algebras and homotopy classes of *-homomorphisms (see for example [15]). We shall denote the homotopy class of a *-homomorphism f by $[f]$.

We may also form a stable homotopy category $\mathbf{S}$ using the suspension functor $S : \mathbf{H} \to \mathbf{H}$, defined by

$$S(A \xrightarrow{[f]} B) = C_0(0,1) \otimes A \xrightarrow{[1\otimes f]} C_0(0,1) \otimes B.$$

The objects of $\mathbf{S}$ are pairs (A,m), where A is a separable C^*-algebra and $m \in \mathbf{Z}$ (we shall write A_m instead of (A,m)). The morphisms are given by:

$$\mathbf{S}(A_m, B_n) = \varinjlim_{K} \mathbf{H}(S^{m+K}A, S^{n+K}B).$$

Here $\mathbf{H}(S^{m+K}A, S^{n+K}B)$ is mapped into $\mathbf{H}(S^{m+K+1}A, S^{n+K+1}B)$ by suspension (we consider only those K such that $K+m \geq 0$ and $K+n \geq 0$). Given a *-homomorphism $f: S^{m+K}A \to S^{n+K}B$, we shall denote the corresponding morphism in $\mathbf{S}(A_m, B_n)$ by $\{f\}$. As is the case with stable homotopy of spaces, $\mathbf{S}$ is an additive category (compare [15, Section 3]).

Now, denote by $\mathcal{K}$ the C^*-algebra of compact operators on a separable Hilbert space. The following facts about $\mathcal{K}$ are well known and easily derived.

2.1. Lemma. (i) *The C^*-algebras $\mathcal{K} \otimes \mathcal{K}$ and $\mathcal{K}$ are isomorphic, and any two *-isomorphisms are homotopic.*

(ii) *Any two rank one projections in $\mathcal{K}$ are homotopic.*

(iii) *If $p \in \mathcal{K}$ is a rank-one projection then the *-homomorphism $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ given by $k \to k \otimes p$ is homotopic to a *-isomorphism.*

Define functors $T: \mathbf{H} \to \mathbf{H}$ and $T: \mathbf{S} \to \mathbf{S}$ by

$$T(A) = A \otimes \mathcal{K}, \qquad T([f]) = [f \otimes 1]$$

and

$$T(A_n) = (A \otimes \mathcal{K})_n, \qquad T(\{f\}) = \{f \otimes 1\}.$$

Fixing a rank-one projection $p \in \mathcal{K}$ and a *-isomorphism $h: \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$, define natural transformations $\eta: 1 \to T$ and $\mu: T^2 \to T$ by means of the *-homomorphisms

$$e_A: A \to A \otimes \mathcal{K},\ e_A(a) = a \otimes p,$$

for η and

$$m_A: A \otimes \mathcal{K} \otimes \mathcal{K} \to A \otimes \mathcal{K},\ m_A(a \otimes k_1 \otimes k_2) = a \otimes h(k_1 \otimes k_2),$$

for μ. Using Lemma 2.1 we obtain the following result:

2.2. Lemma. (i) *The natural transformation μ is an isomorphism, with inverse $T\eta$.*

(ii) *Both η and μ commute with T in the sense that $T\eta = \eta T$ and $T\mu = \mu T$ as natural transformations.*

(By $T\eta$ we mean the natural transformation $T \to T^2$ given by $(T\eta)_A = T(\eta_A)$, and by ηT we mean the natural transformation $T \to T^2$ given by $(\eta T)_A = \eta_{T(A)}$. The natural transformations μT and $T\mu$ are defined similarly.)

Using T, η and μ, define new categories $\mathbf{TH}$ and $\mathbf{TS}$ as follows (we shall spell out the construction for $\mathbf{TH}$; the case of $\mathbf{TS}$ is the same). The objects of $\mathbf{TH}$ are the objects of $\mathbf{H}$, and

$$\mathbf{TH}(A, B) = \mathbf{H}(A, TB).$$

Given $\phi: A \to TB$ and $\psi: B \to TC$ their composition is given by

$$A \xrightarrow{\ \phi\ } TB \xrightarrow{\ T(\psi)\ } T^2C \xrightarrow{\ \mu_C\ } TC.$$

It is easily checked that this composition law gives a category and that $F:\mathbf{H}\to\mathbf{TH}$ given by

$$F(A \xrightarrow{\phi} B) = A \xrightarrow{\eta_B\phi} TB$$

is a functor. (In fact, by Lemma 2.2, T, η and μ form an idempotent monad, and **TH** is the Kleisli category of this monad—see [13, Chapter 6] for the terminology.)

The main feature of **TH** (and similarly for **TS**) is that the functor $F:\mathbf{H}\to\mathbf{TH}$ is *stable*, that is, $F([e_A]):A\to A\otimes\mathscr{K}$ is an isomorphism for every A. Indeed, the identity *-homomorphism on $A\otimes\mathscr{K}$ determines a morphism in **TH** from $A\otimes\mathscr{K}$ to A which is inverse to $F([e_A])$. Furthermore, if $G:\mathbf{H}\to\mathbf{X}$ is any stable functor then there is a unique functor $\hat{G}:\mathbf{TH}\to\mathbf{X}$ such that $G=\hat{G}\circ F$, namely

$$\hat{G}(A \xrightarrow{[f]} B) = G(A) \xrightarrow{G([f])} G(B\otimes\mathscr{K}) \xrightarrow{G([e_B])^{-1}} G(B).$$

(Thus **TH** is obtained from **H** by inverting the morphisms $[e_A]$.)

The group structure on $\mathbf{TS}(A,B)=\mathbf{S}(A,B\otimes\mathscr{K})$ makes **TS** an additive category and $F:\mathbf{S}\to\mathbf{TS}$ an additive functor.

We are mainly interested in the category **TH**, in the sense that we are interested in functors on C^*-**Alg** which are homotopy invariant and stable. However, the category **TS** fits much more easily into the framework of Section 1, first of all because it is additive, and secondly because it is better suited to analyzing the following standard construction.

2.3. Definition. Let $f:A\to B$ be a *-homomorphism. The *mapping cone* of f is the C^*-algebra

$$C_f=\{a\oplus g\in A\oplus C_0[0,1)\otimes B \mid f(a)=g(0)\}.$$

(We identify $C_0[0,1)\otimes B$ with the C^*-algebra of continuous maps from $[0,1]$ to B which vanish at 1. Note that following standard C^*-algebra usage, '$\oplus$' denotes cartesian product—the product in C^*-**Alg**.)

There is a natural map $p:C_f\to A$ given by $p(a\oplus g)=a$. As far as we are concerned, the principal fact about C_f is the following result.

2.4. Theorem. *For any object D_n of* **TS** *and any $m\in\mathbf{Z}$ the sequences*

$$\mathbf{TS}(B_m,D_n) \xrightarrow{\{f\}^*} \mathbf{TS}(A_m,D_n) \xrightarrow{\{p\}^*} \mathbf{TS}((C_f)_m,D_n)$$

and

$$\mathbf{TS}(D_n,(C_f)_m) \xrightarrow{\{p\}_*} \mathbf{TS}(D_n,A_m) \xrightarrow{\{f\}_*} \mathbf{TS}(D_n,B_m)$$

are exact in the middle.

These sequences are nothing more than pieces of the Puppe exact sequences of algebraic topology. For a proof, see [15, Section 3]. (Actually, [15] deals with **S**, not

TS, and so it is necessary to make the simple observation that the mapping cones for $f: A \to B$ and $f \otimes 1: A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ are related by $C_f \otimes \mathcal{K} \cong C_{f\otimes 1}$.)

2.5. Definition. (i) Denote by Σ^{00} the class of morphisms $\phi: A_m \to A'_{m'}$ in **S** of the form $\phi = \{j\}$, where $j: \Sigma^{m+N}A \to \Sigma^{m'+N}A'$ is an injection of $\Sigma^{m+N}A$ onto an ideal of $\Sigma^{m'+N}A'$, such that the quotient C^*-algebra $\Sigma^{m'+N}A'/\Sigma^{m+N}A$ is contractible.

(ii) Denote by Σ^0 the set of morphisms σ in **TS** of the form $\sigma = F(\sigma_0)$, where $F: \mathbf{S} \to \mathbf{TS}$ is the canonical functor, and $\sigma_0 \in \Sigma^{00}$.

(iii) Denote by Σ the class of morphisms in **TS** generated under composition by Σ^0 and the class of all isomorphisms in **TS**.

We will show that Σ is admissible, in the sense of Definition 1.1. The following is one of two technical C^*-algebra results needed (Lemma 2.8 being the other).

2.6. Lemma. *Let A be an ideal in A' and let B be a stable C^*-algebra (that is, $B \cong B \otimes \mathcal{K}$). Any $*$-homomorphism $f_0: A \to B$ is homotopic to a $*$-homomorphism $f_1: A \to B$ which extends to a $*$-homomorphism from A' into the multiplier algebra $M(B)$.*

Proof. By Lemma 2.1 we may assume that B is of the form $B' \otimes \mathcal{K}$ for some B', and that f_0 is of the form

$$A \xrightarrow{e_A} A \otimes \mathcal{K} \xrightarrow{f_0' \otimes 1} B' \otimes \mathcal{K}.$$

Since e_A extends to a map from A' into $M(A \otimes \mathcal{K})$, it suffices to show that $f_0' \otimes 1: A \otimes \mathcal{K} \to B' \otimes \mathcal{K}$ is homotopic to a map which extends to multiplier algebras. Let $v_1 \in M(A \otimes \mathcal{K})$ be the isometry constructed in Lemma 1.4 of [8]. Then $\mathrm{Ad}(v_1)$ maps $M(A \otimes \mathcal{K})$ into $M(\tilde{A} \otimes \mathcal{K}) \subset M(A \otimes \mathcal{K})$, where $\tilde{A}$ denotes A with a unit adjoined (if A has a unit we could set $\tilde{A} = A$, although the result we are trying to prove is trivial in this case anyway). Now, $f_0' \otimes 1$ extends to a map from $M(\tilde{A} \otimes \mathcal{K})$ to $M(B \otimes \mathcal{K})$ (see [8, paragraph 1.3]) and so it suffices to show that $\mathrm{Ad}(v_1): A \otimes \mathcal{K} \to A \otimes \mathcal{K}$ is homotopic to the identity, for then we can define f_1 to be the composition

$$A \xrightarrow{e_A} A \otimes \mathcal{K} \xrightarrow{\mathrm{Ad}(v_1)} A \otimes \mathcal{K} \xrightarrow{f_0' \otimes 1} B' \otimes \mathcal{K}.$$

But according to [8], $\mathrm{Ad}(v_1)$ is homotopic to $1 \otimes \mathrm{Ad}(w_0)$ for some isometry w_0, and then by Lemma 2.1, $1 \otimes \mathrm{Ad}(w_0): A \otimes \mathcal{K} \to A \otimes \mathcal{K}$ is homotopic to the identity map. $\square$

2.7. Proposition. *The class Σ satisfies property* (L1) *of Definition* 1.1.

Proof. Let $\phi: A_m \to B_n$ be a morphism in **TS**, represented by some $*$-homomorphism

$$f: S^{m+M}A \to S^{n+M}B \otimes \mathcal{K},$$

and let $\sigma = F(\sigma_0)$ be a morphism in Σ^0, where $\sigma_0 : A_m \to A'_{m'}$ is a morphism in Σ^{00}, represented by some *-homomorphism

$$j : S^{m+N}A \to S^{m'+N}A'$$

as in Definition 2.5. By taking suspensions of f or j we may assume that $N = M$ (note that a suspension of j is still an inclusion of an ideal, with contractible quotient). By Lemma 2.6 we may assume that f extends to a *-homomorphism from $S^{m'+N}A'$ to $M(S^{n+N}B \otimes \mathcal{K})$, which we shall call $\tilde{f}$. Let D be the subalgebra of $M(S^{n+N}B \otimes \mathcal{K}) \oplus S^{m'+N}A'/S^{m+N}A$ generated by $S^{n+N}B \otimes \mathcal{K} \oplus 0$ and elements of the form $\tilde{f}(x) \oplus \dot{x}$, where $x \in S^{m'+N}A'$ and $\dot{x}$ denotes the image of x in the quotient algebra (this subalgebra is automatically closed, and so is a C^*-algebra). Under the natural inclusion $i : S^{n+N}B \otimes \mathcal{K} \to D$, $S^{n+N}B \otimes \mathcal{K}$ is an ideal in D; the quotient is isomorphic to $S^{m'+N}A'/S^{m+N}A$, and is therefore contractible. The map i thus gives rise to a morphism $\{i\} : (B \otimes \mathcal{K})_n \to D_{-N}$ in **S** which is an element of Σ^{00}; the map $f' : S^{m'+N}A' \to D$ defined by $x \mapsto f(x) \oplus \dot{x}$ gives rise to a morphism $\{f'\} : A'_{m'} \to D_{-N}$ in **S**. The standard map $e : B \to B \otimes \mathcal{K}$ gives rise to a morphism $\{e\} : B_n \to (B \otimes \mathcal{K})_n$ in **S** which is invertible in **TS**. Since the diagram

$$\begin{tikzcd} A_m \arrow[r, "\phi"] \arrow[d, "\sigma"'] & B_n \arrow[d, "F(\{i\})F(\{e\})"] \\ A'_{m'} \arrow[r, "F(\{f'\})"'] & D_{-N} \end{tikzcd}$$

in **TS** is easily seen to commute (using Lemma 2.1), we are done. □

2.8. Lemma. *If $f : B' \to A'$ is any *-homomorphism then there is a homotopy equivalence $h : B'' \to B'$ such that the composition $f \circ h : B'' \to A$ is homotopic to a surjective *-homomorphism.*

Proof. Take B'' to be the free product $B' * (C_0[0,1) \otimes A)$ (this is the coproduct in C^*-**Alg**), and let $h = 1 * 0 : B'' \to B'$. Then $f \circ h : B'' \to A$ is equal to $f * 0 : B'' \to A$, which is homotopic to the surjection $f * e : B'' \to A$, where $e : C_0[0,1) \otimes A \to A$ is evaluation at $0 \in [0,1)$. □

2.9. Proposition. *The class Σ satisfies property* (R1) *of Definition* 1.1.

Proof. Let $\phi' : B'_{n'} \to A'_{m'}$ be represented by some *-homomorphism

$$f : S^{n'+M}B' \to S^{m'+M}A' \otimes \mathcal{K}$$

and let $\sigma = F(\sigma_0) : A_m \to A'_{m'}$ be a morphism in Σ^0, where $\sigma_0 \in \Sigma^{00}$ is represented by a *-homomorphism

$$j : S^{m+N}A \to S^{m'+N}A'$$

as in Definition 2.5. As in Proposition 2.7, we may assume that $M=N$. By Lemma 2.8 there exists a homotopy equivalence $h:B''\to S^{n'+N}B'$ such that $f\circ h$ is homotopic to a surjective *-homomorphism $g:B''\to S^{m'+N}A'\otimes\mathscr{K}$. Let B be the ideal $g^{-1}[S^{m+N}A\otimes\mathscr{K}]$ of B''. The quotient B''/B is *-isomorphic to $S^{m'+N}A'\otimes\mathscr{K}/S^{m+N}A\otimes\mathscr{K}$ and is therefore contractible. The diagram in **TS**

$$\begin{tikzcd}
A'_{m'} & & B'_{n'} \arrow[ll, "\phi'"'] \\
& & B''_{-N} \arrow[u, "\cong"', "F(\{h\})"] \\
A_m \arrow[uu, "\sigma"] & (A\otimes\mathscr{K})_m \arrow[l, "F(\{e\})^{-1}"] & B_{-N} \arrow[l, "F(\{g|_B\})"] \arrow[u, "F(\{i\})"']
\end{tikzcd}$$

commutes, where $i:B\to B''$ is the inclusion map, and so we are done. $\square$

In order to establish that the properties (L2) and (R2) hold, we need to recall some basic facts concerning mapping cones. For the simple proof of the following lemma, compare for example [15, Section 3]. Note that given a *-homomorphism $f:A\to B$ there is a natural map $s:SB\to C_f$ given by $s(g)=0\oplus g$.

2.10. Lemma. *Given a *-homomorphism of C^*-algebras $h:X\to Y$ the compositions*

$$C_h \xrightarrow{p} X \xrightarrow{h} Y$$

and

$$SX \xrightarrow{Sh} SY \xrightarrow{s} C_h$$

are homotopic to zero. If $h:X\to Y$ is homotopic to zero then $[p]:C_h\to X$ is a split epimorphism in **H** *and $[s]:SY\to C_h$ is a split monomorphism.*

2.11. Proposition. *The class Σ satisfies property* (L2) *of Definition* 1.1.

Proof. Let $\phi:A'_{m'}\to B_n$ be represented by $f:S^{m'+M}A'\to S^{n+M}B\otimes\mathscr{K}$ and let $\sigma=F(\sigma_0):A_m\to A'_{m'}$ be a morphism in Σ given by $j:S^{m+N}A\to S^{m'+N}A'$ as in Definition 2.5. We may assume that $M=N$. Furthermore, if $\sigma\phi=0$, then by choosing N large enough we may assume that the composition fj is homotopic to zero. It follows from Lemma 2.10 that the map $s:S^{n+1+N}B\otimes\mathscr{K}\to C_{fj}$ is a split monomorphism in **H**, and so the map $F(\{s\}):(B\otimes\mathscr{K})_n\to(C_{fj})_{-N-1}$ is a split monomorphism in **TS**. The C^*-algebra C_{fj} is an ideal in C_f (via the embedding $a\oplus g\to a\oplus g$); the quotient is isomorphic to $S^{m'+N}A'/S^{m+N}A$, and is therefore contractible. Also, by Lemma 2.10 the composition

$$S^{m'+1+N}A'\to S^{n+1+N}B\otimes\mathscr{K}\to C_f$$

is homotopic to zero. It follows that $\tau \circ \phi = 0$ in **TS**, where τ is the composition

$$B_n \xrightarrow{F(\{e\})} (B \otimes \mathcal{K})_n \xrightarrow{F(\{s\})} (C_{fj})_{-N-1} \xrightarrow{F(\{i\})} (C_f)_{-N-1}.$$

Since $F(\{e\})$ is an isomorphism, $F(\{s\})$ is a split monomorphism, and $F(\{i\}) \in \Sigma^0$, we have that $\tau \in \Sigma_L$ as required. □

2.12. Proposition. *The class satisfies property* (R2) *of Definition* 1.1.

Proof. Suppose that $\sigma\phi = 0$ in **TS**, where $\phi : A_m \to B_n$ is represented by $f : S^{m+M}A \to S^{n+M}B \otimes \mathcal{K}$ and $\sigma \in \Sigma$, $\sigma : B_n \to B'_{n'}$, is given by $j : S^{n+N}B \to S^{n'+N}B'$. We may assume that $M = N$ and that the composition

$$S^{m+N}A \xrightarrow{f} S^{n+N}B \otimes \mathcal{K} \xrightarrow{j \otimes 1} S^{n'+N}B' \otimes \mathcal{K}$$

is homotopic to zero. By Lemma 2.10 the morphism $[p] : C_{(j\otimes 1)f} \to S^{m+N}A$ is a split epimorphism in **H**, and hence $F(\{p\}) : (C_{(j\otimes 1)f})_{-N} \to A_m$ is a split epimorphism in **TS**. On the other hand, the composition

$$C_f \to C_{(j\otimes 1)f} \to S^{m+N}A \xrightarrow{f} S^{n+N}B \otimes \mathcal{K}$$

is homotopic to zero (by Lemma 2.10 again). Also, C_f is included in $C_{(j\otimes 1)f}$ as an ideal; the quotient is isomorphic to $(S^{n'+1+N}B' \otimes \mathcal{K})/(S^{n+1+N}B \otimes \mathcal{K})$, and is hence contractible. Thus if τ is the composition

$$(C_f)_{-N} \xrightarrow{F(\{i\})} (C_{(j\otimes 1)f})_{-N} \xrightarrow{F(\{p\})} A_m$$

then $\tau \in \Sigma_R$ and $\phi\tau = 0$. □

3. *KK*-theory

3.1. Definition. Let **E** be the full subcategory of the category of fractions $\mathbf{TS}[\Sigma^{-1}]$ (with Σ as in Definition 2.5) whose objects are the objects A_0 in **TS**.

We shall identify the objects A_0 of **E** with the class of separable C^*-algebras A. There is of course an obvious functor from C^*-**Alg** to **E**, mapping A_0 to A.

Putting together the results of Sections 1 and 2 we obtain the following result.

3.2. Theorem. *If* $0 \to J \to^{j} A \to^{q} B \to 0$ *is an exact sequence of separable* C^**-algebras then for any separable* C^**-algebra* C *the sequences of abelian groups*

$$\mathbf{E}(B, C) \xrightarrow{q^*} \mathbf{E}(A, C) \xrightarrow{j^*} \mathbf{E}(J, C)$$

and

$$\mathbf{E}(C, J) \xrightarrow{j_*} \mathbf{E}(C, A) \xrightarrow{q_*} \mathbf{E}(C, B)$$

are exact in the middle.

Proof. Since Σ is an admissible class in **TS** (by Propositions 2.7, 2.9, 2.11 and 2.12) it follows from Theorems 1.9 and 2.4 that these sequences would be exact if $j: J \to A$ were replaced with $p: C_q \to A$. There is an inclusion $i: J \to C_q$, of J as an ideal in C_q, namely $i(a) = a \oplus 0$, such that the $p \circ i = j: J \to A$, and so it suffices to show that i maps to an isomorphism in $\mathbf{E}$. But the quotient C_q/J is isomorphic to $C_0[0,1) \otimes B$ and is hence contractible. Therefore, by definition of Σ, i is invertible in $\mathbf{E}$. □

3.3. Remark. Theorem 3.2 asserts that the functors $\mathbf{E}(C, \cdot)$ and $\mathbf{E}(\cdot, C)$ on C^*-**Alg** are *half-exact*. They are in addition stable and homotopy invariant and so by Cuntz's Bott Periodicity Theorem [3, Theorem 4.4], there is a natural six term exact sequence

$$\begin{tikzcd} & \mathbf{E}(C,A) \arrow[r, "q_*"] & \mathbf{E}(C,B) \arrow[dr, "\partial"] & \\ \mathbf{E}(C,J) \arrow[ur, "j_*"] & & & \mathbf{E}(C,SJ) \arrow[dl, "Sj_*"] \\ & \mathbf{E}(C,SB) \arrow[ul, "\partial"] & \mathbf{E}(C,SA) \arrow[l, "Sq_*"] & \end{tikzcd}$$

as well as a similar sequence for $\mathbf{E}(\cdot, C)$ (these are natural with respect to change of the exact sequences, as well as natural in C).

Now, denote by **KK** the category obtained from Kasparov's KK-theory as described in the Introduction. The functor $C^*\text{-}\mathbf{Alg} \to \mathbf{KK}$ is characterized in [8] as follows.

3.4. Theorem. *Let* $\mathbf{A}$ *be an additive category and let* $F: C^*\text{-}\mathbf{Alg} \to \mathbf{A}$ *be a functor with the following properties*:

(i) *F is a homotopy functor*;

(ii) *F is stable*; *and*

(iii) *if* $0 \to J \to^{j} A \leftrightarrows^{s}_{p} B \to 0$ *is a split exact sequence of* C^**-algebras then* $F(A)$ *is the direct sum of* $F(J)$ *and* $F(B)$ *via the maps* $F(j)$ *and* $F(s)$.

There exists a unique functor $\hat{F}: \mathbf{KK} \to \mathbf{A}$ *such that* F *is the composition* $C^*\text{-}\mathbf{Alg} \to \mathbf{KK} \to^{\hat{F}} \mathbf{A}$. (*Furthermore, the functor* $C^*\text{-}\mathbf{Alg} \to \mathbf{KK}$ *itself has properties* (i), (ii) *and* (iii).)

There is a suspension functor $S: \mathbf{KK} \to \mathbf{KK}$,

$$S(A \xrightarrow{\phi} B) = SA \xrightarrow{S\phi} SB,$$

which is compatible with the suspension functor on C^*-**Alg** (see for example [8, Section 4.7], where S is denoted $1_{C_0(0,1)} \boxtimes -$). It follows from the Bott Periodicity Theorem in KK-theory [11, Section 5] that $S: \mathbf{KK} \to \mathbf{KK}$ is full and faithful. Using this, it is a simple matter to compare **KK** and **E**.

3.5. Theorem. *There is a unique (additive) functor* **KK** $\to$ **E** *such that the diagram*

$$\begin{tikzcd} & C^*\text{-}\mathbf{Alg} \arrow[dl] \arrow[dr] & \\ \mathbf{KK} \arrow[rr] & & \mathbf{E} \end{tikzcd}$$

commutes. If A is a K-nuclear C^*-algebra then for any B the homomorphism $KK(A,B) \to \mathbf{E}(A,B)$ is an isomorphism.

Proof. The existence and uniqueness of a functor from **KK** to **E** follows from Theorem 3.4 and Remark 3.2, since the six term exact sequences for a split exact sequence degenerate into split exact sequences of **E**-groups, and these imply that **E** satisfies condition (iii) of Theorem 3.4, whilst the other conditions are obviously satisfied. If A is *any* C^*-algebra then we may define a functor K_A from **S** into abelian groups as follows. On objects we set

$$K_A(B_m) = \varinjlim_N KK(S^N A, S^{m+N} B), \tag{7}$$

where the direct limit is taken using the suspension functor. A morphism $\{f\}$: $B_m \to C_n$ in **S** induces homomorphisms $KK(S^N A, S^{m+N}B) \to KK(S^N A, S^{n+N}C)$, for large enough N, compatible with suspension, and so we obtain a homomorphism $K_A(\{f\}) : K_A(B_m) \to K_A(C_n)$.

Since **KK** is stable it follows that K_A passes to a functor from **TS** to abelian groups (which we will also call K_A). Suppose now that A is K-nuclear (see [17, Definition 3.1]). Then all suspensions $S^m A$ of A are K-nuclear ([17, Proposition 3.5]), and the functors $KK(S^m A, -)$ on C^*-**Alg** are half-exact by [17, Theorem 3.6]. It follows from the resulting long exact sequence that if $0 \to B \to B' \to B'/B \to 0$ is a short exact sequence of C^*-algebras with B'/B contractible then the map $KK(S^m A, B) \to KK(S^m A, B')$ is an isomorphism. Therefore the functor K_A passes to a functor on $\mathbf{TS}[\Sigma^{-1}]$ (with Σ as in Definition 2.5) and so by restricting this we obtain a functor K_A on the category **E**. Since the suspension functor $S : \mathbf{KK} \to \mathbf{KK}$ is fully faithful, the direct limit in (7) is superfluous for objects in **E**: we have $K_A(B) = \mathbf{KK}(A,B)$. Denote by E_A the functor $\mathbf{E}(A,-)$ on **E** or **KK**. The functor $\mathbf{KK} \to \mathbf{E}$ gives a natural transformation $\tau : K_A \to E_A$, and we may define a natural transformation $\sigma : E_A \to K_A$ (considering E_A and K_A as functors on **E** or **KK**) by

$$\sigma_A(\phi) = K_A(\phi)(1_A) \in KK(A,B),$$

where $1_A \in E_A(A) = E(A,A)$ is the identity morphism. It follows from the Yoneda Lemma that σ and τ are mutual inverses. $\square$

We shall conclude by outlining a characterization of **E**, analogous to the characterization of **KK** in Theorem 3.4.

3.6. Theorem. *Let F be any functor from C^*-**Alg** to an additive category* **A** *which satisfies the following conditions*:

(i) *F is a homotopy functor*;

(ii) *F is stable*; *and*

(iii) *if $0 \to J \to A \to B \to 0$ is any short exact sequence of C^*-algebras, and C is any other C^*-algebra, then the sequences of abelian groups*

$$\mathbf{A}(B,C) \xrightarrow{q^*} \mathbf{A}(A,C) \xrightarrow{j^*} \mathbf{A}(J,C)$$

and

$$\mathbf{A}(C,J) \xrightarrow{j_*} \mathbf{A}(C,A) \xrightarrow{q_*} \mathbf{A}(C,B)$$

are exact in the middle.

There exists a unique functor $\hat{F}: \mathbf{E} \to \mathbf{A}$ such that F is equal to the composition C^*-$\mathbf{Alg} \to \mathbf{E} \to^{F} \mathbf{A}$.

By Theorem 3.2 and the properties of **TS**, the functor C^*-**Alg** $\to$ **E** satisfies properties (i), (ii) and (iii). Thus **E** is characterized by Theorem 3.6 as the universal stable cohomology theory on C^*-**Alg**. In distinction to Theorem 3.4, it is *not* obvious that such a universal theory exists.

Proof. By Cuntz's Bott Periodicity Theorem [3, Theorem 4.4], there are natural isomorphisms $\mathbf{A}(F(S^2A), F(B)) \cong \mathbf{A}(F(A), F(B)) \cong \mathbf{A}(F(A), F(S^2B))$ (where we regard $\mathbf{A}(F(A), F(B))$, etc., as a bifunctor on C^*-**Alg**). By property (i), we may regard F as a functor on **H**, and by means of these isomorphisms we may construct from F a functor on **S**. By property (ii) we obtain from this a functor on **TS**. Property (iii) implies that associated to any short exact sequence of C^*-algebras,

$$0 \to J \xrightarrow{j} A \xrightarrow{q} B \longrightarrow 0,$$

there are six term exact sequences in the C and D variables of $\mathbf{A}(F(C), F(D))$. If B is contractible then these exact sequences degenerate to isomorphisms

$$j^*: \mathbf{A}(F(A), F(D)) \to \mathbf{A}(F(J), F(D))$$

and

$$j_*: \mathbf{A}(F(C), F(J)) \to \mathbf{A}(F(C), F(A)).$$

It follows that $f(j): F(J) \to F(A)$ is an isomorphism, and so $F(\sigma)$ is an isomorphism for any σ in the class Σ of Definition 2.5. Thus F passes to a functor on $\mathbf{TS}[\Sigma^{-1}]$, and hence to a functor $\hat{F}$ on **E**. Uniqueness follows from the fact that **E** is generated by the image of $\mathbf{KK} \to \mathbf{E}$ (on which $\hat{F}$ is unique by Theorem 3.4) together with the inverses of certain morphisms in this image. □

As an application, using the approach of [8, Section 4.7] it is straightforward to show the existence of an associative pairing

$$\mathbf{E}(A_1, B_1) \otimes \mathbf{E}(A_2, B_2) \to \mathbf{E}(A_1 \otimes A_2, B_1 \otimes B_2),$$

as long as A_1 and B_1 (or A_2 and B_2) are nuclear, or more generally K-nuclear. In the absence of some nuclearity condition the situation is unclear, the reason being that in general the tensor product

$$0 \to C \otimes J \to C \otimes A \to C \otimes B \to 0$$

of an exact sequence with an auxiliary C^*-algebra C is not necessarily exact.

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