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A Primer on KK-Theory

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Introduction. The purpose of this article is to acquaint the reader with G. G. Kasparov's KK-theory for C^*-algebras. I have tried to avoid writing for the C^*-algebra specialist, and I hope that the paper will serve as a useful introduction to the subject for mathematicians from other areas, particularly those with some familiarity, or at least interest in, the index theory of elliptic operators.

The prerequisites for reading the article are roughly as follows. Not much C^*-algebra theory beyond the very basics (say, the first chapter of [**38**]) is needed. One or two additional topics are rapidly introduced at the beginning of Section 1. (Of course, to develop all the details of the theory somewhat more background in functional analysis is required.) I shall assume that the reader is familiar with K-theory for compact spaces. The first two chapters of Atiyah's book [**1**], together with the appendix, cover the necessary material. In Section 2 of this paper I shall outline the basic properties of C^*-algebra K-theory, using [**1**] as a guide of what to expect. The other main requirement is some knowledge of the theory of pseudodifferential operators, as it relates to index theory for elliptic operators. The reader familiar with, say, Sections 5 and 6 of [**4**] will have all the necessary background. Pseudodifferential operators will not make any explicit appearance in the theory we develop (they appear only in the examples), but all of the techniques are motivated by the constructions involving pseudodifferential operators which appear in the proof of the Atiyah-Singer Index Theorem. In fact one can regard KK-theory as a generalization of these constructions to situations where the underlying space is no longer a smooth closed manifold but is some sort of "singular space," represented by a C^*-algebra (say, for example, the C^*-algebra of a discrete group, representing the space of representations of the group).

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The key idea in KK-theory is that (families of) elliptic operators give rise to maps between K-theory groups. This basic principle is due to Atiyah (see for example [2] and [3]), and has been broadly extended by Kasparov to what might be called "generalized elliptic operators". The construction alone is a very powerful tool in the computation of K-theory groups, but the full power of Kasparov's theory lies in a remarkable "composition formula" for generalized elliptic operators. Thus given generalized operators F_1 and F_2, giving rise to maps $K(A) \to K(B)$ and $K(B) \to K(C)$ in K-theory, Kasparov gives a formula for an operator $F_1 \# F_2$ which induces the composition map $K(A) \to K(C)$. The formula for $F_1 \# F_2$ is rather complicated, but it is nevertheless possible to work it out in many important cases, and so to deduce results in K-theory. For example, in many situations one has $A = C$, and by computing $F_1 \# F_2$ and $F_2 \# F_1$, one can show that the maps between $K(A)$ and $K(B)$ induced by F_1 and F_2 are inverse to one another. The bulk of the article is devoted to explaining these things. In Section 3 I give a description of K-theory (which is also due to Kasparov) in terms of "generalized Fredholm operators." This is a fairly simple context in which to introduce the basic technical devices in Kasparov's theory (Hilbert modules, Fredholm operators, tensor products). Following this, Kasparov's generalized notion of elliptic operator is given in Section 4, and the product construction is described in Section 5. In Section 6 I briefly describe the theory for selfadjoint operators.

My lecture at the New Hampshire meeting was concerned with the homological properties of KK-theory, and in particular with what Paul Baum calls my "polemic against KK-theory." The point, roughly speaking, is that although Kasparov's theory is a powerful calculus for solving questions in K-theory, there are some important applications in which it fails. Some discussion of this, along with speculation on the possibility of improving Kasparov's theory, is given in Section 7. I have also included a brief discussion on how to characterize KK-theory from a homological point of view, since this is a question often asked by newcomers to the subject.

Although it is not possible to include many details in an article such as this (and it would not really be desirable to do so anyway), I have tried to write a paper which could be used as a framework for learning the subject fully. Results marked as "Propositions" are more or less straightforward, and the interested reader can probably find the proofs for himself or herself. In any case, references are given for these as well as the harder results (labelled "Theorems"). I have included a number of examples of the basic constructions of Sections 3, 4, and 5; my hope is that the reader who works through these will come to have a good grasp of what the theory is about. However, this is only a primer, and in particular I have not discussed in any detail at all any of the serious applications of the KK-machinery. The notes in Section 7 are intended to guide the reader who wants to venture into the literature.

I would like to thank the great many people with whom I have had discussions and seminars on KK-theory. In particular I thank the participants

(victims) of the seminar on KK-theory at the University of Warwick in the Fall of 1986, especially my host David Evans. This article is, in parts, a revision of the notes [23] produced from those lectures.

1. C^*-algebras. There are one or two constructions involving C^*-algebras which will come up in the course of these notes. Although they are quite elementary they go beyond the basics of the subject as mentioned in the introduction, and so for the reader's benefit we shall say something about them here. Following these brief remarks we will describe some of the families of C^*-algebras which are of interest from the point of view of K-theory and its applications. We should mention right away that all of the C^*-algebras A, B, etc., that we consider will be assumed to be *separable*. This is a necessary assumption at one or two points later on in the theory.

1.1 *Tensor products.* The tensor product $A_1 \otimes A_2$ of two C^*-algebras (to be precise, the minimal tensor product) is constructed as follows. Represent A_1 and A_2 as algebras of operators on Hilbert spaces H_1 and H_2. There is then a representation of the algebraic tensor product $A_1 \odot A_2$ on the Hilbert space tensor product $H_1 \otimes H_2$, and $A_1 \otimes A_2$ denotes the completion with respect to the operator norm on $H_1 \otimes H_2$. This is independent of the initial choice of representations (see for example [31] for details). If $A_1 = C_0(X)$ then $A_1 \otimes A_2$ is isomorphic to the C^*-algebra of continuous functions on X, with values in A_2, vanishing at infinity. If $A_1 = C_0(X_1)$ and $A_2 = C_0(X_2)$ then $A_1 \otimes A_2$ is isomorphic to $C_0(X_1 \times X_2)$. We shall frequently make these identifications.

1.2 *Multiplier algebras.* If A is any C^*-algebra then the *multiplier algebra* of A, denoted $M(A)$, is the C^*-algebra which is characterized by the following two properties:

(i) $M(A)$ contains A as a (closed, two-sided) ideal; and
(ii) if B is any C^*-algebra containing A as an ideal then there is a unique *-homomorphism from B to $M(A)$ which is the identity on A.

To obtain $M(A)$ more concretely, faithfully, and nondegenerately, represent A as operators on a Hilbert space H; then $M(A)$ is (canonically isomorphic to) the largest C^*-algebra of operators on H which contains A as an ideal. (When we say that an operator "determines" an element of $M(A)$, we shall mean via this characterization.)

Here are the two basic examples. If $A = C_0(X)$ then $M(A)$ is isomorphic to the C^*-algebra of all bounded continuous functions on X. If A is the C^*-algebra of compact operators on a Hilbert space then $M(A)$ is the C^*-algebra of *all* Hilbert space operators. It is generally useful (in K-theory at least) to think of elements of $M(A)$ as being "bounded" operators in some sense, and elements of A as being "compact" (compare for example Paragraph 3.14 below). The C^*-algebras $M(A)$ violate the one rule we have made so far: they are not separable. See [9] and [38] for further details.

1.3 *Stable C^*-algebras.* Denote by $\mathscr{K}$ the C^*-algebra of compact operators on a separable Hilbert space. A C^*-algebra A is said to be *stable* if it is isomorphic to $A \otimes \mathscr{K}$. Since it is easily checked that $\mathscr{K} \otimes \mathscr{K} \cong \mathscr{K}$, every C^*-algebra of the form $A \otimes \mathscr{K}$ is stable. The process of passing from A to $A \otimes \mathscr{K}$ is often referred to as *stabilization*. This turns out to be an operation of intrinsic importance in K-theory. (It is also very closely related to a C^*-algebraic notion of Morita equivalence, which in turn is of importance in analysing the structure of C^*-algebras, particularly those C^*-algebras related to actions of groups on spaces. We shall not make any use of this here, but the reader is referred to [39] for a survey.)

1.4 *Homotopy.* We shall use the following standard notion of homotopy for C^*-algebras. Two *-homomorphisms $f_0, f_1 \colon A \to B$ are said to be *homotopic* if there exists a family of *-homomorphisms $f_t \colon A \to B$ $(t \in [0,1])$, connecting f_0 and f_1, which is continuous in the sense that for each $a \in A$ the map $t \mapsto f_t(a)$ is norm-continuous. This agrees with the usual notion of homotopy if $A = C(X)$ and $B = C(Y)$ (recall that a *-homomorphism from A to B corresponds here to a continuous map from Y to X).

These brief preliminaries dispensed with, we turn to some examples.

1.5 *Families of smoothing operators.* Let X be a smooth, closed manifold and denote by $\mathscr{L}$ the algebra of smoothing operators acting on $L^2(X)$ (we must specify some smooth measure on M). These are operators of the form

$$Kf(x) = \int_X k(x,x')f(x')\,dx'$$

where $k(x,x')$ is a smooth function on $X \times X$. The algebra $\mathscr{L}$ is not complete (in the operator norm on $L^2(X)$). The completion is nothing more than the C^*-algebra of compact operators on $L^2(X)$.

We can generalize this example to include a parameter space, as follows. Let Z be a compact fibre bundle with base space Y and fibre a smooth closed manifold X (as described in [5]). Denote by G the "graph" of Z:

$$G = \{(z_1, z_2) \in Z \times Z \,|\, \pi(z_1) = \pi(z_2)\},$$

where $\pi \colon Z \to Y$ is the projection onto the base space. A smooth function k on G corresponds to a smooth family of smooth kernels $\{k_y(x,x') \mid y \in Y\}$ on the fibres of Z, from which we obtain a family $K = \{K_y \mid y \in Y\}$ of smoothing operators, acting on the family of Hilbert spaces $\{L^2(X_y) \mid y \in Y\}$. These families form an algebra, and the completion of it with respect to the norm

$$\|K\| = \sup_{y \in Y} \|K_y\|$$

is a C^*-algebra $\mathscr{K}_Y$. It might at first seem that $\mathscr{K}_Y$ depends in some interesting way on the topology of the fibration, but this is not so: $\mathscr{K}_Y$ is in fact *-isomorphic to the C^*-algebra $C(Y) \otimes \mathscr{K}$. The basic reason is that the bundle of Hilbert spaces $\{L^2(X_y) \mid y \in Y\}$ over Y is trivializable (the appropriate

structure group is the group of unitary Hilbert space operators in the *strong* operator topology, and this is easily seen to be contractible). The details are left as an interesting exercise for the reader.

The C^*-algebra $\mathcal{K}_Y$ is of interest from the point of view of index theory because if $D = \{D_y \mid y \in Y\}$ is a continuous family of order zero elliptic pseudodifferential operators as in [5, Section 1] then D determines an element of the multiplier algebra of $\mathcal{K}_Y$ (as can be seen by representing $\mathcal{K}_Y$ on the large direct sum $\oplus_{y \in Y} L^2(X_y)$). In fact, by virtue of ellipticity, D is invertible modulo $\mathcal{K}_Y$, in the sense that there exists a "parametrix" $Q \in M(\mathcal{K}_Y)$ such that $1 - QD \in \mathcal{K}_Y$ and $1 - DQ \in \mathcal{K}_Y$. As we shall see in the next section, it follows that D determines an element of the C^*-algebra K-theory group $K(\mathcal{K}_Y)$. We shall also see that since $\mathcal{K}_Y \cong C(Y) \otimes \mathcal{K}$, the K-theory group $K(\mathcal{K}_Y)$ is isomorphic to the topological K-theory group $K(Y)$; the element of $K(Y)$ so obtained from D is the analytic families index of $D = \{D_y\}$ as in [5].

In this example, and in the next two, we could equally well consider operators acting on sections of bundles, rather than scalar differential operators, by replacing L^2-spaces of functions with L^2-spaces of sections of the vector bundles. The C^*-algebras involved would be the same, up to isomorphism (canonical at the level of K-theory).

1.6 *Smoothing operators and foliations.* There is a generalization of the above construction which leads to rather more interesting C^*-algebras. (The reader is referred to **[10]** for details as well as to the recent survey article **[42]**.) Let M be a compact foliated manifold, and for simplicity let us suppose that the foliation has no holonomy (see **[20]**; for example, this is so if M is foliated by the orbits of a free action of a connected Lie group, or if the foliation is analytic). The graph G of the foliation, consisting of all pairs $(m, m') \in M \times M$ such that m and m' are in the same leaf $L \in M/\mathcal{F}$ of the foliation, has a natural smooth manifold structure. A smooth, compactly supported function k on G gives rise to a family of operators $K = \{K_L \mid L \in M/\mathcal{F}\}$ acting on the family of Hilbert spaces $\{L^2(L) \mid L \in M/\mathcal{F}\}$ by the formula

$$K_L f(m) = \int_{m' \in L} k(m, m') f(m')\, dm'.$$

(We are supposing that we have fixed a smoothly varying (in the appropriate sense) family of smooth measures on the leaves of the foliation.) These families form an algebra, and the completion with respect to the norm $\|K\| = \sup_L \|K_L\|$ is a C^*-algebra, denoted $C^*(M, \mathcal{F})$. It is the foliation C^*-algebra of A. Connes. If the foliation is obtained from a fibre bundle as in Paragraph 1.2, then the graph G is exactly the graph of the fibre bundle as above and $C^*(M, \mathcal{F})$ is equal to the C^*-algebra $\mathcal{K}_Y \cong C(Y) \otimes \mathcal{K}$. However, if the leaves L are not closed subsets of M, then the operators K_L will no longer be necessarily compact, and more interesting C^*-algebras arise. For example, in the case of the Kronecker foliation of the Torus $\mathbb{R}^2/\mathbb{Z}^2$ by lines of irrational

slope θ, $C^*(M,\mathcal{F})$ is $*$-isomorphic to $A_\theta \otimes \mathcal{K}$ where A_θ is the C^*-algebra generated by two unitaries U and V which satisfy the relation $UV = e^{2\pi i\theta} VU$ (this is called the "irrational rotation algebra").

The C^*-algebra $C^*(M,\mathcal{F})$ plays a role in index theory analogous to the role of $\mathcal{K}_Y$ above. Thus an order zero pseudodifferential operator D operating in the leaf direction and leafwise elliptic (see **[10]**) determines an element of the multiplier algebra of $C^*(M,\mathcal{F})$ which is invertible modulo $C^*(M,\mathcal{F})$, and so determines an element of the K-theory group $K(C^*(M,\mathcal{F}))$. This is the analytic index of D; the Longitudinal Index Theorem of **[11]** gives a topological formula for it in the spirit of the proof of the Index Theorem in **[4]**.

1.7 *Invariant smoothing operators.* Suppose that N is a Galois covering of a compact manifold M, with covering group π. Consider the C^*-algebra of operators on the Hilbert space $L^2(N)$ generated by operators given by kernel functions $k(x,x')$ which are (i) π-invariant: $k(gx,gx') = k(x,x')$ for $g \in \pi$; and (ii) supported within a finite distance of the diagonal in $N \times N$ (as measured by some π-invariant metric on N). It turns out that this C^*-algebra is $*$-isomorphic to $C_r^*(\pi) \otimes \mathcal{K}$, where $C_r^*(\pi)$ denotes the completion of the complex group algebra $\mathbb{C}\pi$, considered as an algebra of operators on the Hilbert space $\ell^2\pi$ (this is the "reduced C^*-algebra of π"). An order zero elliptic operator D on M lifts to a π-invariant operator on N, and in a manner similar to the above examples, D determines an element of the K-theory group $K(C_r^*(\pi) \otimes \mathcal{K}) \cong K(C_r^*(\pi))$.

Taking for example N to be the universal covering space of M, we see that an elliptic operator on M has not only an integer-valued index but also an "index" in the group $K(C_r^*(\pi_1 M))$. The operation of assigning to an elliptic symbol the index of the associated elliptic operator gives a homomorphism

$$\mathrm{Index}_\pi \colon K(T^*M) \to K(C_r^*(\pi_1 M))$$

(compare the construction of the analytic index map $K(T^*M) \to \mathbb{Z}$ in **[4]**). The analysis of this construction is one of the most important applications of Kasparov's KK-theory, for the reasons we shall indicate at the end of the next paragraph.

1.8 *Vector A-bundles.* Apart from arising as algebras of smoothing operators, there is an interesting geometric construction which more directly involves C^*-algebras (it is however closely related to the discussion above). Let A be a unital C^*-algebra. A vector A-bundle over a space X is a locally trivial bundle of finitely generated projective right A-modules (the structure group is the group of A-module automorphisms of the fibre). One should think of such a bundle as being a sort of generalized vector bundle, where the scalars are now elements of A instead of complex numbers. Their theory is worked out by A. Mishchenko and A. Fomenko in **[36]** (see also **[35]** and **[25]**). In particular, if E_1 and E_2 are smooth vector A-bundles over a smooth closed

manifold then there is a natural notion of "pseudodifferential A-operator"

$$D\colon \Gamma^\infty(E_1) \to \Gamma^\infty(E_2),$$

mapping smooth sections of E_1 to smooth sections of E_2 (reducing to the usual notion if $A = \mathbb{C}$). An elliptic pseudodifferential A-operator has an index in the group $K(A)$.

Here is the basic example in the theory. Let $A = C_r^*(\pi_1 M)$ and let $\pi = \pi_1 M$ act on A by left multiplication; we can then form the bundle

$$E_\pi = \widetilde{M} \times_\pi A$$

($\widetilde{M}$ is the universal covering space and E_π is the quotient of $\widetilde{M} \times A$ by the diagonal action of π). This is a vector A-bundle over M; in fact E_π is a *flat* bundle over M. Whilst flat vector bundles tend to be topologically trivial, this "infinite-dimensional" bundle is definitely nontrivial, and we can use it to construct in a different way the K-theory homomorphism Index_π of the previous paragraph. Indeed, starting with an elliptic operator D on M we can lift D to act on sections of E_π, and taking the index of this elliptic pseudodifferential $C_r^*(\pi)$-operator we obtain the element $\mathrm{Index}_\pi(D)$ of $K(C_r^*(\pi_1 M))$.

Now, if π is any discrete group then there is a canonical map

$$\beta\colon K_0(B\pi) \to K(C_r^*(\pi))$$

from the K-homology of the classifying space $B\pi$ to the K-theory group of $C_r^*(\pi)$, introduced by Kasparov [26] and Mishchenko [35]. If $B\pi$ is a manifold M then this is related to the map Index_π by a Poincaré duality isomorphism between $K_0(M)$ and $K^0(T^*M)$. There are various conjectures about this map, of which we shall mention two:

> *Strong Novikov Conjecture.* The map β is injective, modulo torsion.
> *Baum-Connes Conjecture.* If π is torsion free then the map β is an isomorphism.

See [29], [6]. The Strong Novikov conjecture would imply the generalized Novikov conjecture on higher signatures [29], [30], whilst the Baum-Connes Conjecture would have other consequences as well [6], [42].

For an interesting application of these ideas to geometry, which uses in a direct way the flatness of E_π, see [40].

2. K-theory For C^*-algebras. We shall begin by introducing K-theory in a way which is ultimately not the most useful in the study of elliptic operators, but which is well suited to establishing the basic features of the K-theory groups. Elegant and concise proofs of most of the propositions of this section can be found in the survey article [15] of J. Cuntz. See also [7].

Let A be a C^*-algebra and denote by $M_k(A)$ the C^*-algebra of $k \times k$ matrices over A. We shall regard $M_k(A)$ as the subalgebra of $M_{k+1}(A)$ consisting of those matrices whose $(k+1)$st column and row are zero, and we shall denote the direct limit $\bigcup_k M_k(A)$ by $M_\infty(A)$.

2.1. DEFINITION. Let A be a C^*-algebra with unit. The group $K(A)$ (also denoted $K_0(A)$) is the abelian group generated by the symbols $[p]$, where p is a projection in the algebra $M_\infty(A)$ (that is, a selfadjoint idempotent), subject to the following relations:

(i) $[p] = [q]$ if there exists an element v of $M_\infty(A)$ such that $p = vv^*$ and $q = v^*v$; and

(ii) if $pq = 0$ then $[p] + [q] = [p + q]$.

If A does not have a unit then we denote by $A^\sim$ the C^*-algebra obtained from A by adjoining a unit, and we define $K(A)$ to be the kernel of the homomorphism $K(A^\sim) \to K(\mathbb{C})$ induced from the *-homomorphism $A \to \mathbb{C}$ which maps the unit of $A^\sim$ to $1 \in \mathbb{C}$.

2.2. REMARKS. (a) Since in a C^*-algebra, every idempotent is similar to a projection, we could have defined $K(A)$ using these instead of projections.

(b) If A has a unit then $K(A)$ as defined above is isomorphic to the algebraic K_0-group of A, that is, the Grothendieck group of the category of finitely generated projective right A-modules. Indeed, if p is a projection in $M_\infty(A)$ then $pM_\infty(A)$ is in a natural way a right A-module, and it is finitely generated and projective. Part (i) of Definition 2.1 corresponds to isomorphism of modules, whilst part (ii) corresponds to direct sum of modules. The reader can easily supply the details for himself or herself.

By (b) and by Swan's Theorem, if Y is a compact space then $K(C(Y))$ is naturally isomorphic to the Atiyah-Hirzebruch K-theory group $K(Y)$. Taking this as a guide. we shall develop the basic properties of C^*-algebraic K-theory in parallel with, say, [**1**] (actually it is in some respect easier to work with C^*-algebras and projections than spaces and vector bundles).

First of all, since the image under a *-homomorphism of a projection is a projection, $K(A)$ is a covariant functor from C^*-algebras to abelian groups.

2.3. PROPOSITION. *The functor K is* homotopy invariant. *Thus if $f_0, f_1 : A \to B$ are homotopic *-homomorphisms then*

$$f_{0*} = f_{1*} : K(A) \to K(B) \quad \square$$

2.4. PROPOSITION. *Let J be a (closed, two sided) ideal in A. The functor K is* half-exact, *meaning that the sequence of abelian groups*

$$K(J) \to K(A) \to K(A/J)$$

is exact in the middle. $\square$

These are extensions of Lemmas 1.4.3 and 2.4.2 of [**1**]. Following Section 2.4 of [**1**], for $n \geq 0$ we define groups $K_n(A)$ by

$$K_n(A) = K(C_0(\mathbb{R}^n) \otimes A).$$

As in K-theory of spaces, using the above two results we can link the groups K_n together in a long exact sequence.

2.5. PROPOSITION. *Let J be an ideal in A. There is a natural long exact sequence*

$$\cdots \to K_{n+1}(A) \to K_{n+1}(A/J) \to K_n(J) \to K_n(A) \to \cdots$$

(*where $n \geq 0$*). □

Continuing the development of C^*-algebra K-theory in parallel with topological K-theory, we come next to the Bott Periodicity Theorem. It is in fact rather a straightforward matter to extend to extend the proof of the Periodicity Theorem in [**1**, Section 2.2] to the present context (see for example [**7**, Section 8]). We obtain:

2.6. THEOREM. *There is a natural isomorphism $K_n(A) \cong K_{n+2}(A)$.* □

There is however a beautiful extension of the Bott Periodicity Theorem, due to J. Cuntz [**13**], which is remarkable not only for its generality but also for the simplicity of its proof. In order to state it we need to make note of one further property of C^*-algebra K-theory. If e is a rank one projection in the C^*-algebra $\mathscr{K}$ of compact operators then denote by $e_A: A \to A \otimes \mathscr{K}$ the $*$-homomorphism $a \mapsto a \otimes e$.

2.7. DEFINITION. A functor F on C^*-algebras is said to be *stable* if the morphism $F(e_A)$ is an isomorphism for every A.

2.8. PROPOSITION. *The functor K is stable.* □

This is easily proved from the fact that $A \otimes \mathscr{K}$ is the completion of $M_\infty(A)$ with respect to the (unique) C^*-norm on this algebra.

2.9. THEOREM. *If E is* any *functor from C^*-algebras to abelian groups which is homotopy invariant, half-exact, and stable* (*as in* 2.3, 2.4, *and* 2.8) *then $E(A)$ is naturally isomorphic to $E(C_0(\mathbb{R}^2) \otimes A)$.* □

The proof of this in [**13**] is closely related to, but not exactly the same as, the proof of the Bott Periodicity Theorem in [**2**]. In fact it is really even simpler than this proof. It should be mentioned that it is also closely related to the constructions of the next several sections, via the notion of quasihomomorphism described in Section 7 below.

As a result of the Bott Periodicity Theorem, the long exact sequence of Proposition 2.5 becomes a cyclic six-term exact sequence, and the functors K_0 and K_1 form a sort of $\mathbb{Z}/2$-graded homology theory on the category of C^*-algebras. As with Atiyah-Hirzebruch K-theory, there is a useful concrete description of $K_1(A)$.

2.10. PROPOSITION. *Let A be a C^*-algebra with unit and denote by $U_k(A)$ the topological group of unitary $k \times k$ matrices over A. The group $K_1(A)$ is naturally isomorphic to the group of connected components of the direct limit $U_\infty(A) = \lim_k U_k(A)$.* □

2.11. REMARKS. (a) In the proposition we could equally well work with invertible matrices and the stable general linear group $\mathrm{GL}_\infty(A)$, since by polar decomposition, $U_\infty(A)$ is a deformation retract of $\mathrm{GL}_\infty(A)$.

(b) An explicit isomorphism is given by the formula (2.1) below for the boundary map, applied to the short exact sequence of C^*-algebras

$$0 \to C_0(\mathbb{R}) \otimes A \to C_0(\mathbb{R} \cup \{+\infty\}) \otimes A \to A \to 0.$$

(c) For $A = C(Y)$ the Proposition, and the above isomorphism, is consistent with the definition $K^{-1}(Y)$ in topological K-theory (see [**1**, Lemma 2.4.6]).

Using Proposition 2.10 one can obtain explicit formulas for the boundary maps in the K-theory exact sequence of Proposition 2.5. Consider first the map $\partial \colon K_1(A/J) \to K_0(J)$. For convenience, suppose that the C^*-algebra A has a unit. A class in $K_1(A/J)$ is determined by the connected component in $U_\infty(A/J)$ of some unitary matrix $w \in U_k(A/J)$. The matrix

$$\begin{pmatrix} w & 0 \\ 0 & w^* \end{pmatrix} \in U_{2k}(A/J)$$

is easily seen to be path connected to the identity in $M_{2k}(A/J)$, and so by a simple exercise in C^*-algebra theory it lifts to some unitary matrix $U \in U_{2k}(A)$. We then define $\partial([w])$ to be the element

$$\partial([w]) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] - \left[U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^*\right] \tag{2.1}$$

in $K_0(J) = \mathrm{kernel}(K_0(J^\sim) \to K_0(\mathbb{C}))$. Whilst this "clutching construction" is rather complicated in general, it boils down to something simple and illuminating in the following case.

2.12. PROPOSITION. *Suppose that a unitary $w \in U_k(A/J)$ lifts to a partial isometry $v \in M_k(A)$. The element $\partial([w])$ of $K_0(J)$ is equal to* $[1 - v^*v] - [1 - vv^*]$. □

(Note that since the image of v in the quotient C^*-algebra is unitary, the projections $1 - v^*v$ and $1 - vv^*$ are elements of J.) Suppose for example that A is the C^*-algebra of all bounded operators on some Hilbert space and that J is the ideal of compact operators. Then by Atkinson's Theorem [**18**, Theorem 5.17], invertible elements of A/J correspond to Fredholm operators in A. It follows from Proposition 2.12 that if F is a Fredholm operator then the image under $\partial \colon K_1(A/J) \to K_0(J)$ of the class determined by F is the Fredholm index of F in $K_0(J) \cong \mathbb{Z}$.

Using the Bott Periodicity Theorem we obtain a boundary map $\partial \colon K_0(A/J) \to K_1(J)$. It is given (up to sign) by the formula

$$\partial([p]) = \exp(2\pi i p^\sim),$$

where $p \in M_k(A/J)$ is a projection and $p^\sim \in M_k(A)$ is any lifting of p to a selfadjoint element.

2.13. *Operations.* To complete our treatment of C^*-algebra K-theory in parallel with [1], we might ask what the possibilities are for operations in C^*-algebra K-theory. Unfortunately there are none (no nontrivial ones), for the following reason. Suppose that $\alpha_A\colon K(A) \to K(A)$ is a natural transformation. It is clear from Definition 2.1 and Proposition 2.8 that α is determined by the maps $\alpha_{A\otimes\mathscr{K}}$, where A is a unital C^*-algebra. But for these it is easy to see that $K(A\otimes\mathscr{K})$ is generated by elements of the form $[p]$, where p is a projection in $A\otimes\mathscr{K}$. Defining a $*$-homomorphism $f\colon \mathbb{C} \to A\otimes\mathscr{K}$ by mapping $1\in\mathbb{C}$ to $p\in A\otimes\mathscr{K}$, and using the facts that the diagram

$$\begin{array}{ccc} K(\mathbb{C}) & \xrightarrow{f_*} & K(A\otimes\mathscr{K}) \\ {\scriptstyle\alpha_{\mathbb{C}}}\downarrow & & \downarrow{\scriptstyle\alpha_{A\otimes\mathscr{K}}} \\ K(\mathbb{C}) & \xrightarrow[f_*]{} & K(A\otimes\mathscr{K}) \end{array}$$

commutes, and that $f_*([1]) = [p]$, we see that in fact α is completely determined by its action on the generator [1] of $K(\mathbb{C}) = \mathbb{Z}$. Hence α is trivial. (This argument uses the assumption that α is additive, but it is not hard to show that any α is automatically additive.)

It follows from this that, for example, there is a unique Bott Periodicity isomorphism, up to sign. With only slightly more work one can show that the boundary maps in the long exact sequence of Proposition 2.5 are unique, up to sign.

2.14. *Indices of elliptic operators.* We shall close this section with a quick description of the way in which an elliptic operator may be assigned an "index" in a C^*-algebra K-theory group, as asserted in the examples in Section 1. To pick a specific example, an order zero leafwise elliptic operator D on a foliated manifold determines an element of the multiplier algebra $M(C^*(M,\mathscr{F}))$ of the foliation C^*-algebra, which is invertible modulo the ideal $C^*(M,\mathscr{F})$. We therefore obtain an invertible element of the quotient C^*-algebra $M(C^*(M,\mathscr{F}))/C^*(M,\mathscr{F})$ and so an element of the K-theory group $K_1M(C^*(M,\mathscr{F}))/C^*(M,\mathscr{F})$. The boundary homomorphism in the K-theory exact sequence

$$\cdots \to K_1M(C^*(M,\mathscr{F})) \to K_1M(C^*(M,\mathscr{F}))/C^*(M,\mathscr{F}) \to K_0C^*(M,\mathscr{F}) \\ \to K_0M(C^*(M,\mathscr{F})) \to \cdots$$

associated with the exact sequence

$$0 \to C^*(M,\mathscr{F}) \to M(C^*(M,\mathscr{F})) \to M(C^*(M,\mathscr{F}))/C^*(M,\mathscr{F}) \to 0,$$

maps this element to an element $\mathrm{Index}(D) \in K_0C^*(M,\mathscr{F})$.

3. Generalized Fredholm operators. Our goal is to recast the definition of K-theory given in the previous section in terms of Fredholm operators. (For a review of the basic theory of Fredholm operators the reader is referred to

[18, Chapter 5].) The prototypical result in this direction is the following theorem of Atiyah and Janich [1, Appendix].

3.1. THEOREM. *Let X be a compact space. The group $K(X)$ is isomorphic to the group of homotopy classes of maps from X into space of Fredholm operators on a Hilbert space (topologized using the* norm *topology on the algebra of bounded Hilbert space operators).* □

We shall develop a closely related description of K-theory, although roughly speaking, we shall consider more general continuous families of Fredholms than those in the theorem, along with a correspondingly more general notion of homotopy. In fact the new framework is broad enough to "include" the definition of K-theory in Section 2, and we shall see (in Proposition 3.27) that in it, a Fredholm operator becomes actually homotopic to its index!

We begin with a generalization of the notion of Hilbert space, in which the scalars $\mathbb{C}$ are replaced by a C^*-algebra A. The following definition is due to W. Paschke [37], but most of the subsequent results, along with the idea of developing K-theory from this point of view, are due to Kasparov [27], [28] and A. Mishchenko [35].

3.2. DEFINITION. A *Hilbert A-module $\mathscr{E}$* is a right A-module, equipped with an A-valued form $\langle\ ,\ \rangle : \mathscr{E}\times\mathscr{E} \to A$ which satisfies the following axioms:

(i) $\langle \eta, \xi_1 + \xi_2\rangle = \langle \eta, \xi_1\rangle + \langle \eta, \xi_2\rangle$;
(ii) $\langle \eta, \xi a\rangle = \langle \eta, \xi\rangle a$;
(iii) $\langle \eta, \xi\rangle^* = \langle \xi, \eta\rangle$;
(iv) $\langle \eta, \eta\rangle \geq 0$;
(v) $\langle \eta, \eta\rangle = 0$ if and only if $\eta = 0$; and
(vi) $\mathscr{E}$ is complete with respect to the norm $\|\eta\| = \|\langle \eta, \eta\rangle\|^{1/2}$.

3.3. REMARKS. (a) The positivity in (iv) is positivity in the sense of C^*-algebras.

(b) The norm in (vi) does indeed satisfy the triangle inequality. This is a consequence of the following generalization of the Cauchy-Schwarz inequality (see for example [27, Lemma 1]).

3.4. PROPOSITION. *If $\langle\ ,\ \rangle$ is a form on a right A-module satisfying axioms* (i)–(iv) *above then* $\|\langle \eta, \xi\rangle\|^2 \leq \|\langle \eta, \eta\rangle\|\, \|\langle \xi, \xi\rangle\|$. □

It is not part of the definition that $\mathscr{E}$ be a unital A-module (if A is unital), but it is easy to check that this is a consequence of part (v) of the definition. More generally, if $\{u_n\}$ is an approximate unit for A then considering the u_n as functions on $\mathscr{E}$, $\{u_n\}$ converges pointwise to the identity on $\mathscr{E}$. It follows from this that $\mathscr{E}$ has a natural vector space structure, and is a Banach space with respect to the norm of part (vi) of the definition.

Like the physicists, we require our inner products to be linear in the second variable and conjugate linear in the first variable. This is the choice most compatible with our choice of *right* actions of A, which is in turn most compatible with the usual convention that linear operators act on the left.

3.5. EXAMPLES. (a) If $A = \mathbb{C}$ then Definition 3.2 reduces to the definition of a Hilbert space. The reader is warned that only part of the basic theory of Hilbert space carries over to the theory of Hilbert A-modules, a circumstance which can be traced to the fact that the Riesz Representation Theorem fails for Hilbert modules.

(b) Of course, every C^*-algebra A can be regarded as a Hilbert module over itself, with the inner product $\langle a, a' \rangle = a^* a'$. We shall make use of this frequently below in connection with tensor products. (Note that if J is an ideal in A then the same inner product makes J into a Hilbert A-module, and note that the Riesz Representation Theorem fails for this module.)

(c) There is a *standard* Hilbert A-module H_A, modelled on the Hilbert space $l^2\mathbb{N}$. It consists of all sequences $\{a_n\}$ in A such that the series $\sum a_n^* a_n$ converges in A, along with the inner product

$$\langle \{a_n\}, \{b_n\} \rangle = \sum a_n^* b_n.$$

The *Kasparov Stabilization Theorem* [27, 33] which we shall state in a slightly simplified form, justifies the term "standard." It is an analogue of the result that there is a unique separable, infinite dimensional Hilbert space, up to isomorphism.

3.6. THEOREM. *If $\mathcal{E}$ is any separable Hilbert A-module (separable as a Banach space) then the direct sum $\mathcal{E} \oplus H_A$ (defined in the obvious way) is isomorphic, as a Hilbert A-module, to H_A.* □

From now on, all our Hilbert modules will be assumed to be separable.

3.7. *Finitely Generated Modules.* If A is unital then any finitely generated projective module over A may be equipped with an inner product so as to make it a Hilbert A-module. Indeed, it is easily checked that in the case of a trivial module $A^{(n)} = A \oplus A \oplus \cdots \oplus A$ the inner product

$$\langle \{a_i\}, \{b_i\} \rangle = \sum a_i^* b_i$$

as in 3.5(c) makes $A^{(n)}$ into a Hilbert module. A general $\mathcal{E}$ can be embedded as a complemented submodule of such a trivial module, from which it inherits an inner product.

Here are two simple but useful results. For the first, see [32, Section 1] (the result follows from Lemmas 1.3 and 1.6 there, together with the Kasparov Stabilization Theorem). For the second, compare [25], [35], [36].

3.8. PROPOSITION. *Let A be a C^*-algebra with unit. Every Hilbert A-module which is finitely generated (in the* algebraic *sense) is projective.* □

3.9. PROPOSITION. *Let A be a C^*-algebra with unit.*

(i) *Every finite generated projective A-module has a unique Hilbert A-module structure, up to unitary isomorphism.*

(ii) *The group $K(A)$ is isomorphic to the Grothendieck group of unitary isomorphism classes of finitely generated Hilbert A-modules.* □

The reader should compare this result with the fact (which is a special case of the proposition) that every complex vector bundle over a compact space has a unique Hermitian structure, up to unitary isomorphism. From now on all the finitely generated projective modules that we encounter will be assumed to be equipped with a Hilbert A-module structure.

3.10. *Bundles of Hilbert Spaces.* Turning to more geometric examples, let Y be a locally compact space and let E be a locally trivial bundle of Hilbert spaces over Y. The structure group here is the unitary group of a Hilbert space, with the strong operator topology. The space $\Gamma(E)$ of continuous sections of E vanishing at ∞, is a $C_0(Y)$-module, and in fact a Hilbert module with respect to the inner product

$$\langle \eta, \xi \rangle(y) = \langle \eta(y), \xi(y) \rangle.$$

This is really the basic example in the theory, and one should regard Definition 3.2 as extending this to noncommutative C^*-algebras A.

As mentioned in Paragraph 1.5, assuming the fibres of E are infinite dimensional, E is isomorphic to the trivial bundle $Y \times H$, and a check of topologies shows that consequently the Hilbert module $\Gamma(E)$ is unitarily isomorphic to $\Gamma(Y \times H)$, which is easily seen to be isomorphic to the standard module H_A. However, there is often no natural trivialization of E, and so it is not always very convenient to identify $\Gamma(E)$ with H_A.

Incidentally, every Hilbert $C_0(Y)$-module can be identified with the space of sections of some bundle of Hilbert spaces, although this bundle is not necessarily locally trivial.

3.11. *L^2-sections of vector A-bundles.* Let M be a compact manifold with a fixed smooth measure, let E be a vector A-bundle over M as described in Paragraph 1.8, and suppose that the fibres have Hilbert A-module structures (which smoothly, or at least continuously, vary from point to point). The space of continuous sections of E is an A-module, and we can define an A-valued inner product on it as follows

$$\langle \eta, \xi \rangle = \int_M \langle \eta(m), \xi(m) \rangle \, dm.$$

This inner product satisfies all of the axioms of Definition 3.2 except that of completeness. Thus we define $L^2(E)$ to be the completion of the space of continuous sections with respect to the norm of Definition 3.2(vi).

Note, by the way, that the space of continuous sections of E is also a Hilbert $C(M) \otimes A$-module; it is finitely generated. The reader can formulate the appropriate generalization of Swan's Theorem connecting vector A-bundles and projective $C(M) \otimes A$-modules.

3.12. *Foliations.* A careful description in geometric terms of Hilbert modules over a foliation algebra is given in **[10]**.

We consider now operators on Hilbert A-modules. It is very convenient to consider not all bounded A-module maps as operators, but only a subclass.

3.13. DEFINITION. Let $\mathscr{E}_1$ and $\mathscr{E}_2$ be Hilbert A-modules. An *operator* from $\mathscr{E}_1$ into $\mathscr{E}_2$ is a function $T\colon \mathscr{E}_1 \to \mathscr{E}_2$, for which there exists an adjoint $T^*\colon \mathscr{E}_2 \to \mathscr{E}_1$ such that

$$\langle \eta, T\xi\rangle = \langle T^*\eta, \xi\rangle$$

for all $\eta \in \mathscr{E}_1$ and all $\xi \in \mathscr{E}_2$. The set of all operators from $\mathscr{E}_1$ to $\mathscr{E}_2$ is denoted $\mathscr{L}(\mathscr{E}_1, \mathscr{E}_2)$.

(It is not necessary to mention that T be A-linear because that is a simple consequence of the existence of an adjoint operator T^*. Furthermore, by the Uniform Boundedness Principle, the existence of T^* guarantees that T is a bounded Banach space operator. Also, the operator T^* is uniquely determined by T, and $(T^*)^* = T$, so that $T^* \in \mathscr{L}(\mathscr{E}_2, \mathscr{E}_1)$.)

3.14. *Algebras of operators.* The space of operators from a Hilbert module $\mathscr{E}$ into itself is a Banach algebra (under composition of operators), and in fact a C^*-algebra with the *-operation $T \mapsto T^*$. If the module $\mathscr{E}$ is simply the C^*-algebra A itself then $\mathscr{L}(\mathscr{E})$ is isomorphic to the multiplier algebra $M(A)$ (it is clear that an element of $M(A)$ gives an operator, and a simple calculation shows that all operators arise this way).

3.15. *Polar decomposition.* If the range of $T\colon \mathscr{E}_1 \to \mathscr{E}_2$ is a *closed* submodule of $\mathscr{E}_2$, then some of the elementary theory of Hilbert space operators carries over to T. For example, T has a polar decomposition $T = V|T|$, and there are direct sum decompositions

$$\mathscr{E}_1 = \mathrm{range}(T^*) \oplus \mathrm{kernel}(T)$$

and

$$\mathscr{E}_2 = \mathrm{range}(T) \oplus \mathrm{kernel}(T^*).$$

If the range of T is not closed then a polar decomposition need not exist (as the simplest examples show). If T is invertible as a Banach space operator then T is invertible as an operator; the operator T^* is then also invertible and $(T^{-1})^* = (T^*)^{-1}$. Compare [32], [33].

3.16. EXAMPLE. For the Hilbert modules of Paragraph 3.10 there is a concrete characterization of the operators. For simplicity let us consider operators on the Hilbert $C(Y)$-module $\mathscr{E}$ of continuous sections of the trivial bundle over Y. Elements of $\mathscr{E}$ are continuous functions on Y with values in a fixed Hilbert space H. The operators on $\mathscr{E}$ are given by bounded functions $T\colon Y \to \mathscr{B}(H)$ such that for each $v \in H$ the functions $y \mapsto T(y)v$ and $y \mapsto T^*(y)v$ are continuous (in the norm topology of H). Such functions are said to be "bounded and *-strongly continuous" or "strictly continuous."

We consider next the analogue of the compact Hilbert space operators. Recall that in the case of Hilbert space, the class of compact operators is exactly the norm closure of the class of finite rank operators.

3.17. Definition. An operator $T\colon \mathscr{E}_1 \to \mathscr{E}_2$ is of *finite rank* if it is of the form

$$T(\eta) = \sum_1^n \xi_i \langle \eta_i, \eta \rangle$$

for vectors $\eta_1, \dots, \eta_n$ in $\mathscr{E}_1$ and vectors $\xi_1, \dots, \xi_n$ in $\mathscr{E}_2$. An operator is called a *generalized compact operator* if it can be approximated in the norm topology of $\mathscr{L}(\mathscr{E}_1, \mathscr{E}_2)$ by finite rank operators. The class of all generalized compact operators (a closed subspace of $\mathscr{L}(\mathscr{E}_1, \mathscr{E}_2)$) is denoted $\mathscr{K}(\mathscr{E}_1, \mathscr{E}_2)$.

Since we shall refer to them so often, we shall abbreviate "generalized compact" to "compact." As is the case for Hilbert space, the adjoint of a compact operator is a compact operator, and the composition of an arbitrary operator with a compact operator is compact. Thus $\mathscr{K}(\mathscr{E})$ is a (closed, two sided) ideal in $\mathscr{L}(\mathscr{E})$.

3.18. Definition. We shall call an operator $F\colon \mathscr{E}_1 \to \mathscr{E}_2$ a *generalized Fredholm operator* if there exists an operator $S\colon \mathscr{E}_2 \to \mathscr{E}_1$ (a *parametrix*) such that $1 - SF$ and $1 - FS$ are compact.

This definition is of course in the spirit of Atkinson's Theorem on Fredholm Hilbert space operators.

3.19. Examples. (a) If $\mathscr{E} = A$ then as pointed out above, $\mathscr{L}(\mathscr{E}) = M(A)$. The ideal of compact operators in $\mathscr{L}(\mathscr{E})$ is simply A. Thus for example, if A is the C^*-algebra of continuous functions on a locally compact space which vanish at infinity, then an operator on A is generalized Fredholm if and only if it is given by a function f which is bounded away from zero outside of a compact subset.

(b) In the case of the standard Hilbert module H_A, the ideal of compact operators can be shown to be isomorphic, as a C^*-algebra, to $A \otimes \mathscr{K}$. The full C^*-algebra of operators on H_A is *-isomorphic to $M(A \otimes \mathscr{K})$. See **[27]**.

(c) Continuing Paragraph 3.16, an operator T on $\mathscr{E}$ is compact if and only if T is a *norm continuous* map from Y to the compact operators on the Hilbert space H. This is not quite the same thing as T being strictly continuous and compact-operator-valued. Similarly, for F to be a generalized Fredholm operator it is not quite sufficient that each $F(y)$ be Fredholm since there may not exist a strictly continuous parametrix for F. On the other hand, a *norm*-continuous family of Fredholm operators (these are the objects of consideration in Theorem 3.1) is easily seen to be a generalized Fredholm operator. Thus, for example, a continuous family of order zero elliptic pseudodifferential operators parameterized by a compact space Y, as in [**5**, Section 1], gives rise to a generalized Fredholm operator over $C(Y)$.

(d) If E_1 and E_2 are smooth vector A-bundles over a smooth, closed manifold then an order zero elliptic pseudodifferential A-operator (in the sense of Mishchenko-Fomenko **[36]**) from smooth sections of E_1 to smooth sections of E_2 extends to a generalized Fredholm operator from $L^2(E_1)$ to $L^2(E_2)$.

3.20. *Operators on finitely generated modules.* If A is a unital C^*-algebra and $\mathscr{E}_1$ and $\mathscr{E}_2$ are finitely generated Hilbert modules then it is easily checked that *every* operator $F: \mathscr{E}_1 \to \mathscr{E}_2$ is compact. Therefore every operator from $\mathscr{E}_1$ to $\mathscr{E}_2$, even $F = 0$, is a generalized Fredholm operator.

Given a generalized Fredholm operator F we would like to define the index of F to be the difference in $K(A)$ of the classes determined by the A-modules $\ker(F)$ and $\ker(F^*)$. Unfortunately it is not necessarily true that $\ker(F)$ and $\ker(F^*)$ are finitely generated, and so this does not immediately make sense (this problem also arises in the proof of Theorem 3.1). However, it is possible to perturb F in such a way that the perturbed operator does have a well defined index:

3.21. THEOREM. *Let A be a C^*-algebra with unit and let $F: \mathscr{E}_1 \to \mathscr{E}_2$ be a generalized Fredholm operator between Hilbert A-modules. If the range of F is a closed submodule of $\mathscr{E}_2$ then both* $\ker(F)$ *and* $\ker(F^*)$ *are finitely generated Hilbert A-modules (and are therefore also projective). In general there is a compact perturbation F' of the direct sum $F \oplus 1: \mathscr{E}_1 \oplus H_A \to \mathscr{E}_2 \oplus H_A$ such that F' has closed range.* □

For a proof of this see for example [**32**, Section 1] (the proof of our theorem is not quite contained in the results of [**32**], but follows easily from them).

Given Theorem 3.21, it is more or less clear that in order to obtain a version of Theorem 3.1 for generalized Fredholm operators all that is needed is an appropriate notion of homotopy. There are in fact one or two different solutions to this problem. We shall obtain our definition of homotopy from an important tensor product construction, which will also be needed for other purposes in the next two sections.

3.22. DEFINITION. Let $\mathscr{E}$ be a Hilbert A-module and let $\mathscr{E}'$ be a Hilbert B-module which is equipped with a $*$-representation (not necessarily faithful or unital) of A as operators on it. The Hilbert B-module $\mathscr{E} \otimes_A \mathscr{E}'$ is defined as follows. The algebraic tensor product (over $\mathbb{C}$) of $\mathscr{E}$ and $\mathscr{E}'$ is a right B-module. Define a B-valued bilinear form on $\mathscr{E} \odot_{\mathbb{C}} \mathscr{E}'$ by

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \xi_i, \langle \eta_1, \eta_2 \rangle \xi_2 \rangle$$

(note that $\langle \eta_1, \eta_2 \rangle$, being an element of A, acts on $\mathscr{E}'$, so this makes sense). This satisfies properties (i)–(iv) of Definition 3.2. The quotient of $\mathscr{E} \odot_{\mathbb{C}} \mathscr{E}'$ by the submodule consisting of elements ν for which $\langle \nu, \nu \rangle = 0$ satisfies in addition the property (v) of Definition 3.2, and the Hilbert B-module $\mathscr{E} \otimes_A \mathscr{E}'$ is defined to be the completion of this quotient.

This really is a tensor product over A as the notation suggests since a simple computation reveals that elements of the form $\eta a \otimes \xi - \eta \otimes a\xi$ are in the submodule that is divided out.

3.23. EXAMPLES. (a) Regarding the Hilbert space $\ell^2\mathbb{N}$ as a Hilbert $\mathbb{C}$-module, and letting $\mathbb{C}$ act on any B as multiples of the identity, we may form the tensor product $\ell^2\mathbb{N} \otimes_{\mathbb{C}} B$; it is isomorphic to the standard module H_B.

(b) Let M be a compact manifold and let $\mathscr{E}$ be the module of continuous sections of some Hermitian bundle E over M. With respect to the natural action of $C(M)$ on $L^2(M)$, the Hilbert space $\mathscr{E} \otimes_{C(M)} L^2(M)$ is isomorphic to $L^2(E)$.

(c) Let M and N be manifolds; let $\Gamma(E)$ be the Hilbert $C_0(N)$-module of continuous sections (vanishing at infinity) of the trivial Hilbert space bundle $E = L^2(M)\times N$ as in 3.10. Considering $L^2(N)$ as a Hilbert $\mathbb{C}$-module with the obvious action of $C_0(N)$, the tensor product $\Gamma(E)\otimes_{C(N)} L^2(N)$ is isomorphic to $L^2(M \times N)$. Examples of this type (generalized to include various vector bundles and families of vector bundles over M and N) are important in applications of the theory of the next two sections.

(d) If $\mathscr{E}$ is any Hilbert A-module and $f\colon A \to B$ is a *-homomorphism, then f provides a *-representation of A as operators on B and so we may form $\mathscr{E} \otimes_A B$. This *extension of scalars* construction is familiar from algebra.

(e) As an illustration of (d), suppose that $A = C(X), B = C(Y)$, and f is induced by $\varphi\colon Y \to X$. If $\mathscr{E} = \Gamma(E)$ for some bundle E over X as in 3.10, then $\Gamma(E) \otimes_{C(X)} C(Y)$ is isomorphic to the Hilbert $C(Y)$-module of continuous sections of the pullback bundle $\varphi^* E$ over Y.

3.24 *Operators on tensor products.* An operator $T\colon \mathscr{E}_1 \to \mathscr{E}_2$ gives rise to an operator $T\otimes 1\colon \mathscr{E}_1 \otimes_A \mathscr{E}' \to \mathscr{E}_2 \otimes_A \mathscr{E}'$, by the formula

$$T\otimes 1(\eta\otimes\xi) = T\eta\otimes\xi.$$

Let us consider the situation of Example 3.23 (e) above. If $\varphi\colon Y \to X$ is an inclusion of Y as a subset of X then the tensor product is obtained from the restriction of E to Y. An operator T on $\Gamma(E)$ is determined by a family $\{T_x\}$ of Hilbert space operators on the fibres of E as in Example 3.16. The operator $T\otimes 1$ on $\Gamma(E|_Y)$ is the operator determined by the restriction of $\{T_x\}$ to Y.

Bearing this example in mind, the following definition is quite natural.

3.25. Definition. (i) Two Fredholm operators $F_0\colon \mathscr{E}_0^{(0)} \to \mathscr{E}_1^{(0)}$ and $F_1\colon \mathscr{E}_0^{(1)} \to \mathscr{E}_1^{(1)}$ between Hilbert A-modules are said to be *unitarily equivalent* if there exist unitary operators $U_0\colon \mathscr{E}_0^{(0)} \to \mathscr{E}_0^{(1)}$ and $U_1\colon \mathscr{E}_1^{(0)} \to \mathscr{E}_1^{(1)}$ such that $F_0 = U_1^* F_1 U_0$.

(ii) Two Fredholm operators as in (i) are said to be *homotopic* if there exists a Fredholm operator $F\colon \widetilde{\mathscr{E}}_0 \to \widetilde{\mathscr{E}}_1$ between Hilbert $A\otimes C[0,1]$-modules such that $F\otimes 1\colon \widetilde{\mathscr{E}}_0 \otimes_{A\otimes C[0,1]} A^{(i)} \to \widetilde{\mathscr{E}}_1 \otimes_{A\otimes C[0,1]} A^{(i)}$ and $F_i\colon \mathscr{E}_0^{(i)} \to \mathscr{E}_1^{(i)}$ are unitarily equivalent for $i = 0, 1$, where $A^{(i)}$ denotes the Hilbert A-module A, with action of $A\otimes C[0,1]$ given by evaluation at $i \in [0,1]$.

3.26. Example. If $F_t\colon \mathscr{E}_0 \to \mathscr{E}_1$ ($t \in [0,1]$), is a norm-continuous family of Fredholm operators, then F_0 and F_1 are homotopic. Indeed, the space $\widetilde{\mathscr{E}}_0$ (resp. $\widetilde{\mathscr{E}}_1$) of continuous functions from $[0,1]$ into $\mathscr{E}_0$ (resp. $\mathscr{E}_1$) is in a natural

way a Hilbert $A \otimes C[0,1]$-module, and F defined by

$$(F\eta)(t) = F_t(\eta(t))$$

is easily checked to be a generalized Fredholm operator on this module, giving a homotopy between F_0 and F_1. In this special situation we say that F_0 and F_1 are *operator homotopic*. The following result shows that much more drastic deformations are possible.

3.27. PROPOSITION. *Let A be a C^*-algebra with unit and suppose that $F: \mathscr{E}_0 \to \mathscr{E}_1$ is a generalized Fredholm operator between Hilbert A-modules which has closed range. Then F is homotopic to the zero operator* $0: \ker(F) \to \ker(F^*)$.

SKETCH OF THE PROOF. First, recall from Theorem 3.21 that $\ker(F)$ and $\ker(F^*)$ are finitely generated, and so by 3.20, $0: \ker(F) \to \ker(F^*)$ is indeed a generalized Fredholm operator. Since F has closed range, it has a polar decomposition, and by deforming $|F|$ to the identity (an operator homotopy) we easily reduce to the case that F is a partial isometry. Define Hilbert $A \otimes C[0,1]$-modules as follows:

$$\widetilde{\mathscr{E}}_0 = \{\eta: [0,1] \to \mathscr{E}_0 \mid \eta(1) \in \ker(F)\},$$

and

$$\widetilde{\mathscr{E}}_1 = \{\eta: [0,1] \to \mathscr{E}_1 \mid \eta(1) \in \ker(F^*)\}.$$

Define an operator $\widetilde{F}: \widetilde{\mathscr{E}}_0 \to \widetilde{\mathscr{E}}_1$ by the obvious formula: $(\widetilde{F}\eta)(t) = F(\eta(t))$. The adjoint of $\widetilde{F}$ is obtained by a corresponding formula from the adjoint of F and so the operators $1 - \widetilde{F}^*\widetilde{F}$ and $1 - \widetilde{F}\widetilde{F}^*$ are the projections onto the (complemented) submodules of $\widetilde{\mathscr{E}}_0$ and $\widetilde{\mathscr{E}}_1$ consisting of functions taking *all* their values in $\ker(F)$ and $\ker(F^*)$, respectively. These are finitely generated, and from this it follows that $\widetilde{F}$ is a generalized Fredholm operator. It is clear that $\widetilde{F}$ gives the desired homotopy. □

A similar argument shows the following result (which doesn't require that A be unital).

3.28. PROPOSITION. *If F is an invertible operator then F is homotopic to the zero operator* $0: 0 \to 0$.

PROOF. Defining $\widetilde{\mathscr{E}}_0, \widetilde{\mathscr{E}}_1$, and $\widetilde{F}$ as above (there is no need here for A to be unital or for F to be a partial isometry) we see that $\widetilde{F}$ is *invertible*, and so in particular Fredholm. It gives the desired homotopy. □

Note that this is a very formal argument, using only "trivial" homotopies. The reader should compare this with the proof of Theorem 3.1, which requires Kuiper's Theorem on the contractibility of the unitary group of a Hilbert space in the norm topology.

3.29. Definition. Denote by $K'(A)$ the set of homotopy classes of generalized Fredholm operators on Hilbert A-modules.

Given the above the results, the reader's first guess might be that $K'(A) = 0$! This is not so however: we shall see very shortly that $K'(A)$ is isomorphic to the K-theory $K(A)$ of Definition 2.1. Having done so, we will change notation from $K'(A)$ to $K(A)$. To begin with, $K'(A)$ is indeed an abelian group.

3.30. Proposition. (i) *By forming two generalized Fredholm operators* $F: \mathscr{E}_0 \to \mathscr{E}_1$ *and* $F': \mathscr{E}_0' \to \mathscr{E}_1'$ *their* direct sum $F \oplus F': \mathscr{E}_0 \oplus \mathscr{E}_0' \to \mathscr{E}_1 \oplus \mathscr{E}_1'$, *we make* $K'(A)$ *into an abelian group. The identity element of* $K'(A)$ *is given by the zero operator* $0: 0 \to 0$, *or by any invertible operator; the inverse of the element determined by* $F: \mathscr{E}_0 \to \mathscr{E}_1$ *is given by any parametrix* $S: \mathscr{E}_1 \to \mathscr{E}_0$ *for* F.

(ii) *The sum of the classes given by the operators* $F: \mathscr{E}_0 \to \mathscr{E}_1$ *and* $F': \mathscr{E}_1 \to \mathscr{E}_2$ *is also given by the composition* $F'F: \mathscr{E}_0 \to \mathscr{E}_2$. □

The fact that S gives the inverse of F follows from the fact that $F \oplus S$ is operator homotopic to an invertible element. For example, $F \oplus S$ is equal, modulo compacts (and hence is homotopic to), the following composition of invertible operators from $\mathscr{E}_0 \oplus \mathscr{E}_1$ to $\mathscr{E}_1 \oplus \mathscr{E}_2$:

$$\begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F & 1 \end{pmatrix} \begin{pmatrix} -1 & S \\ 0 & 1 \end{pmatrix}.$$

If $f: A \to B$ is a *-homomorphism then f induces a map $f_*: K'(A) \to K'(B)$ using the tensor product construction of paragraph 3.23 (d), because if $F: \mathscr{E}_0 \to \mathscr{E}_1$ is a generalized Fredholm operator between Hilbert A-modules then the operator $F \otimes 1: \mathscr{E}_0 \otimes_A B \to \mathscr{E}_1 \otimes_A B$ is also Fredholm (this follows from the easily verified fact that if K is compact then so is $K \otimes 1$). Thus $K'(A)$ is a covariant functor on C^*-algebras.

Now, for unital C^*-algebras, define a map $J: K(A) \to K'(A)$ by sending $[E] - [F]$ (where E and F are finitely generated Hilbert A-modules) to the class of the generalized Fredholm operator $0: E \to F$. It is clear that J is a natural transformation.

3.31. Proposition. *The natural transformation* $J: K \to K'$ *is an isomorphism, and extends to an isomorphism* $J: K \to K'$ *on the category of all* C^**-algebras* (*unital or not*).

Sketch of proof. The point is that by virtue of Theorem 3.21, we can define an inverse map Index: $K'(A) \to K(A)$. Starting with a generalized Fredholm operator F, obtain from it as in Theorem 3.21 a generalized Fredholm operator F' for which $\ker(F')$ and $\ker(F'^*)$ are finitely generated modules, and then map $[F] \in K'(A)$ to the class

$$\mathrm{Index}([F]) = [\ker(F')] - [\ker(F'^*)]$$

in $K(A)$. That this is well defined follows from computations similar to those of [**32**], together with the following trick: given homotopic Fredholm operators F_0 and F_1, and a homotopy $\widetilde{F}$, we can find a corresponding $\widetilde{F}'$ as in Theorem 3.21, and so form $\mathrm{Index}([\widetilde{F}]) \in K(C[0,1] \otimes A)$. It is then easily verified that $e_{i*}(\mathrm{Index}([\widetilde{F}])) = \mathrm{Index}([F_i])$, where $e_i\colon C[0,1] \otimes A \to A$ is evaluation at $i = 0, 1$. Therefore $\mathrm{Index}([F_0]) = \mathrm{Index}([F_1])$, since $e_0* = e_1*$ by the homotopy invariance of K-theory (Proposition 2.3). It is obvious that the composition

$$K(A) \xrightarrow{J} K'(A) \xrightarrow{\mathrm{Index}} K(A)$$

is the identity; the other composition is the identity by Proposition 3.27.

The fact that the isomorphism of $K(A)$ with $K'(A)$ extends to non-unital C^*-algebras follows from the computation that $K'(A)$ maps isomorphically to the kernel of the map $K'(\widetilde{A}) \to K'(\mathbb{C})$, which we omit (see for example [**28**]). $\square$

The final topic we want to touch upon in this section is that of products in K-theory. As with most of the rest of the section, we are simply setting the stage for the elliptic operator theory to come, and so we shall be rather brief. If p and q are projections in C^*-algebras A and B respectively then $p \otimes q$ is a projection in $A \otimes B$. Given the definition of the K-theory groups as in Section 2, it is straightforward to show that the construction of $p \otimes q$ from p and q gives rise to a homomorphism

$$K(A) \otimes K(B) \to K(A \otimes B). \tag{3.1}$$

(If A is commutative then the multiplication map $A \otimes A \to A$ is a *-homomorphism and so induces a map $K(A \otimes A) \to K(A)$. We obtain then a ring structure on $K(A)$, which is of course the same as the ring structure in [**1**].) We wish to describe the pairing (3.1) at the level of generalized Fredholm operators. This is based on the following definition of "external" tensor product.

3.32. Definition. Let $\mathscr{E}$ be a Hilbert $\mathscr{A}$-module and let $\mathscr{E}'$ be a Hilbert B-module. Denote by $\mathscr{E} \boxtimes \mathscr{E}'$ the Hilbert $A \otimes B$-module obtained by completing the algebraic tensor product $\mathscr{E} \odot \mathscr{E}'$ with respect to the norm associated with the $A \otimes B$-valued inner product

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle \otimes \langle \xi_1, \xi_2 \rangle.$$

The reader can easily compute some examples, starting with Hilbert modules as in Paragraphs 3.7, 3.10, and 3.11.

In order to most simply describe the K-theory product, it is necessary to introduce the following notion.

3.33. Definition. (i) A $\mathbb{Z}/2$-*grading* on a Hilbert module $\underline{\mathscr{E}}$ is a decomposition of the module as an orthogonal direct sum $\underline{\mathscr{E}} = \mathscr{E}_0 \oplus \mathscr{E}_1$. Elements of $\mathscr{E}_0$ and $\mathscr{E}_1$ are said to be *homogeneous*, of *grading degree zero* and *one*, respectively (or to be of *even* and *odd degree*, respectively). We write $\partial \eta = 0$ and $\partial \eta = 1$.

(ii) The *$\mathbb{Z}/2$-graded tensor product* $\underline{\mathscr{E}}\hat{\boxtimes}\underline{\mathscr{E}}'$ of two Hilbert modules as in Definition 3.32 is the tensor product $\underline{\mathscr{E}} \boxtimes \underline{\mathscr{E}}'$ equipped with the grading

$$\underline{\mathscr{E}} \boxtimes \underline{\mathscr{E}}' = (\mathscr{E}_0 \boxtimes \mathscr{E}_0' \oplus \mathscr{E}_1 \boxtimes \mathscr{E}_1') \oplus (\mathscr{E}_1 \boxtimes \mathscr{E}_0' \oplus \mathscr{E}_0 \boxtimes \mathscr{E}_1').$$

(We shall use underlined letters to denote graded modules, as well as operators on graded modules.)

The motivation for this definition is, on one level at least, straightforward. If $\underline{\mathscr{E}}$ is finitely generated then think of it as representing the *difference* $[\mathscr{E}_0] - [\mathscr{E}_1]$ in $K(A)$. Then the graded tensor product $\underline{\mathscr{E}}\hat{\boxtimes}\underline{\mathscr{E}}'$ represents the product of $[\mathscr{E}_0] - [\mathscr{E}_1]$ and $[\mathscr{E}_0'] - [\mathscr{E}_1']$ under the pairing (3.1).

Now, given a generalized Fredholm operator $F\colon \mathscr{E}_0 \to \mathscr{E}_1$ we can form the operator

$$\underline{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} \tag{3.2}$$

on the graded Hilbert module $\underline{\mathscr{E}} = \mathscr{E}_0 \oplus \mathscr{E}_1$. This is itself a generalized Fredholm operator. We wish to describe $K(A)$ and the product map in terms of these gadgets, and in order to do so it is convenient to introduce the following notion.

3.34. Definition. An operator $\underline{F}$ on $\underline{\mathscr{E}}$ is said to be *homogeneous of degree $\partial\underline{F}$* if for every homogeneous element η of $\underline{\mathscr{E}}$,

$$\partial(\underline{F}\eta) = \partial\underline{F} + \partial\eta \pmod 2.$$

Thus the operator $\underline{F}$ above is homogeneous of degree one (or just "degree one," for short). On the other hand, suppose that $\underline{F}$ is an operator on a $\mathbb{Z}/2$-graded module $\underline{\mathscr{E}}$ such that:

(i) $\underline{F}$ is of degree one;
(ii) $\underline{F}$ is a selfadjoint; and
(iii) $\underline{F}$ is a generalized Fredholm operator.

Then with respect to the direct sum decomposition of $\underline{\mathscr{E}}$ given by the grading, $\underline{F}$ is of the form (3.2) above, with F a generalized Fredholm operator. It follows very easily from these observations that we may describe $K(A)$ as the group of homotopy classes of degree one, selfadjoint Fredholm operators on $\mathbb{Z}/2$-graded Hilbert A-modules (the correct notion of homotopy is of course given by operators on Hilbert $A \otimes C[0,1]$-modules satisfying (i)–(iii) above). For the rest of this section $\underline{F}$, $\underline{G}$, etc., will denote such operators.

Operators $\underline{G}$ and $\underline{F}$ on $\underline{\mathscr{E}}$ and $\underline{\mathscr{E}}'$ clearly give rise to operators $\underline{G} \boxtimes \underline{1}$ and $\underline{1} \boxtimes \underline{F}$ on $\underline{\mathscr{E}}\hat{\boxtimes}\underline{\mathscr{E}}'$. We need the following slight refinement.

3.35. Definition. Denote by $\underline{G}\hat{\boxtimes}\underline{1}$ simply the operator $\underline{G} \boxtimes \underline{1}$ as above. Denote by $\underline{1}\hat{\boxtimes}\underline{F}$ the operator $\underline{1} \boxtimes \underline{F}$ "twisted" by the grading as follows:

$$\underline{1}\hat{\boxtimes}\underline{F}(\eta \otimes \xi) = (-1)^{\partial\underline{F}\partial\eta}\eta \otimes \underline{F}\xi.$$

The crucial property of $\underline{G}\hat{\boxtimes}\underline{1}$ and $\underline{1}\hat{\boxtimes}\underline{F}$ is that they *anti-commute*:

$$(\underline{G}\hat{\boxtimes}\underline{1})(\underline{1}\hat{\boxtimes}\underline{F}) + (\underline{1}\hat{\boxtimes}\underline{F})(\underline{G}\hat{\boxtimes}\underline{1}) = \underline{0}.$$

We are interested in the operator

$$\underline{G}\#\underline{F} = \underline{G}\hat{\boxtimes}\underline{1} + \underline{1}\hat{\boxtimes}\underline{F} \tag{3.3}$$

on $\underline{\mathscr{E}}\hat{\boxtimes}\underline{\mathscr{E}}'$. If we recover from this a generalized Fredholm operator between Hilbert $A \otimes B$-modules as in (3.2) then we obtain

$$G\#F = \begin{pmatrix} G \boxtimes 1 & -1 \boxtimes F^* \\ 1 \boxtimes F & G^* \boxtimes 1 \end{pmatrix} \tag{3.4}$$

(we are expressing $G\#F$ as a matrix with respect to the direct sum decomposition of part (ii) of Definition 3.33). The reader can easily verify that if F and G are Hilbert space operators then $G\#F$ is a Fredholm operator and

$$\ker(G\#F) = \ker(G) \otimes \ker(F) \oplus \ker(G^*) \otimes \ker(F^*)$$

and

$$\ker(G\#F) = \ker(G^*) \otimes \ker(F) \oplus \ker(G) \otimes \ker(F^*)$$

so that $G\#F$ is a Fredholm whose index is the *product* of the indices of F and G. In general, we have:

3.36. PROPOSITION. *The operator $\underline{G}\#\underline{F}$ above is a generalized Fredholm operator. The map $[\underline{G}] \otimes [\underline{F}] \mapsto [\underline{G}\#\underline{F}]$ from $K(A) \otimes K(B)$ to $K(A \otimes B)$ is well-defined and gives the pairing* (3.1). □

The proof is easy: once it is verified that $\underline{G}\#\underline{F}$ is a generalized Fredholm operator (a simple C^*-algebra exercise) and that the pairing is well-defined (more or less obvious) we can reduce to zero operators on finitely generated modules by Proposition 3.31, for which the result is clear.

4. Generalized elliptic operators. The purpose of this section is to show how "generalized elliptic operators" play an important role in the calculation and manipulation of K-theory groups. The basic ideas originate in Atiyah's articles [2] and [3], and are developed extensively by Kasparov in [28]. We shall begin with Atiyah's constructions in topological K-theory and the theory of elliptic operators, and then describe the general setup of Kasparov's theory.

Let M be a smooth, closed manifold and let D be an elliptic operator on M. It is convenient to assume that D is an order zero operator, and so D is a pseudodifferential, rather than differential, operator. It is permissible that D act on sections of some bundle, but for the sake of a slight simplification in notation we shall ignore the bundle and pretend that D acts on functions. Since it is of order zero we can and will think of D as a bounded operator on $L^2(M)$.

The basic object of interest for us here is a group homomorphism

$$\mathrm{Index}_D \colon K(M) \to \mathbb{Z}$$

with generalizes the Fredholm index of D. It is defined as follows. If E is a smooth complex vector bundle over M then we construct from D an operator

$$D_E \colon L^2(M, E) \to L^2(M, E),$$

acting on sections of E. If D was, say, a first-order differential operator then we could get D_E by choosing a connexion for the bundle E and then letting D act on sections of E via this connexion. In general we define D_E by using the local triviality of E together with a partition of unity argument. Thus we choose a partition of unity $\{f_1, \ldots, f_k\}$ for M such that each f_i is supported within an open set U_i over which the bundle E is trivializable. Choosing trivializations, and hence isomorphisms $L^2(U_i, E \mid U_i) \cong L^2(U_i) \otimes \mathbb{C}^k$ (where k is the dimension of the bundle), we define operators $(f_i^{1/2} D f_i^{1/2})_E$ on $L^2(U_i, E \mid U_i)$ by pulling back the operators $f_i^{1/2} D f_i^{1/2} \otimes 1$ on $L^2(U_i) \otimes \mathbb{C}^k$ via these isomorphisms. Finally, define D_E to be the operator

$$D_E = \sum_{i=1}^{k} \left(f_i^{1/2} D f_i^{1/2}\right)_E$$

on $L^2(M, E)$. The operator we obtain in this way obviously depends on the choice of partition of unity, and so on. However, whatever the choices, D_E is a Fredholm operator, and its index does not depend on the choices. In fact if we define

$$\mathrm{Index}_D(E) = \mathrm{Index}(D_E)$$

then this passes to a homomorphism from $K(M)$ to $\mathbb{Z}$, as desired. Note that if E is the trivial line bundle over M then $\mathrm{Index}_D(E)$ is simply $\mathrm{Index}(D)$.

The homomorphism $\mathrm{Index}_D \colon K(M) \to \mathbb{Z}$ can be thought of in two different (although closely related) ways. On the one hand, it gives additional information about the index theory of D beyond $\mathrm{Index}(D)$ itself. There is thus the possibility of studying $\mathrm{Index}(D)$ via the homomorphism Index_D using techniques from K-theory. This is roughly the point of view of the proof of the Atiyah-Singer Index Theorem in [4] (if M is an even-dimensional spin^c-manifold and if D is the Dirac operator on M, then modulo the Thom Isomorphism $K(M) \cong K(TM)$, the map Index_D is exactly the analytic index map a-ind: $K(TM) \to \mathbb{Z}$ of [4]). On the other hand we might turn things around and try to use $\mathrm{Index}_D \colon K(M) \to \mathbb{Z}$, and generalizations of it, to study K-theory. This idea is illustrated by Atiyah's proof of the Bott Periodicity Theorem in [2]. In this article we are more or less concerned with extensions of this latter approach.

What is it exactly about D that makes the construction of Index_D work? Why is D_E Fredholm and why is $\mathrm{Index}(D_E)$ well defined? It is very easy to see that it is not enough to start off with any random Fredholm operator in place of D. Without going through great lengths to motivate the answer, we shall simply state that the crucial and in fact sufficient property of D (apart from it being Fredholm) is that it is "pseudolocal" in the following C^*-algebraic sense: if $f \in C(M)$ and we regard f as acting on $L^2(M)$ by pointwise multiplication then the commutator $fD - Df$ is a compact operator. The importance of this condition should be clear from the formula for D_E given above. To justify the use of the term "pseudolocal," we mention

the following observation of Kasparov [**26**]: an operator T on $L^2(M)$ is pseudolocal in this sense if and only if for every two disjoint closed subsets C and C' in M the compression of T to an operator from $L^2(C)$ to $L^2(C')$ is a compact operator. Note that D is pseudolocal by this characterization since its distributional kernel is a smooth function off the diagonal of $M \times M$: the compression is therefore compact by Paragraph 1.5. The reader can check that if D is any Fredholm operator which is pseudolocal then the operator D_E constructed as above is indeed Fredholm, and its index is independent of the choices made in the construction (we shall prove this in a more general context in a moment).

The conclusion to be drawn from this is that if we wish to generalize the notion of elliptic operators for the purposes of K-theory, then the operators involved should have the following properties:

(i) the should be pseudolocal, in some sense; and
(ii) they should be Fredholm.

Given the work of the previous section we can proceed very quickly to the following definition.

4.1. DEFINITION. Let A and B be C^*-algebras. A *generalized elliptic operator* over A with coefficients in B consists of the following data:

(i) An operator $F\colon \mathscr{E}_0 \to \mathscr{E}_1$ between Hilbert B-modules; and
(ii) *-representations of A as operators on $\mathscr{E}_0$ and $\mathscr{E}_1$ such that for every $a \in A$ the operators $aF - Fa$, $a(F^*F - 1)$ and $a(FF^* - 1)$ are generalized compact operators.

Our elliptic operator D acting on $L^2(M)$ fits into this example if we take $A = C(M)$ and $B = \mathbb{C}$, providing that D^* is a parametrix for D (this is so if the principal symbol of D is unitary valued).

The condition that $aF - Fa \in \mathscr{K}$ for all $a \in A$ is clearly a general version of the pseudolocal property of D given above.

The condition that $a(F^*F-1)$ and $a(FF^*-1)$ be compact is a slight weakening of the requirement that F be a generalized Fredholm operator which is "unitary modulo compact operators." The unitarity is simply a small technical convenience and could be omitted (by a C^*-algebraic polar decomposition argument, we can in these circumstances always reduce to "essentially unitary" operators). The fact that we require F to be only "approximately" a generalized Fredholm operator is useful in the construction of examples (particularly examples associated with noncompact manifolds).

Finally the fact that we allow in this definition the possibility that $\mathscr{E}_0 \neq \mathscr{E}_1$ (in our basic example, both of the modules are $L^2(M)$) is very useful in most applications. For example, if an elliptic operator D operates on sections of one bundle, with values in a different bundle, then the associated Hilbert spaces of sections are isomorphic, even with the left $C(M)$-module structure taken into account, but as there is no natural isomorphism it is usually convenient not to make the identification.

Let us see how a generalized elliptic operator F gives rise to a group homomorphism

$$\mathrm{Index}_F\colon K(A) \to K(B),$$

extending our construction in the case of the elliptic operator D on M. Suppose for the sake of simplicity that A is a unital C^*-algebra (the nonunital case can be handled by, for example, a 2×2 matrix trick which converts any generalized elliptic operator over A to a generalized elliptic operator over the C^*-algebra $\widetilde{A}$). We shall regard $K(A)$ as being generated by finitely generated projective Hilbert A-modules. Given such a module $\mathcal{E}$ and a generalized elliptic operator $F\colon \mathcal{E}_0 \to \mathcal{E}_1$ we wish to define an operator $F_{\mathcal{E}}\colon \mathcal{E}\otimes_A\mathcal{E}_0 \to \mathcal{E}\otimes_A\mathcal{E}_1$ which extends the construction of the operator D_E. This is accomplished using the following notion of connexion.

4.2. Definition. An operator $F'\colon \mathcal{E}\otimes_A \mathcal{E}_0 \to \mathcal{E}\otimes_A \mathcal{E}_1$ is said to be an F-connexion if for every $\eta\in\mathcal{E}$ the operators $F'T_\eta - T_\eta F$ and $T_\eta^* F' - FT_\eta^*$ are compact, where the operators T_η are defined by $T_\eta(\xi) = \eta\otimes\xi$.

The point behind the definition is made clear by looking at a simple example.

4.3. *Free modules.* If $\mathcal{E} = A^{(n)}$ is the direct sum of n copies of A, then the tensor product $\mathcal{E}\otimes\mathcal{E}_i$ is the direct sum of n copies of the range $\mathcal{E}_i'$ of $1\in A$, considered as an operator on $\mathcal{E}_i$. Any operator on $\mathcal{E}\otimes_A\mathcal{E}_i'$ is therefore given by an $n\times n$ matrix of operators on $\mathcal{E}_i'$, and applying the condition of Definition 4.2 we see that any F-connexion must be of the form

$$F^{(n)} = \begin{pmatrix} 1F1 & & & \\ & 1F1 & & 0 \\ 0 & & \ddots & \\ & & & 1F1 \end{pmatrix},$$

modulo compact operators (note that the operator commutators of Definition 4.2 are zero for this particular operator $F^{(n)}$). From this computation, and the fact that every finitely generated $\mathcal{E}$ may be embedded isometrically as a complemented submodule of a free module, we obtain the following result. (We should point out that the pseudolocality of F is used to show that the operator $F^{(n)}$ above commutes, modulo compact operators, with the projection operators $A^{(n)}\otimes_A\mathcal{E}_i \to \mathcal{E}\otimes_A\mathcal{E}_i$ associated with the complemented submodule $\mathcal{E}$ of $A^{(n)}$.)

4.4. Proposition. *For any $\mathcal{E}$ and any generalized elliptic operator F there exists an F-connexion F'. The operator F' is unique up to compact perturbation and is a generalized Fredholm operator.* □

4.5 *Comparison with Atiyah's construction.* To compare this with the construction of D_E, note that if $\mathcal{E}$ is the module of sections of some bundle E over M, then as pointed out in Paragraph 3.23 (b), the tensor product $\mathcal{E}\otimes_{C(M)} L^2(M)$ is equal to the Hilbert space $L^2(M,E)$. The maps $T_\eta\colon L^2(M) \to$

$L^2(M, E)$ are given by multiplying L^2-functions by fixed continuous sections η of E, and it is easily checked that the operator D_E is a D-connection.

It is a straightforward matter to verify the following result.

4.6. PROPOSITION. *The operation which assigns* $\mathrm{Index}(F') \in K(B)$ *(as in Proposition* 3.31*) to the module* $\mathscr{E}$ *induces a group homomorphism* $\mathrm{Index}_F\colon K(A) \to K(B)$.

Here are a couple of examples.

4.7. *Bott Periodicity.* The basic example in the whole theory is Atiyah's elliptic operator proof of Bott Periodicity. This is fundamental not only because of the central importance of the Bott Periodicity Theorem per se, but because in a great many of the significant applications of the theory, the key issue is the construction of generalizations of the "Dirac" and "Dual Dirac" operators of Atiyah. In the present context, these are as follows.

The Dirac operator $D' = d/dx + i\,d/dy$ on $\mathbb{R}^2$ is not a bounded operator, but the operator $D = D'(1 + D'^*D')^{-1/2}$ on $H = L^2(\mathbb{R}^2)$ is. It is easily seen (by Fourier analysis) to be an elliptic operator over $C_0(\mathbb{R}^2)$ (with coefficients in $\mathbb{C}$). If A is any C^*-algebra then the Hilbert A-module $\mathscr{E} = H \boxtimes A$ has a natural left action of A, and the operator $F = D \boxtimes 1$ on $\mathscr{E}$ is a generalized elliptic operator over $C_0(\mathbb{R}^2) \otimes A$ with coefficients in A.

Let $g(z) = z(1+|z|)^{-1}$. This is a bounded continuous function on $\mathbb{R}^2 (= \mathbb{C})$ and so defines an operator on $C_0(\mathbb{R}^2)$, considered as a Hilbert $C_0(\mathbb{R}^2)$-module. In fact g is a generalized Fredholm operator (compare Paragraph 3.19(a)) and so defines a generalized elliptic operator over $\mathbb{C}$, with coefficients in $C_0(\mathbb{R}^2)$. The operator $G = g \boxtimes 1$ on the Hilbert $C_0(\mathbb{R}^2) \otimes A$ module $C_0(\mathbb{R}^2) \otimes A$ is a generalized elliptic operator over A with coefficients in $C_0(\mathbb{R}^2) \otimes A$.

The maps $\mathrm{Index}_F\colon K(C_0(\mathbb{R}^2)\otimes A) \to K(A)$ and $\mathrm{Index}_G\colon K(A) \to K(C_0(\mathbb{R}^2)\otimes A)$ are inverse to one another. One way to show this is to use the Kasparov product to be developed in the next section. However, as Atiyah observes in [2], there is a simple "rotation trick" by means of which the problem is reduced to a single simple computation (the technique of [2] works as well for C^*-algebra K-theory as for the K-theory of compact spaces). Thus in this case simply the construction of the index maps is sufficient to obtain an important result: the Bott Periodicity Theorem.

4.8. *Foliations.* Let M be a smooth, closed manifold, equipped with a free action of the Lie group $\mathbb{R}^2$. Let $\mathscr{F}$ be the foliation on M whose leaves are the orbits of this action. We can generalize the constructions in the above paragraph as follows.

Using the action, each leaf L of the foliation can be identified with $\mathbb{R}^2$ canonically, up to translation. Since the operator D of the above paragraph is translation invariant, we obtain via this identification a family of operators $\{D_L \mid L \in M/\mathscr{F}\}$ on the family of Hilbert spaces $\{L^2(L) \mid L \in M/\mathscr{F}\}$. It is not hard to see that the family determines an element F of the multiplier

algebra of $C^*(M,\mathscr{F})$. In fact, considering $C^*(M,\mathscr{F})$ as a Hilbert module over itself, F is a generalized Fredholm operator. Furthermore there is an action of $C(M)$ as operators on $C^*(M,\mathscr{F})$, since a continuous function on M acts on each $L^2(L)$ by pointwise multiplication, and one can easily show that in this way it determines an element of the multiplier algebra of $C^*(M,\mathscr{F})$. With this additional structure, F becomes a generalized elliptic operator over $C(M)$ with coefficients in $C^*(M,\mathscr{F})$.

The construction of G is just slightly more complicated. The Hilbert $C(M)$-module on which G acts is obtained as in Paragraph 3.10 from a bundle E of Hilbert spaces over M. The fibre of E_x over a point $x \in M$ is $L^2(L_x)$, where L_x is the leaf of the foliation containing x; these fibres glue together in a natural way (which we won't describe) to form a locally trivial bundle over M. The operator G acting on the fibre $E_x = L^2(L_x)$ (see Paragraph 3.16) is pointwise multiplication by the function g as above, where we identify L_x with $\mathbb{R}^2$ so that x is the origin. The C^*-algebra $C^*(M,\mathscr{F})$ acts on each $L^2(L_x)$ in the natural way, and we obtain a generalized elliptic operator over $C^*(M,\mathscr{F})$ with coefficients in $C(M)$.

Once again, these two operators give mutually inverse maps at the level of K-theory. Once again, there is a trick (this time the trick uses the harmonic analysis of $\mathbb{R}^2$) which reduces the work involved in showing this substantially, so that the machinery of the next section is not really needed. See **[19]**. (For the benefit of the reader who turns to this reference we point out that the C^*-algebra $C^*(M,\mathscr{F})$ is *-isomorphic to the C^*-algebra crossed product $C(M) \times \mathbb{R}^2$. Furthermore, although it is not necessary to do so, it is most natural to deal with $\mathbb{R}$, not $\mathbb{R}^2$, and this requires the theory of selfadjoint elliptic operators sketched in Section 6.)

More complicated examples of "Dirac" and "Dual Dirac" operators arise in the context of examples 1.3 and 1.4. (We should remark that the elliptic pseudodifferential A-operators of Mishchenko and Fomenko **[36]** are generalized elliptic operators over $C(M)$ with coefficients in A.) The tricks of the above examples are not available, and more machinery is needed to deal with them.

5. The Kasparov product. We shall first define Kasparov's KK-groups. As was the case with products in K-theory, it is convenient to introduce the language of $\mathbb{Z}/2$-gradings. Thus we introduce the following objects.

5.1. DEFINITION. A *Kasparov (A,B)-cycle* consists of the following data:

(i) a $\mathbb{Z}/2$-graded Hilbert B-module $\underline{\mathscr{E}}$ and a generalized Fredholm operator $\underline{F}\colon \underline{\mathscr{E}} \to \underline{\mathscr{E}}$ of grading degree 1; and

(ii) a *-representation of A as operators on $\underline{\mathscr{E}}$ of grading degree zero, such that

(iii) for every $a \in A$ the operators $a(\underline{F} - \underline{F}^*)$, $a(\underline{F}^2 - 1)$, and $a\underline{F} - \underline{F}a$ are compact.

We shall write such a cycle as $(\underline{F}, \underline{\mathscr{E}})$; in keeping with the discussion at the end of Section 3, we use underlined letters to denote $\mathbb{Z}/2$ graded objects. The definition is just a repackaging of the definition of generalized elliptic operator. Indeed, given a generalized elliptic operator $F: \mathscr{E}_0 \to \mathscr{E}_1$ we can form the Kasparov (A, B)-cycle

$$\underline{\mathscr{E}} = \mathscr{E}_0 \oplus \mathscr{E}_1, \ \underline{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}, \tag{5.1}$$

whilst on the other hand the hypotheses involving the grading in Definition 5.1 imply that every Kasparov (A, B)-cycle decomposes like this with respect to the direct sum decomposition of $\underline{\mathscr{E}}$ induced by the grading. The condition (ii) of Definition 4.1 is exactly condition (iii) of Definition 5.1 under this correspondence.

We need to extend the notion of connexion introduced in the previous section to this setup. Let $\underline{\mathscr{E}}'$ be a $\mathbb{Z}/2$-graded Hilbert A-module (not necessarily finitely generated and projective any more).

5.2. Definition. An operator $\underline{F}'$ on $\underline{\mathscr{E}}'\hat{\otimes}_A \underline{\mathscr{E}}$ of odd grading degree is called an $\underline{F}$-connexion if for every $\eta \in \underline{\mathscr{E}}'$ the operators $\underline{T}_\eta \underline{F} - (-1)^{\partial\eta} \underline{F}' \underline{T}_\eta$ and $\underline{T}_\eta^* \underline{F} - (-1)^{\partial\eta} \underline{F}' \underline{T}_\eta^*$ are compact, where the $\underline{T}_\eta$ are as in Definition 4.2.

Thus we require T_η to essentially intertwine or "anti-intertwine" the operators $\underline{F}$ and $\underline{F}'$, according as η is of odd or even grading degree. The reason for this of course is that we want to model not "$\underline{1} \otimes \underline{F}$" but a graded tensor product "$\underline{1}\hat{\otimes}\underline{F}$".

The existence of connexions can easily be verified by appealing to the Kasparov Stabilization Theorem (Theorem 3.6), as follows. Simplifying things just a little, let us consider the case of a unital B and a unital action of B on $\underline{\mathscr{E}}$. If $\underline{\mathscr{E}}'$ is the standard ($\mathbb{Z}/2$-graded) Hilbert B-module $\underline{H}_B = H_B \oplus H_B$ then $\underline{H}_B \hat{\otimes}_A \underline{\mathscr{E}}$ is isomorphic to $\underline{H}_{\mathbb{C}} \hat{\boxtimes} \underline{\mathscr{E}}$ and we may define $\underline{F}'$ to be $\underline{1}\hat{\boxtimes}\underline{F}$. In general we can realize $\underline{\mathscr{E}}'$ as a direct summand of $\underline{H}_B$ and then compress the connexion just constructed to the direct summand $\underline{\mathscr{E}}'\hat{\otimes}_A\underline{\mathscr{E}}$ of $\underline{H}_B\hat{\otimes}_A\underline{\mathscr{E}}$. A connexion constructed in this way is called a *Grassman connexion.* As Proposition 4.4 showed, for our former notion of connexion, up to compact perturbation every connexion was of this Grassman type. Here however, where we are allowing "infinite dimensional" Hilbert A-modules, the notion is considerably more general. Before reading on, the reader might want to consider the possibilities in the case of a tensor product as in Paragraph 3.23 (c), and a generalized elliptic operator F given by a family of elliptic order zero pseudodifferential operators parameterized by N.

As a warm-up for what is to follow, the reader can verify the following version of Proposition 4.5. If $\underline{F}'$ is a Fredholm operator of grading degree one on a $\mathbb{Z}/2$-graded Hilbert B-module then we define $\mathrm{Index}(\underline{F}') \in K(B)$ to be the index of the part F' of $\underline{F}'$ which maps even vectors to odd vectors.

5.3. PROPOSITION. *Let A be a unital C^*-algebra. If $\underline{F}$ and F are as in (5.1) and if $\underline{\mathscr{E}}' = \mathscr{E}_0' \oplus \mathscr{E}_1'$ is finitely generated and projective, then every $\underline{F}$-connexion $\underline{F}'$ is a generalized Fredholm operator, and if $\underline{F}'$ is of odd grading degree then* $\mathrm{Index}_F([\mathscr{E}_0'] - [\mathscr{E}_1']) = \mathrm{Index}(\underline{F}')$ *in $K(B)$.* □

Thus the factor $(-1)^{\partial\eta}$ in Definition 5.2 neatly allows us to incorporate the main construction of the previous section into the framework of $\mathbb{Z}/2$-gradings and Kasparov cycles.

There is a natural notion of isomorphism of Kasparov cycles: an isomorphism is obtained from a grading degree zero unitary isomorphism between the underlying Hilbert modules which intertwines (exactly) the actions of A and the generalized Fredholm operators. We can also extend the notion of homotopy from Fredholm operators to Kasparov cycles in a natural way: a homotopy of Kasparov (A, B)-cycles is simply a Kasparov $(A, B \otimes C[0,1])$-cycle. As we did with Fredholm operators in Section 3 we pass from cycles to homotopy classes of cycles.

5.4. DEFINITION. Denote by $KK(A, B)$ the set of homotopy classes of Kasparov (A, B)-cycles.

Adapting the arguments of Proposition 3.30, we obtain the following result.

5.5. PROPOSITION. *The operator of direct sum of Kasparov cycles makes $KK(A, B)$ into an abelian group. The zero element is represented by any Kasparov cycle for which the compact operators in part (iii) of Definition 5.1 are all identically zero. The additive inverse of a cycle is obtained by reversing the grading on the module.* □

Note that in terms of generalized elliptic operators rather than Kasparov cycles, the effect of "reversing the grading" is to pass from the operator F to the operator F^*. Here is one more exercise for the reader.

5.6. PROPOSITION. *The index map construction of Section 4 and Proposition 5.3 passes to a well defined group homomorphism*

$$\otimes_A \colon K(A) \otimes KK(A, B) \to K(B). \quad \square$$

Now, it is (almost) immediate from the Definitions 3.29 and 5.1 that $K(A) = KK(\mathbb{C}, A)$. Indeed if $F \colon \mathscr{E}_0 \to \mathscr{E}_1$ is any Fredholm operator then the action of $\mathbb{C}$ on the Hilbert modules by multiples of the identity operator makes F into a generalized elliptic operator over $\mathbb{C}$, with coefficients in A. The only distinction between cycles for the groups $K(A)$ and $KK(\mathbb{C}, A)$ is that in the case of a generalized elliptic operator over $\mathbb{C}$, the action of $\mathbb{C}$ need not be unital, but this difference disappears at the level of homotopy. Thus the pairing of the proposition could be rewritten as

$$KK(\mathbb{C}, A) \otimes KK(A, B) \to KK(\mathbb{C}, B).$$

Kasparov generalizes this to a pairing

$$KK(A,B) \otimes KK(B,C) \to KK(A,C) \tag{5.2}$$

based on the following construction (the particular form of it given here is due to Skandalis [44], [11]).

5.7. DEFINITION. Let $(\underline{G}, \underline{\mathscr{E}'})$ be a Kasparov (A,B)-cycle and let $(\underline{F}, \underline{\mathscr{E}})$ be a Kasparov (B,C)-cycle. A *product* of these two cycles is any Kasparov (A,C)-cycle of the form $(\underline{G\#F}, \underline{\mathscr{E}'}\hat{\otimes}_B \underline{\mathscr{E}})$, where $\underline{G\#F}$ has the following properties:

(i) the operator $\underline{G\#F}$ is an $\underline{F}$-connexion on $\underline{\mathscr{E}'}\hat{\otimes}_B\underline{\mathscr{E}}$; and
(ii) for every $a \in A$ the operator

$$a^* \left\{ (\underline{G}\hat{\otimes}\underline{1})\,(\underline{G\#F}) + (\underline{G\#F})\,(\underline{G}\hat{\otimes}\underline{1}) \right\} a$$

is a positive on $\underline{\mathscr{E}'}\hat{\otimes}_B\underline{\mathscr{E}}$, modulo compact operators (thus it is of the form {positive operator}+{compact operator}).

In the next several paragraphs we will try to motivate this definition.

5.8. *Comparison with Section* 4. To begin with, suppose we consider the simple case where $A = \mathbb{C}$, and where the cycle $(\underline{G}, \underline{\mathscr{E}'})$ is of the form $(0, \mathscr{E}_0 \oplus \mathscr{E}_1)$, the Hilbert B-modules $\mathscr{E}_0$ and $\mathscr{E}_1$ being finitely generated. Then as pointed out in Proposition 5.3, any $\underline{F}$-connexion $\underline{F'}$ is automatically a generalized Fredholm operator and hence automatically an elliptic operator over $\mathbb{C}$. Furthermore, condition (ii) of Definition 5.7 is trivially satisfied by any $\underline{F'}$, and so every $\underline{F}$-connexion is a product $\underline{G\#F}$. It follows from Proposition 5.3 then that the class of any product $(\underline{G\#F}, \underline{\mathscr{E}'}\hat{\otimes}_B\underline{\mathscr{E}})$ in $KK(\mathbb{C}, B) \cong K(B)$ is equal to the image under the map

$$\mathrm{Index}_F \colon K(A) \to K(B)$$

of the class of $(\underline{G}, \underline{\mathscr{E}'}) \in KK(\mathbb{C}, A) \cong K(A)$ (here F is the generalized elliptic operator associated to $\underline{F}$ as in (5.1)). If we take for granted for the moment the fact that the KK-theory class of the product of two Kasparov cycles passes to a well-defined map at the level of KK-groups, then we see that the product contains the construction of the Index map in the last section. Furthermore, it extends the construction to generalized Fredholm operators not necessarily of the elementary form $0\colon \mathscr{E}_0 \to \mathscr{E}_1$. It is clear that this will be of importance in situations where a generalized Fredholm operator G is given and it is not so easy to explicitly deform the operator to one of this simple type.

5.9. *Products of order zero pseudodifferential operators.* Having seen that Definition 5.7 does incorporate the basic construction tying together elliptic operators and K-theory, let us see how we might arrive at the definition in the general case. Taking the K-theory product described at the end of Section 3 as a guide (and perhaps considering also the formula for the "product" of

elliptic operators computed at the level of symbols) we look for a formula for the product of $(\underline{G}, \mathscr{E}')$ and $(\underline{F}, \mathscr{E})$ of the form

$$\underline{G}\#\underline{F} = \underline{G}\hat{\otimes}\underline{1} + \underline{F}'$$

where $\underline{F}'$ is, say, a Grassman connexion as described above. The trouble with this is that the operator so defined is not a generalized elliptic operator. The problem, along with its solution, is illustrated by the following rather informally phrased proposition (as indicated by Kasparov in [**26**], this is the motivating example for his construction of the product).

5.10. PROPOSITION. *Let D_1 and D_2 be order zero elliptic pseudodifferential operators on smooth closed manifolds X_1 and X_2 and suppose that the symbols of D_1 and D_2 are unitary valued (this last condition is just so that the operators D_1 and D_2 determine generalized elliptic operators in the sense of Definition* 4.1). *The operator*

$$\underline{D}_1\hat{\boxtimes}\underline{1} + \underline{1}\hat{\boxtimes}\underline{D}_2 = \begin{pmatrix} D_1 \otimes 1 & -1 \otimes D_2^* \\ 1 \otimes D_2 & D_2^* \otimes 1 \end{pmatrix}$$

is not (in general) a pseudodifferential operator on $X_1 \times X_2$. However, there exist operators $\underline{M}$ and $\underline{N}$ such that:

(i) *$\underline{M}$ and $\underline{N}$ are positive, of grading degree zero, and $\underline{M} + \underline{N} = \underline{1}$; and*

(ii) *the operator $\underline{M}^{1/2}(\underline{D}_1\hat{\boxtimes}\underline{1}) + \underline{N}^{1/2}(\underline{1}\hat{\boxtimes}\underline{D}_2)$ is an elliptic order zero pseudodifferential operator (with unitary valued symbol), or at least a norm limit of such operators.*

PROOF. The problem is that if D is order zero then $D \otimes 1$ is not (in general) pseudodifferential; in fact it is not even pseudolocal in the sense of Section 4. This problem is familiar from the proof of the Atiyah-Singer Index Theorem in [**4**]; the solution is to pass from order zero operators to, say, order one operators, for if D is of order greater than zero then $D \otimes 1$ *is* a pseudodifferential operator, of the same order as D, or at least it is in the closure of this class (see [**4**, Section 5]). Given this, if Δ_1 and Δ_2 are positive, order two elliptic pseudodifferential operators on X_1 and X_2 then we can form $\underline{M}$ and $\underline{N}$ as diagonal matrices with entries the operators $(\Delta_1 \otimes 1 + 1 \otimes \Delta_2)^{-1}\Delta_1 \otimes 1$ and $(\Delta_1 \otimes 1 + 1 \otimes \Delta_2)^{-1} 1 \otimes \Delta_2$. □

In the light of this we try to modify our construction of the operator $\underline{G}\#\underline{F}$ to something of the form

$$\underline{G}\#\underline{F} = \underline{M}^{1/2}(\underline{G}\hat{\otimes}\underline{1}) + \underline{N}^{1/2}\underline{F}' \tag{5.3}$$

for appropriate "generalized pseudodifferential operators" $\underline{M}$ and $\underline{N}$. In fact this is exactly the prescription for the product given by Kasparov in [**28**]. It is a remarkable fact that in general, without any reference to order one pseudodifferential operators and so on as in Proposition 5.10, appropriate operators $\underline{M}$ and $\underline{N}$ can be constructed. This is the so-called Technical Theorem of Kasparov [**28**, Section 3] (see [**21**] for a simpler proof). The reader

is referred to [22, Appendix] for an exposition of the technical details of Kasparov's construction.

What then is the relationship between the formula (5.3) and Definition 5.7? The crucial observation is that for the particular choices of operators specified in Kasparov's formulation of the product, the *whole* of the expression (5.3) is an $\underline{F}$-connexion. This is because the operator $\underline{M}$ has the property that $\underline{T_\eta M}$ is a compact operator for every $\eta \in \mathscr{E}'$. (One might compare this with the property of the operator $(\Delta_1 \otimes 1 + 1 \otimes \Delta_2)^{-1}\Delta_1 \otimes 1$ constructed in Proposition 5.10, that the product of it with any operator of the form

$$\{\text{compact operator}\} \otimes \{\text{bounded operator}\}$$

is compact.) This justifies to some extent condition (i) of Definition 5.7. As for the second condition, note that since the definition of connexion takes into account the $\mathbb{Z}/2$-grading, we expect that the operators $\underline{G}\hat{\boxtimes}\underline{1}$ and $\underline{F}'$ should more or less anticommute (compare with the operators $\underline{G}\hat{\boxtimes}\underline{1}$ and $\underline{1}\hat{\boxtimes}\underline{F}$ considered in Section 3). Hence the expression in part (ii) of Definition 5.7 should more or less amount to $a^*(\underline{G}\hat{\boxtimes}\underline{1})^2 a$ (let us ignore for the moment the operators $\underline{M}$ and $\underline{N}$). Since $\underline{G}$ is essentially selfadjoint (see condition (iii) of Definition 5.1) this quantity should be more or less positive. The extent to which these statements are "more or less" true depends of course on the exact properties of the operators $\underline{M}$ and $\underline{N}$, but checking the details one finds that condition (ii) of Definition 5.7 is indeed satisfied by Kasparov's operator (5.3).

The upshot of all this is that the operators $\underline{G}\#\underline{F}$ of Definition 5.7 include the construction (5.3) of Kasparov, which in turn can be understood in terms of already known product constructions in K-theory and index theory. The important advantage of Definition 5.7 is that it allows for more general representations of the product, and in particular it is not necessary to represent the module $\mathscr{E}'$ as a direct summand of a standard Hilbert B-module in order to construct a Grassman connexion for $\underline{F}$. In applications this is often a significant point.

5.11. THEOREM. *There always exist product operators $\underline{G}\#\underline{F}$ as in Definition 5.7. The product is well defined at the level of KK-groups and gives rise to a homomorphism*

$$KK(A,B) \otimes KK(B,C) \xrightarrow{\otimes_B} KK(A,C).$$

SKETCH OF THE PROOF. The existence part of the theorem is proved as we have indicated: one constructs an operator $\underline{G}\#\underline{F}$ of the form (5.3). As for well-definedness, it has to be shown first that for fixed $(\underline{G}, \mathscr{E}')$ and $(\underline{F}, \mathscr{E})$ all possible choices for $\underline{G}\#\underline{F}$ are homotopic, and second that homotopic Kasparov (A,B)-cycles have homotopic products. The second part turns out to be very straightforward: we need only observe that a product of homotopies gives rise to a homotopy between the products. The first part is a little more

complicated. The heart of the matter is this: if J_1 and J_2 are generalized elliptic operators (acting on the same Hilbert modules) and if $J_1^* J_2$ is positive, modulo compacts, then J_1 and J_2 are homotopic. Indeed J_2 is homotopic to $(J_1 J_1^*) J_2$ and then following the straight line path from $J_1^* J_2$ to the identity operator gives a homotopy from $J_1(J_1^* J_2)$ to J_1. Somewhat extending this basic idea, one can show that if $\underline{J}_1$ and $\underline{J}_2$ are Kasparov (A, C)-cycles (these are the operators; we suppose that they act on the same module) and if the quantity $a^*\{\underline{J}_1\underline{J}_2 + \underline{J}_2\underline{J}_1\}a$ is positive, modulo compacts, then $\underline{J}_1$ and $\underline{J}_2$ are homotopic [**44**, Lemma 11]. Finally, condition (ii) of Definition 5.7 implies that if $\underline{G}\#\underline{F}$ is any product and if $(\underline{G}\#\underline{F})_{\mathrm{Kasp}}$ denotes the particular product (5.3) then the quantity

$$a^* \left\{(\underline{G}\#\underline{F})\,(\underline{G}\#\underline{F})_{\mathrm{Kasp}} + (\underline{G}\#\underline{F})_{\mathrm{Kasp}}\,(\underline{G}\#\underline{F})\right\} a$$

is positive, modulo compacts. (To be more accurate, it is necessary to choose $\underline{M}$ and $\underline{N}$ somewhat carefully, but one can easily show that $(\underline{G}\#\underline{F})_{\mathrm{Kasp}}$ is independent of the choice of $\underline{M}$ and $\underline{N}$, up to homotopy.) Thus every $\underline{G}\#\underline{F}$ is homotopic to $(\underline{G}\#\underline{F})_{\mathrm{Kasp}}$, and so any two products are homotopic. □

As one might expect, in applications, existence of the product is not the issue: the problem is to show that a reasonable candidate operator satisfies the conditions of Definition 5.7, and then (usually) to deform this candidate operator to a more easily understandable one. However, the abstract construction of the product (the Technical Theorem) is often useful in this, and of course it is needed to develop the properties of the theory.

The group $KK(A, B)$ is contravariantly functorial in A: if $f\colon A' \to A$ is a *-homomorphism then by composing with f we obtain from an action of A as operators on a Hilbert module an action of A'. Thus by composing with f we obtain from any Kasparov (A, B)-cycle a Kasparov (A', B)-cycle. The group $KK(A, B)$ is covariantly functorial in B using the extension of scalars construction of Example 3.23 (d). It is simple matter to check that the pairing (5.2) is functorial.

5.12. THEOREM. *The pairing* (5.2) *is associative, in the sense that if* $\alpha \in KK(A, B)$, $\beta \in KK(B, C)$, *and* $\gamma \in KK(C, D)$ *then*

$$\alpha \otimes_B (\beta \otimes_C \gamma) = (\alpha \otimes_B \beta) \otimes_C \gamma. \quad \square$$

For a proof of this important property, see [**44**, Proposition 20]. Actually, as we shall indicate in Section 7, it is possible to prove this theorem in an indirect way, by studying the algebraic properties of the KK-groups. On the other hand, it is not so hard to show directly that the two iterated products $(\underline{G}\#\underline{F})\#\mathbf{E}$ and $\underline{G}\#(\underline{F}\#\underline{E})$ are operator homotopic.

5.13. PROPOSITION. *Let A be any C^*-algebra and denote by 1_A the class in $KK(A, A)$ of the cycle associated to the generalized elliptic operator* $0\colon A \to 0$ (*we consider A as a Hilbert A-module in the usual way, together with the*

obvious action of A as operators on itself). Then for any $\alpha \in KK(A,B)$, and any $\beta \in KK(B,A)$, $1_A \otimes_A \alpha = \alpha$ and $\beta \otimes_A 1_A = \beta$. □

Given the almost trivial form of the cycle representing 1_A, this is an almost trivial direct computation (see [**44**, Proposition 17]).

Now let us return to generalized elliptic operators and K-theory. Suppose that we have constructed generalized elliptic operators F and G, giving rise to elements $\alpha \in KK(A,B)$ and $\beta \in KK(B,A)$, respectively, and suppose we wish to show that the maps $\mathrm{Index}_F: K(A) \to K(B)$ and $\mathrm{Index}_G: K(B) \to K(A)$ are inverse to one another. Identifying $K(A)$ with $KK(\mathbb{C},A)$ and $K(B)$ with $KK(\mathbb{C},B)$, we can attack this problem using the Kasparov product and its properties. Indeed, by Paragraph 5.8, we have that $\mathrm{Index}_F(\gamma) = \gamma \otimes_A \alpha$ and $\mathrm{Index}_G(\gamma) = \gamma \otimes_B \beta$, and so by the associativity of the product, the compositions of the two index maps are given by Kasparov product with $\alpha \otimes_B \beta$ and $\beta \otimes_A \alpha$. If these elements are 1_A and 1_B respectively, then by Proposition 5.12 the two compositions are the identity maps on $K(A)$ and $K(B)$. This suggests the following terminology.

5.14. DEFINITION. If there exist elements $\alpha \in KK(A,B)$ and $\beta \in KK(B,A)$ such that $\alpha \otimes \beta = 1$ and $\beta \otimes \alpha = 1$ then the C^*-algebras A and B are said to be *K-equivalent.* The elements α and β are said to be *invertible.*

From this we see that it is of some importance to be able to show that a given generalized elliptic operator is homotopic to the operator in Proposition 5.13, and so represents the element 1_A. Unfortunately this is very often a very difficult problem, at the heart of many applications of KK-theory. Here is one rather simple result (the proof is a simple modification of the argument in Proposition 3.27).

5.15. PROPOSITION. *If V is a Fredholm operator of index one on a Hilbert space H, then the generalized elliptic operator $F = V \boxtimes 1$ on the Hilbert A-module $H \boxtimes A$ (with the obvious action of A) is homotopic to the operator of Proposition* 5.13. □

It is a very useful exercise to compute the four products possible in Examples 4.7 and 4.8 and to reduce them to this proposition. The computation of the index of the operator V obtained ultimately boils down to the same computation as in Section 8 of [**4**], but there is a fair amount of work to be done to get to the point of computing $\mathrm{Index}(V)$.

6. The groups $KK_1(A,B)$. We shall (very) briefly indicate the construction and properties of the odd KK-groups $KK_1(A,B)$. They are constructed in the same manner as the groups $KK(A,B)$ out of homotopy classes of generalized elliptic operators, except that we consider not arbitrary generalized elliptic operators but only those which are *selfadjoint.* We can, as in Section 4, motivate this by starting with a selfadjoint elliptic pseudodifferential operator D and constructing from it a map

$$\mathrm{Index}_D: K^1(M) \to \mathbb{Z}$$

in the following manner. Denote by P the spectral projection of D corresponding to the interval $[0,\infty)$; since D is elliptic this is also a pseudodifferential operator. An element of $K^1(M)$ is given by a unitary $n\times n$ matrix $U=[u_{ij}]$ of continuous functions on M (see Section 2) and considering this as an operator on $L^2(M)\otimes\mathbb{C}^n$ we may form the operator

$$U_p=[Pu_{ij}P+(1-P)]$$

on this space. As a result of the pseudolocality of P it is easy to see that this is a Fredholm operator, and we define

$$\mathrm{Index}_D([U])=\mathrm{Index}(U_P).$$

This construction may be generalized as follows: if F is any selfadjoint generalized elliptic operator over A with coefficients in B then we can construct from F a homomorphism

$$\mathrm{Index}_F\colon K_1(A)\to K_0(B)$$

by means of the above formula. In the general context of Hilbert modules the Spectral Theorem is not applicable, and so we cannot directly define the operator P as above, but we may take, for example $P=1/2(F+1)$. Since the definition of generalized elliptic operator requires that F be "essentially unitary," if F is in addition selfadjoint then it is not hard to show that the operator P so defined is "essentially" a projection.

Generalizing things a little, using for example Bott Periodicity (or other explicit constructions), one can show that a generalized elliptic operator gives rise to homomorphisms

$$\mathrm{Index}_F\colon K_*(A)\to K_*(B)\qquad (*=0,1)$$

whilst a *selfadjoint* generalized elliptic operator gives rise to homomorphisms

$$\mathrm{Index}_F\colon K_*(A)\to K_{*+1}(B)\qquad (*=0,1).$$

We can ask then for a calculus for these more general maps along the lines of Section 5.

The most convenient way to incorporate the hypothesis of selfadjointness into the formalism of Section 5 is to make further use of $\mathbb{Z}/2$-gradings. Thus, given a Kasparov (A,B)-cycle $(\underline{F},\underline{\mathcal{E}})$, suppose that there is a supplementary unital action of the C^*-algebra $\underline{C}_1=\mathbb{C}\oplus\mathbb{C}$ on the Hilbert module $\underline{\mathcal{E}}$ such that

(i) $1\oplus -1$ acts as a grading degree 1 operator;
(ii) the action commutes with the action of A; and
(iii) the operator $1\oplus -1$ anticommutes with $\underline{F}$.

The effect of the action of $\underline{C}_1$ is that if we identify the even and odd parts of $\underline{\mathcal{E}}$ via $1\oplus -1$ then by virtue of (iii) the operator $\underline{F}$ (as in (5.1)) is selfadjoint. Thus this does capture the selfadjointness condition on $\underline{F}$.

Taking a small leap forward in abstraction, the above structure added to the notion of Kasparov (A,B)-cycle can be summarised in the notion of

Kasparov $(A, B \otimes \underline{C}_1)$-cycle, where we regard $\underline{C}_1$, and hence $A \otimes \underline{C}_1$, as a $\mathbb{Z}/2$-graded C^*-algebra by assigning the grading degree 1 to $1 \oplus -1$. We shall not go into this (the reader can refer to, say, [7] for the exact definition) except to say that by introducing the notions of $\mathbb{Z}/2$-graded C^*-algebras and "graded commutators" one can treat the two types of Index map described above on a completely equal footing. In particular, using the definition for the product as in Section 5, *mutatis mutandis*, we can extend the product to a pairing

$$KK_i(A, B) \otimes KK_j(B, C) \to KK_{i+j}(A, C),$$

where the indices are in $\mathbb{Z}/2$, and KK_0 denotes KK as in Section 5, whilst KK_1 denotes the group of homotopy classes of Kasparov $(A, B \otimes \mathbf{C}_1)$-cycles. The pairing has all the various possible functoriality and associativity properties of the product of Section 5.

This extension of the theory to cover the selfadjoint case is important for at least two reasons. First of all, many important operators in practical examples turn out to be selfadjoint (the prototype being the Dirac operator on the smooth closed spin^c-manifold of odd dimension) and it is very convenient to be able to incorporate them into K-theory. Secondly, just as it is useful to consider both the even and odd topological K-theory groups for a space, when analysing K-theory from a cohomological point of view, the same phenomenon occurs in KK-theory (see for example the next section).

Some final remarks:

(i) As one might expect, if F is selfadjoint and if we forget about this extra structure and regard F as determining an element of $KK(A, B)$, then the element is trivial.

(ii) We have that $K_1(B) \cong KK_1(\mathbb{C}, B)$. Thus, parallel to Section 3, $K_1(B)$ may be described in terms of selfadjoint Fredholm operators.

(iii) There is an invertible element of $KK_1(C_0(\mathbb{R}), \mathbb{C})$ which gives rise to a Bott Periodicity isomorphism

$$KK_1(C_0(\mathbb{R}) \otimes A, B) \cong KK_0(A, B).$$

(iv) The group KK_1 plays an important role in the theory of C^*-algebra extensions (see [**28**, Section 7]).

7. Characterization of KK-theory. In this section we want to comment on two questions concerning the KK-groups considered as functors on C^*-algebras. The first is about the extent to which KK-theory really is a systematization of the calculus of elliptic operators acting on K-theory groups. In the course of answering this, one is led to study the homological properties of the KK-groups, and from this the second question arises: is it possible to characterize KK-theory by some system of axioms, in the spirit of the Eilenberg-Steenrod axioms for ordinary homology and cohomology?

I shall begin by explaining exactly what is meant by the first question. The groups $KK(A, B)$ have been introduced by putting a natural equivalence relation on generalized elliptic operators, in order to analyze the maps

$\mathrm{Index}_F\colon K(A) \to K(B)$ more fully, and in particular to analyze the composition of two such maps. Thus we study the composition not at the level of the homomorphisms of the K-theory groups, but at the level of the elliptic operators inducing the homomorphisms. Since we are interested in, for example, deciding when the composition of two Index maps is the identity on K-theory, it is natural to ask whether or not we can always answer this problem by analysis at the level of the operators. Formulating the question more mathematically, we ask the following question: to what extent is the homomorphism

$$KK_*(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \tag{7.1}$$

injective? (We have been for the most part concerned with the even KK-groups, but with a little thought one can see that it is more natural for the purposes of the present question to consider both even and odd groups). It is also interesting to consider whether or not the map is *surjective*. For example if we are studying a map $\mathrm{Index}_F\colon K(B) \to K(A)$ which we believe to be an isomorphism, is it reasonable to seek an inverse map of the form $\mathrm{Index}_G\colon K(A) \to K(B)$?

Of course, (7.1) immediately suggests the idea of formulating some sort of Universal Coefficient Theorem for KK-theory. Considering the KK-groups from a homological point of view, it is then natural to begin by studying the excision properties of the KK-groups. (This line of reasoning does not represent the historical order of events, where questions concerning C^*-algebra extension theory made the computation of KK-groups important.) In fact for a large class of C^*-algebras the situation is very satisfactory, as the following result indicates.

7.1. THEOREM. *Suppose that*

$$0 \to J \to A \to A/J \to 0$$

is a short exact sequence of nuclear *C^*-algebras (see below). Then for any C^*-algebra D the sequences of abelian groups*

$$KK(D,J) \to KK(D,A) \to KK(D,A/J)$$

and

$$KK(A/J,D) \to KK(A,D) \to KK(J,D)$$

are exact in the middle. □

This result is due to Kasparov [**28**, Section 7]. The theorem is generalized and the proof simplified by Cuntz and Skandalis in [**17**], and then further extended in [**45**]. All the various generalizations involve some hypothesis related to nuclearity. (This condition on a C^*-algebra A is most easily characterized as the property that for any C^*-algebra B there be a unique C^*-norm ($\|x^*x\| = \|x\|^2$) on the algebraic tensor product $A \otimes B$ which is compatible ($\|a \otimes b\| = \|a\|\,\|b\|$) with the norms on A and B.) Although the class covered

in [45] is broad enough to include a great many C^*-algebras of interest (certainly including all commutative C^*-algebras, for example) it also does not include *all* the C^*-algebras which are of interest from the point of view of geometry and topology, and furthermore, it seems very likely that in fact the theorem is *not true* for all of these C^*-algebras. Before considering this in more detail, let us return to the Universal Coefficient Theorem. For a precise statement of the following result the reader is referred to [43].

7.2. THEOREM. *For a large class of C^*-algebras there is a natural exact sequence*

$$0 \to \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK_*(A,B) \to \mathrm{Hom}(K_*(A), K_*(B)) \to 0. \quad \square$$

The idea behind the proof is simple enough: we verify the exact sequence first for elementary C^*-algebras such as $C_0(\mathbb{R}^n)$, where the result follows from Bott Periodicity; then we show that the class of C^*-algebras for which the result is true is closed under various operations—for example, if two out of three C^*-algebras in a short exact sequence of nuclear C^*-algebras is in the class then so is the third, and so on. Finally we can obviously close the class under K-equivalence (see Definition 5.14).

Theorem 7.2 gives on the face of it a very satisfactory answer to the questions we posed at the beginning: for the class of C^*-algebras to which it applies, the theorem shows that, modulo perhaps some torsion, the calculus of the maps Index_F is exactly the same as Kasparov's calculus for the elliptic operators F. Unfortunately Skandalis has produced a sobering example [45] which shows that in fact the class of C^*-algebras to which Theorem 7.2 applies is limited and definitely does *not* contain certain C^*-algebras of importance in applications. In particular, if Π is a discrete, torsion-free, finite covolume subgroup of $\mathrm{Sp}(n,1)$ then the reduced group C^*-algebra $C_r^*(\Pi)$ is an example. Skandalis's argument in [45] (very crudely summarized) is as follows. He shows by careful analytic arguments that the C^*-algebra $C_r^*(\Pi)$ does not have a certain nuclearity property: in the terminology of [45] it is not *K-nuclear*. On the other hand every C^*-algebra which is K-equivalent to a K-nuclear C^*-algebra is itself K-nuclear, and since every commutative C^*-algebra is K-nuclear, $C^*(\Pi)$ is not K-equivalent to a commutative C^*-algebra. Finally, by a sort of geometric resolution argument it is not hard to show that the class of C^*-algebras for which Theorem 7.2 holds is exactly the class of C^*-algebras which are K-equivalent to commutative C^*-algebras.

This is of a great deal of significance in connection with the Baum-Connes conjecture mentioned in Paragraph 1.8. For given Π as above we can construct "Dirac" and "Dual Dirac" elements $\beta \in KK(C_0(T\Pi), C_r^*(\Pi))$ and $\alpha \in KK(C_r^*(\Pi), C_0(T\Pi))$ (for a suitable space $T\Pi$, as described in, say, [6]). One can show that

$$\beta \otimes_{C_r^*(\Pi)} \alpha = 1_{C_0}(T\Pi). \tag{7.2}$$

The conjecture is that β induces an isomorphism

$$\text{(7.3)} \qquad \_ \otimes \beta \colon K_*(C_0(T\Pi)) \to K_*(C_r^*(\Pi))$$

and one would like to prove this by showing that as well as the relation (7.2) one also has

$$\text{(7.4)} \qquad \alpha \otimes_{C_0(T\Pi)} \beta = 1_{C^*(\Pi)}.$$

However, (7.2) and (7.4) together would show that $C_r^*(\Pi)$ and $C_0(T\Pi)$ are K-equivalent, which, as Skandalis shows, they are not. The conclusion is that, assuming the Baum-Connes conjecture is true in this case, the product technique summarized by Definition 5.1 is not adequate for proving it. Of course, one cannot rule out the possibility of a proof that (7.3) is an isomorphism within the framework of KK-theory, but this example does suggest that beyond the class of K-nuclear C^*-algebras KK-theory may be of limited value.

Let us take the optimistic point of view that the map (7.2) *is* in fact as isomorphism. We might consider the possibility of some sort of alternative calculus for K-theory groups, more suited to dealing with non–K-nuclear C^*-algebras. In connection with this there is at least the following positive result (see [24] for an exact statement).

7.3. THEOREM. *There exists a bifunctor* $\mathbf{E}(A, B)$ *on* C^**-algebras with the following properties*

(i) $\mathbf{E}$ *is homotopy invariant and stable;*
(ii) *there is a product structure on* $\mathbf{E}$ *as in* 5.11 *and* 5.12;
(iii) *the sequences of Theorem* 7.1 (*with* $\mathbf{E}$ *replacing* KK) *are exact in the middle for an arbitrary short exact sequence of* C^**-algebras; and*
(iv) *there is a natural transformation* $KK \to \mathbf{E}$ *which is compatible with all the above structure and which is an isomorphism on the category of* K*-nuclear* C^**-algebras.*

Thus there *does* exist one alternative calculus for K-groups, namely this "E-theory" (by virtue of part (iii) of the theorem, except in the unlikely event that KK-theory is generally half-exact, "E-theory" really is distinct from KK-theory). Furthermore it is at least conceivable that the relation (7.4) does hold in $\mathbf{E}(C_r^*(\Pi), C_r^*(\Pi))$. Unfortunately $\mathbf{E}(A, B)$ is constructed in [24] by very abstract homotopy-theoretic and category-theoretic arguments, which offer no clue as to how to resolve such concrete questions. The mere existence of the theory does however lend credence to the idea that there are, waiting to be constructed, more viable alternative frameworks for K-theory groups, which might play a role in the resolution of problems such as the Baum-Connes Conjecture.

Having given the reader the impression that beyond the world of K-nuclear C^*-algebras the structure and behaviour of the KK-groups is very mysterious, it may seem paradoxical that there is in fact an extremely simple axiomatic

characterization of KK-theory on the category of all C^*-algebras, K-nuclear or not. This is based on the following simple but clever (and also somewhat surprising) observation of Joachim Cuntz: every Kasparov (A,B)-cycle $(\underline{F},\mathscr{E})$ is homotopic to one of the form $(\underline{T},\underline{H}_B)$ where $\underline{H}_B$ denotes the standard $\mathbb{Z}/2$-graded Hilbert B-module and $\underline{T}$ denotes the transposition operator

$$\underline{T}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

(see [12] or [22, Lemma 3.6]). The information contained in the cycle is thus compressed into the representations of A as operators on the two summands of $\underline{H}_B$; the elliptic operator has disappeared! Now, bearing in mind the isomorphism mentioned in Paragraph 3.19 (b) we see that the information contained in the cycle $(\underline{T},\underline{H}_B)$, namely the actions of A, amounts to a pair of *-homomorphisms

$$f_+, f_-\colon A\to M(B\otimes\mathscr{K}) \tag{7.5}$$

such that

$$f_+(a)-f_-(a)\in B\otimes\mathscr{K}\quad\text{for every } a\in A. \tag{7.6}$$

(The condition on f_+-f_- comes from the fact that $\underline{T}$ intertwines the action of A on $\underline{H}_B$.) A set of data of this new form is (more or less) what Cuntz calls a *quasihomomorphism* [12]. We shall not go into details (the reader is referred to [12], [13], and [22]), but simply state that from this one can construct in a rather algebraic fashion a homomorphism

$$\{f_+,f_-\}_*\colon K(A)\to K(B)$$

which turns out to be equal to the map Index_F constructed from the Kasparov (A,B)-cycle we started with. Analysis of this basic construction shows that Index_F, and in fact the general Kasparov product, is determined by the functorial properties of the KK-groups:

7.4. THEOREM. *There is a* unique *functorial pairing*

$$\mu\colon KK(A,B)\otimes KK(B,C)\to KK(A,C),$$

associative or not, subject to the normalization condition that $\mu(1_B,1_B)=1_B$ for every C^*-algebra B. □

For a proof of this, see [22]. The argument does not use the associativity of the Kasparov product, and in fact it is a simple exercise to derive associativity from the theorem. Pursuing the analysis a little further, we arrive at the following characterization of KK-theory (see again [22] for the proof, as well as a more precise statement).

7.5. THEOREM. *Regard KK-theory as a category* **KK** *in which the objects are C^*-algebras; the morphisms from A to B are the elements of $KK(A,B)$; and the composition law is the Kasparov product. Denote by F the functor from*

C^*-algebras to **KK** *which maps a *-homomorphism $f\colon A \to B$ to $f_*(1_A) \in KK(A,B)$. This functor F is the universal functor from C^*-algebras to an additive category which is:*

(i) *homotopy invariant;*

(ii) *stable (see Definition 2.7); and*

(iii) *split exact: if $0 \to J \to A \to A/J \to 0$ is split by a *-homomorphism from A/J to A, then $F(A) \to F(J) \to F(A/J)$ is split exact.* □

As an application of this, it is a simple matter to define the general product

$$KK(A_1, B_1 \otimes D) \otimes KK(D \otimes A_2, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2)$$

of **[28]** from the basic product of Section 5, as well as to establish its various associativity properties (see **[22**, Section 4.7**]**). Of course, since we have moved ourselves quite far away from the basic objects of interest, namely elliptic operators, such a construction of the product is of much more theoretical interest than practical use.

The overall significance (if any) of KK-theory having such a trivial algebraic characterization is not clear. However we should at least mention in closing that a very important aspect of the algebraic point of view on KK (which, as mentioned, is due to Cuntz) is its close connection with cyclic cohomology. This goes as follows: Cuntz constructs in **[14]** a C^*-algebra qA which has the property that homotopy classes of *-homomorphisms from qA to $B \otimes \mathscr{K}$ correspond to homotopy classes of pairs of *-homomorphisms as in (7.5) and (7.6). It follows easily that the set $[qA, B \otimes \mathscr{K}]$ of such homotopy classes (which happens to have a natural abelian group structure) is isomorphic to $KK(A,B)$. On the other hand, for any algebra A (meaning any algebra over the complex numbers, not necessarily with a norm of any sort) there is an algebraic version of qA, and the cyclic cohomology of A can be completely described in terms of traces on (ideals of) qA. The relationship of this to K-homology and the Chern Character is analyzed in **[16]**.

8. Notes. The purpose of this section is to give a short guide to the literature for those who want to pursue the subject further. I had intended to cover rather more material in these notes, in particular some sample computations of the Kasparov product, but a lack of space, time, and energy has prevented me from doing so. I hope nevertheless that these notes will be of some use, and that after looking through them the reader will be in a good position to turn to some of the papers mentioned below.

The basic reference for the theory described in these notes is Kasparov's article **[28]**: just about everything I have covered originates there. This paper is very difficult to read, mostly because a great many challenging new ideas are compressed into a comparatively small number of pages, but there are now a number of articles which clarify the results of **[28]** as well as extend them in certain directions. Here is a short account of this work (the section numbers below refer to **[28]**).

Sections 1 & 2. There is a very short and elegant proof of the Stabilization Theorem (Theorem 1.12) in [33]. The "Generalized Theorems of Voiculescu and Stinespring" (Theorems 1.16 and 1.17) are important from the point of view of C^*-algebra extension theory, but do not play a role in the applications of KK-theory to topology and geometry. Although the notions of $\mathbb{Z}/2$-graded C^*-algebra, $\mathbb{Z}/2$-graded Hilbert module, and so on are very important, the rather confusing material on orientations of Clifford algebras does not play a great role in the theory. (What is at stake here are $+/-$ signs: conventions of orientation).

Section 3. A simplified account of this section is given in [21].

Section 4. This material is covered in the article [44] of Skandalis. This is the approach to the product we followed in Section 5.

Section 6. Theorem 1 of this section is Theorem 19 of [44]. The material on $K(B)$ is roughly what was covered in Section 3 of this article. See for example [32] for details.

Section 7. For a treatment of C^*-algebra extension theory (a subject that was not touched upon at all here) the reader is referred to [8], [41], and [7]. The excision results are proved and generalized in [17] and [45].

As for applications of the theory, the article [29] gives a very rapid overview. The basic reference for applications to the Novikov conjecture is now Kasparov's article [30]. A systematic treatment of the Atiyah-Singer Index Theorem (along the lines of the proof in [4], [5]), using the KK-machine, is given in [11] which goes on to extend the theorem to the case of operators elliptic along the leaves of a foliation.

Apart from the references in these papers, the reader might also peruse the bibliography of [7].

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