Contents

1 Introduction 1
   1.1 Math versus Physics 1
   1.2 What This Book Is About 2
   1.3 A Physical versus a Mathematical Solution: An Example 6
   1.4 Acknowledgments 8

2 The Pythagorean Theorem 9
   2.1 Introduction 9
   2.2 The “Fish Tank” Proof of the Pythagorean Theorem 9
   2.3 Converting a Physical Argument into a Rigorous Proof 12
   2.4 The Fundamental Theorem of Calculus 14
   2.5 The Determinant by Sweeping 15
   2.6 The Pythagorean Theorem by Rotation 16
   2.7 Still Water Runs Deep 17
   2.8 A Three-Dimensional Pythagorean Theorem 19
   2.9 A Surprising Equilibrium 21
   2.10 Pythagorean Theorem by Springs 22
   2.11 More Geometry with Springs 23
   2.12 A Kinetic Energy Proof: Pythagoras on Ice 24
   2.13 Pythagoras and Einstein? 25

3 Minima and Maxima 27
   3.1 The Optical Property of Ellipses 28
   3.2 More about the Optical Property 31
   3.3 Linear Regression (The Best Fit) via Springs 31
   3.4 The Polygon of Least Area 34
   3.5 The Pyramid of Least Volume 36
   3.6 A Theorem on Centroids 39
   3.7 An Isoperimetric Problem 40
   3.8 The Cheapest Can 44
   3.9 The Cheapest Pot 47
vi CONTENTS

3.10 The Best Spot in a Drive-In Theater 48
3.11 The Inscribed Angle 51
3.12 Fermat’s Principle and Snell’s Law 52
3.13 Saving a Drowning Victim by Fermat’s Principle 57
3.14 The Least Sum of Squares to a Point 59
3.15 Why Does a Triangle Balance on the Point of Intersection of the Medians? 60
3.16 The Least Sum of Distances to Four Points in Space 61
3.17 Shortest Distance to the Sides of an Angle 63
3.18 The Shortest Segment through a Point 64
3.19 Maneuvering a Ladder 65
3.20 The Most Capacious Paper Cup 67
3.21 Minimal-Perimeter Triangles 69
3.22 An Ellipse in the Corner 72
3.23 Problems 74

4 Inequalities by Electric Shorting 76
4.1 Introduction 76
4.2 The Arithmetic Mean Is Greater than the Geometric Mean by Throwing a Switch 78
4.3 Arithmetic Mean ≥ Harmonic Mean for n Numbers 80
4.4 Does Any Short Decrease Resistance? 81
4.5 Problems 83

5 Center of Mass: Proofs and Solutions 84
5.1 Introduction 84
5.2 Center of Mass of a Semicircle by Conservation of Energy 85
5.3 Center of Mass of a Half-Disk (Half-Pizza) 87
5.4 Center of Mass of a Hanging Chain 88
5.5 Pappus’s Centroid Theorems 89
5.6 Ceva’s Theorem 92
5.7 Three Applications of Ceva’s Theorem 94
5.8 Problems 96

6 Geometry and Motion 99
6.1 Area between the Tracks of a Bike 99
6.2 An Equal-Volumes Theorem 101
6.3 How Much Gold Is in a Wedding Ring? 102
6.4 The Fastest Descent 104
CONTENTS

6.5 Finding $\frac{d}{dt}\sin t$ and $\frac{d}{dt}\cos t$ by Rotation 106
6.6 Problems 108

7 Computing Integrals Using Mechanics 109
7.1 Computing $\int_0^1 \frac{x \, dx}{\sqrt{1-x^2}}$ by Lifting a Weight 109
7.2 Computing $\int_0^3 \sin \pi \, dt$ with a Pendulum 111
7.3 A Fluid Proof of Green’s Theorem 112

8 The Euler-Lagrange Equation via Stretched Springs 115
8.1 Some Background on the Euler-Lagrange Equation 115
8.2 A Mechanical Interpretation of the Euler-Lagrange Equation 117
8.3 A Derivation of the Euler-Lagrange Equation 118
8.4 Energy Conservation by Sliding a Spring 119

9 Lenses, Telescopes, and Hamiltonian Mechanics 120
9.1 Area-Preserving Mappings of the Plane: Examples 121
9.2 Mechanics and Maps 121
9.3 A (Literally!) Hand-Waving “Proof” of Area Preservation 123
9.4 The Generating Function 124
9.5 A Table of Analogies between Mechanics and Analysis 125
9.6 “The Uncertainty Principle” 126
9.7 Area Preservation in Optics 126
9.8 Telescopes and Area Preservation 129
9.9 Problems 131

10 A Bicycle Wheel and the Gauss-Bonnet Theorem 133
10.1 Introduction 133
10.2 The Dual-Cones Theorem 135
10.3 The Gauss-Bonnet Formula Formulation and Background 138
10.4 The Gauss-Bonnet Formula by Mechanics 142
10.5 A Bicycle Wheel and the Dual Cones 143
10.6 The Area of a Country 146

11 Complex Variables Made Simple(r) 148
11.1 Introduction 148
11.2 How a Complex Number Could Have Been Invented 149
11.3 Functions as Ideal Fluid Flows 150
11.4 A Physical Meaning of the Complex Integral 153
11.5 The Cauchy Integral Formula via Fluid Flow 154
11.6 Heat Flow and Analytic Functions 156
11.7 Riemann Mapping by Heat Flow 157
11.8 Euler’s Sum via Fluid Flow 159

Appendix. Physical Background 161
A.1 Springs 161
A.2 Soap Films 162
A.3 Compressed Gas 164
A.4 Vacuum 165
A.5 Torque 165
A.6 The Equilibrium of a Rigid Body 166
A.7 Angular Momentum 167
A.8 The Center of Mass 169
A.9 The Moment of Inertia 170
A.10 Current 172
A.11 Voltage 172
A.12 Kirchhoff’s Laws 173
A.13 Resistance and Ohm’s Law 174
A.14 Resistors in Parallel 174
A.15 Resistors in Series 175
A.16 Power Dissipated in a Resistor 176
A.17 Capacitors and Capacitance 176
A.18 The Inductance: Inertia of the Current 177
A.19 An Electrical-Plumbing Analogy 179
A.20 Problems 181

Bibliography 183

Index 185