# Compressive Radar Imaging Using White Stochastic Waveforms

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***Abstract*—In this paper, we apply the principles of compressive sampling to ultra-wideband (UWB) stochastic waveform radar. The theory of compressive sampling says that it is possible to recover a signal that is parsimonious when represented in a particular basis, by acquiring few projections on to an appropriate basis set. Drawing on literature in compressive sampling, we develop the theory behind stochastic waveform-based compressive imaging. We show that using stochastic waveforms for radar imaging, it is possible to estimate target parameters and detect targets by sampling at a rate that is considerably slower than the Nyquist rate and recovering using compressive sensing algorithms. Thus, it is theoretically possible to increase the bandwidth (and hence the spatial resolution) of an ultra-wideband radar system using stochastic waveforms, without significant additions to the data acquisition system. Further, there is virtually no degradation in the performance of a UWB stochastic waveform radar system that employs compressive sampling. We present numerical simulations to show that the performance guarantees provided by theoretical results are achieved in realistic scenarios.**

## I. Introduction

Radar systems can be classified to be of two types [1], (1) those employing an analog system for detecting the targets using an analog matched filter, and (2) systems that sample and quantize the reflected radar signal and process it as digital signals. With the advent of fast analog to digital converters (ADC), radar systems of the latter type are gaining popularity for high resolution radar imaging. While digital radar systems provide operational flexibility, their resolution has been limited by the sampling rate of the fastest available ADCs. The performance of digital radar systems is further degraded by the trade-off that exists between the rate of sampling and the number of quantization levels of an ADC.

Compressive sensing is a new paradigm in signal processing with far-reaching implications. In its most useful form, it presents a new approach to digital signal processing by enabling signal recovery from under-sampled measurements. It has been shown that exact recovery of sparse signals is possible by acquiring as few as $O(S\log(N))$ ($N$ is the length of the vector that is also $S$-sparse in some representation) projections of a signal into an appropriate *measurement* basis. While it is possible to recover signals that are sparse in a particular basis of representation, from under-sampled measurements by searching for the vector that, at the same time, is most consistent with the measurements and also minimizes the number of non-zero values, as given by the $l_0$ norm, the problem of finding such a vector is intractable. The most important results concerning compressive sensing were derived in [7] and [8]. It was shown that the minimization of the $l_0$ norm is equivalent, under certain easily verifiable conditions (the restricted isometry property (RIP)), to the tractable convex optimization problem of minimizing the $l_1$ norm. The original papers on compressive sensing proved that random measurement bases drawn from sub-Gaussian and Gaussian distributions almost always satisfied RIP for a large class of representation vectors. Thus, compressive sensing has the potential to significantly reduce the sampling requirements while maintaining and in some cases, improving the performance of the systems.

Random noise radar [5] involves transmitting waveforms that are generated as ultra-wideband stochastic processes. The ultra-wideband property implies that high range resolutions can be obtained. Random noise waveforms have the ability to achieve approximately thumbtack ambiguity functions due to the statistical independence of the waveform at different time intervals. Further, the stochastic and non-repetitive nature of continuous time random waveforms means that they cannot be intercepted or detected by an adversary. Early stochastic waveform radars used analog processing to detect targets [5]. However, with advances in ADC technology, digital systems have become popular in recent times [6] for random waveform radar. In digital stochastic waveform radar systems, a high-rate ADC is employed to acquire samples of the reflected signal, which are then stored and processed as quantized discrete vectors to extract information about the targets.

The randomness of the transmit waveform makes stochastic waveforms suitable for compressive sensing systems. We undertake a systematic study of the performance of compressive stochastic waveform radar systems. Applying the state of the art in compressive sensing to radar systems, we: (1) show theoretical results regarding target estimation, (2) study the robustness of compressive radar systems to uncorrelated additive noise; (3) study the performance of compressive estimators, (4) simulate receiver operating characteristics of compressive radar using phase transition diagrams [10] and receiver operating curves. We demonstrate the suitability of stochastic waveforms (*noise waveforms*) for building practical radar systems based on compressive sensing. The paper is organized as follows. In section 2, we present the theory of compressive sensing with circulant/Toeplitz matrices [9] and outline its implications to white stochastic waveform radar. The estimation performance of compressively sampled random waveforms was shown in [4] to be better than the best-case least-squares estimators in the context of channel sensing. We state this result to emphasize the advantages of compressive sensing. Then, we outline the matched-pursuit-based compressive detection scheme presented in [11] and apply it to radar systems. In section 3, we present numerical results on the performance of compressive-stochastic-waveform radar systems. We detail the application of Donoho-Tanner phase transition diagrams [10] for characterizing the performance of radar systems, and show that even with stochastic waveforms and under-sampled reflected radar signals, it is possible to recover sufficiently sparse signals with adequate accuracy. Using Monte-Carlo simulations, we study the receiver operating characteristics of the proposed compressive radar system. We conclude the paper in section 4 with comments on the scope for future work.

## II. Basics of Compressive Stochastic Waveform Radar

### A. Compressive Radar

We consider one-dimensional radar imaging in our study of compressive radar. The problem of radar imaging involves sensing and processing a transmitted signal that has been linearly filtered by a transfer function representing the target scene. In the most general case, the reflected signal y(t) can be written as a convolution of the transmitted signal x(t) with the target scene impulse response denoted by s(t). In the continuous domain, this can be written as,

$$y(t) = \int_{-\infty}^{+\infty} x(t)s(t-\tau)d\tau + \eta(t). \tag{1}$$

When discretized, this becomes,

$$y[n] = \sum_{k=1}^{N} x[k]s[n-k] + \eta[n], \tag{2}$$

$$y = Xs + \eta, \tag{3}$$

with $s, y \in \mathbb{R}^N$. Furthermore we assume that the vector $s$, representing the target scene is sparse, i.e, $supp(s) = S << N$, where $supp(.)$ is the count of the number of non-zero entries. The matrix $X \in \mathbb{R}^{N\times N}$ is Toeplitz (or circulant). The objective of the radar imaging problem is to accurately recover the target characteristics $s$ from the reflected waveform $y$. Conventionally, the problem can be cast as the minimization of the $l_2$ norm of the estimation error, i.e., $s^* = \arg\min_{s\in\mathbb{R}^N} ||y - Xs||_{l_2}$. The solution to this problem is well known and is given by the estimator, $s^* = (X^TX)^{-1}X^Ty$. However, in the absence of any model order selection, the vector $y$ needs to be at least as long as $s$, in other words, the system should be fully, or over-determined. Compressive sensing answers the question- is it possible to blindly acquire fewer measurements, without using any adaptive, model order selection? In terms of radar imaging, this question can be restated as whether it is possible to acquire fewer measurements while still achieving the resolution limits imposed by the bandwidth. Compressive sensing proposes an estimator based on measuring the vector $y$ using a collection of $M < N$ measurement basis in $\mathbb{R}^N$. Rather than the traditional least squares approach, the solution to the estimation problem is,

$$s^* = \arg\min_{s\in\mathbb{R}^N} ||s||_{l_1} \ subject \ to \ ||z - \Psi Xs||_{l_2} \leq \xi. \tag{4}$$

In its most general form, the reconstruction theorem of compressive sensing states that this problem yields an accurate solution when the matrix $A = \Psi X$ satisfies the restricted isometry property (RIP) with as few as $O(S \log N)$ measurements, i.e., for some $\delta_S$, and $u \in \mathbb{R}^N$, $(1-\delta_S)||u||_{l_2} \leq ||Au||_{l_2} \leq (1+\delta_S)||u||_{l_2}$. RIP and incoherence properties have been shown to hold for different pairs of matrices, such as the random matrices $\Psi$ and arbitrary $X$. Examples of incoherent pairs include spike bases and Fourier bases. The most general procedure proven to recover exactly is when each element of $\Psi$ is drawn randomly from Gaussian and sub-Gaussian distributed random variables. Implementing this would need the acquisition of $M$ random measurements of a time domain signal. The measurement hardware should then consist of $M$ channels of correlation with random measurements. Designing such hardware would be problematic for the sampling rates typically required for ultra-wideband radar systems. The simple system we envision for compressive radar systems should consist only of sampling the reflected signal at a rate lower than the Nyquist rate. Stochastic waveforms are a natural candidate, as we outline in the following sub-section.

### B. Stochastic Waveforms and RIP

Proving the uniqueness of the solution to Equation ( 4) requires that $X$ satisfies RIP. We outline that a circulant or Toeplitz matrix generated from a random, Gaussian, vector satisfies RIP. Particular forms of such matrices have been studied in [3], [4], [9]. In [3], circulant/Toeplitz matrices were constructed such that they represent the operation of *random demodulation* and were shown to satisfy RIP. The results of [3] however, do not generalize to random circulant matrices which are generated from a sequence of independent random variables as is needed in stochastic waveform radar imaging. Random circulant matrices with the first row made up of a sequence of independent random variables were studied by [4] in the context of channel estimation. Using a method primarily based on the Gershgorin disk theorem, the authors stated that RIP and subsequently, recovery using an $l_1$ minimization procedure, is guaranteed when $M$ is at least $O(S^2 \log N)$. Numerical simulations in the present paper (phase transition diagrams in Section 3.2) and elsewhere suggest that this bound is too conservative, as we see in numerical simulations that with as few as $O(S \log N)$ samples, reconstruction is accurate. The theoretical result was improved upon in [9], for the particular case where the first row of X is a sequence of Bernoulli random variables taking values $\pm 1$ with equal probability. The main result of [9] presents a stronger bound for RIP than [4], with the minimum number of samples needed being $O(S\log^{\alpha} N)$ $(\alpha \geq 1)$ using a signal modeled such that the signs of the nonzero entries are Bernoulli random. The results of [9] also hold for circulant and Toeplitz matrices constructed from Gaussian random variables. We restate as Theorem 1, the main theorem of [9] for the case of a circulant matrix that is constructed with the first row as entries drawn at random from a Gaussian distribution of variance $1/\sqrt{N}$ and mean $0$.

*Theorem 1:* Let $s \in \mathbb{R}^N$ and $X \in \mathbb{R}^{N\times N}$, $\Psi \in R^{M\times N}$ be as defined earlier, and $z = \Psi X s$. There exist constants C and $\alpha$, such that recovery of the vector $s$ is guaranteed with probability exceeding $1-\epsilon$ by solving the problem in Equation 4 if, $M \geq CS\log^{\alpha}\frac{N}{\epsilon}$.

### C. Comparison with Best Model Selection

Conventionally, model order selection methods have been used to reduce the dimensionality of problems in radar signal processing. When viewed as a method of estimation, the $l_1$ minimization problem can be interpreted as selecting the model order of the estimator. It has been shown [2] that the performance of compressive sensing is within a $log$ factor of the so called 'oracle' estimator- an estimator with complete knowledge of the location of non-zero members of s. This suggests that even the unrealistic estimator that works with full knowledge of the indices of the non-zero elements, would still only provide performance that is better than compressive sensing by a log factor [2], [4].

## III. Numerical Simulations

In order to understand the implications of the theoretical results underlying compressive radar imaging using stochastic waveforms, it is necessary to look at numerical simulations involving basic detection and estimation. In our simulations, we used discrete sequences of arbitrary lengths to simulate the performance of the compressive detector, and compressive estimator. Numerically, the difference between the compressive radar imaging system proposed above and other applications of compressive sensing arises due to the fact that the matrix that maps the target scene to the under-sampled reflected signal is a random Toeplitz (or circulant) matrix. To a limited extent, such a measurement system has been studied for channel estimation [4] and proposed for use in Fourier optics and synthetic aperture radar imaging [3]. Adopting the theory of compressive sensing for radar systems begs a thorough numerical study of the resolution limitations compressive recovery, detection performance of a compressive receiver, and robustness to noise.

### A. Receiver Operating Characteristics for Compressive Estimation

We study the detection performance of compressive radar in the context of detecting the presence or absence of a single target, using the greedy 'incoherent detection algorithm' described in [11]. The two hypothesis considered are as in conventional binary hypothesis testing, $\mathcal{H}_0 : z = \Phi\eta$ (target absent) and $\mathcal{H}_1 : z = \Phi X s + \Phi\eta$ (target present). The algorithm is essentially the traditional matching pursuit algorithm, with the modification that the pursuit stops when the $l_\infty$ norm, defined as the maximum element of the estimated vector is found to be above a fixed threshold. The algorithm has only been studied empirically, without a thorough theoretical analysis. In this paper, we present receiver operating characteristics of this algorithm in the present context. The matching pursuit algorithm, which the incoherent detection algorithm is based on is guaranteed to converge to a solution when RIP is satisfied, as is the case here. Thus, a convergence to the solution would be seen in the presence of a target, with the subsequently recovered vector lying above the detection threshold. Figure 1 shows the curves for the probability of error and false alarm for an arbitrarily fixed threshold. In the absence of theoretical results, we study the application of this algorithm using Monte-Carlo simulations for computing the probabilities of false alarm and detection. The performance of the above compressive detector in the context of radar systems is seen in Figure 2 to be reasonably good at even low signal to noise ratios.

### B. Phase Transition Diagrams

Conventional radar systems employ receiver operating characteristics (RoC) curves to calibrate the performance of radar

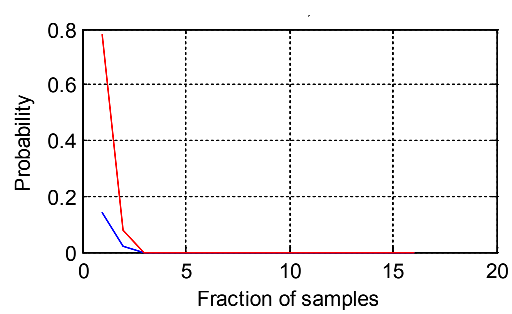



Fig. 1: The probability of error (red curve) and false alarm (blue curve) plotted as a function of the rate of fraction of samples acquired.

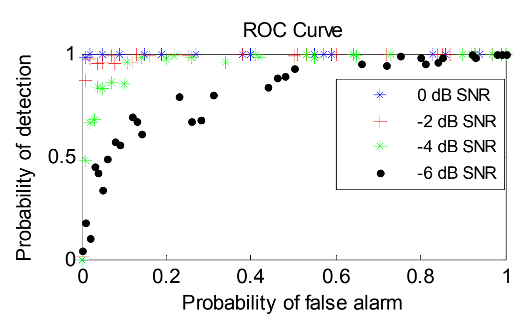



Fig. 2: The receiver operating curves for different values of signal to noise ratio.

detection systems. However, the performance of compressive radar systems cannot be adequately described using RoC curves due to the additional parameters of sparsity and rate of under-sampling affecting receiver performance. In this section, we study phase transition diagrams [10] in the context of compressive sensing to verify the performance of circulant random matrices with random under sampling. Phase transition diagrams present a numerical study of the $l_0$-$l_1$ equivalence that forms the basis of compressive sensing. They are an indirect way of numerically verifying RIP. In our simulations, we used a $256$ length transmit vector and discretized the $(\delta, \rho)$ phase space into a $64 \times 64$ grid. Over several realizations, we computed the average miss rate, which is the average percentage of non-zero samples ($l_0$ norm) that are recovered with a sample-wise error that is greater than a threshold that is related to the variance of the additive noise. The recovery was performed using the spectral projected gradient algorithm [12], [13]. The suitability of stochastic waveforms for compressive radar systems is indicated by the fact that the phase transition diagrams are close to the phase transition diagrams that arise from the general compressive sensing setting of acquisition using random measurements [10]. The phase transition diagram of Figure 3 was generated by acquiring samples of the reflected signal at random intervals. Similar, desirable phase transitions are seen when the reflected signal is uniformly under-sampled as shown in Figure 4, suggesting that a further simplification of the hardware will not degrade the performance significantly. The boundary line between

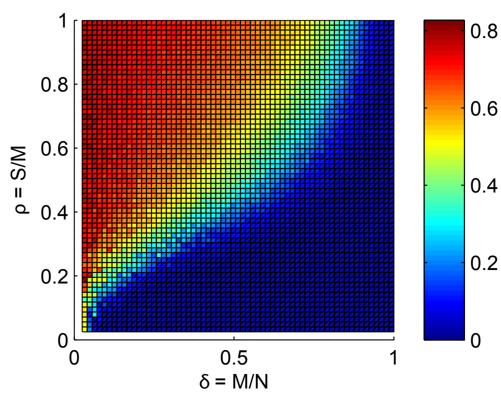



Fig. 3: Phase transition diagram for a Gaussian random circulant matrix with signal to noise ratio of 20 dB and random under-sampling.

the red and blue areas is similar to the case of random measurement matrices [10] and indicates that RIP holds with $M = O(S \log N)$. In the context of compressive sensing, phase transition diagrams are a representation of the success of compressive recovery in terms of the probability of success plotted in the phase space of the pair $(\delta, \rho)$ where, $\delta = M/N$ corresponds to the number of samples acquired and $\rho = S/M$, is a ratio of the sparsity of the signal and the number of samples acquired. The performance of compressive estimation

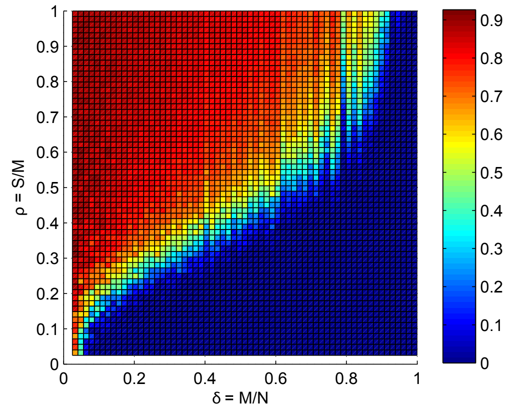
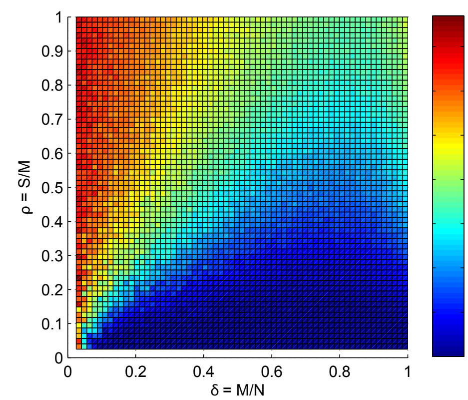
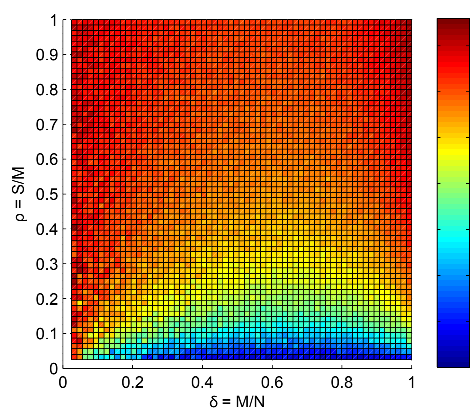



Fig. 4: Phase transition diagram for a Gaussian random circulant matrix with signal to noise ratio of 20 dB and uniform under-sampling.

is seen to deteriorate for low SNRs in Figure 5(a) (10 dB SNR) and Figure 5(b) (0 dB SNR). At 0 dB SNR, recovery is only




(a) SNR 10 dB




(b) SNR 0 dB

Fig. 5: Phase transitions at low SNRs.

possible when a large number of samples are acquired and that too only for very sparse target scenes. This suggests that there is scope for developing compressive recovery algorithms that are more robust to noise. Further, it is seen that lower the SNR, the further the deviation from RIP and $l_0$-$l_1$ equivalence. From a radar systems perspective, phase transition diagrams are a way of calibrating compressive radar systems. Compressive radar imaging is only non-blind in the sense that the operator of the system needs to have an approximate idea of the sparsity of the target scene. The uncertainty in the knowledge of the approximate sparsity of the target scene can be quantified in terms of the phase transition diagrams. For instance, it can be inferred from Figure 3 that if a radar system is intended to operate in an environment with 20 dB SNR, and with 2% of the range cells populated by point scatterers, almost all the targets can be recovered by acquiring as few as 10% of the total samples.

## IV. Conclusion

In this paper, through theoretical arguments and numerical simulations, we showed the suitability of stochastic waveforms for compressive radar systems. It is possible to recover target information and perform detection by sampling the reflected signal at rates far lower than the Nyquist rate. The restricted isometry property of circulant matrices generated from a Gaussian random vector was shown to compare well with that of the fully random matrix by looking at phase transition diagrams, confirming theoretical results. We also outlined how phase transition diagrams can be used to build and operate compressive radar systems.

We are currently working on extending this paper by developing the complete proof for the RIP for Gaussian circulant matrices with $M$ as close to the experimental asymptotic bound of $O(S \log N)$ and by studying the performance of different algorithms for estimation and detection.

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