TOPOLOGICAL COMPLETELY POSITIVE ENTROPY IS NO SIMPLER IN $\mathbb{Z}^2$-SFTS

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ABSTRACT. We construct $\mathbb{Z}^2$-SFTs at every computable level of the hierarchy of topological completely positive entropy (TCPE), answering Barbieri and García-Ramos, who asked if there was one at level 3. Furthermore, we show the property of TCPE in $\mathbb{Z}^2$-SFTs is coanalytic complete. Thus there is no simpler description of TCPE in $\mathbb{Z}^2$-SFTs than in the general case.

1. INTRODUCTION

Despite their similar definitions, the shifts of finite type (SFTs) over $\mathbb{Z}$ and the SFTs over $\mathbb{Z}^2$ often display very different properties. For example, there is an algorithm to determine whether a $\mathbb{Z}$-SFT is empty, but the corresponding problem for $\mathbb{Z}^2$-SFTs is undecidable [2]. Similarly, the possible entropies achievable by a $\mathbb{Z}$-SFT have an algebraic characterization (see e.g. [11, Chapter 4]), but for a $\mathbb{Z}^2$-SFT, the possible entropies are exactly the non-negative numbers obtainable as the limit of *computable* decreasing sequences of rationals [9]. The appearance of computation in both cases is explained in part by the original insight of Wang [17] that an arbitrary Turing computation can be forced to appear in any symbolic tiling of the plane that obeys a precisely crafted finite set of local restrictions.

The paper of Hochman and Meyerovitch revived the idea that superimposing these computations on an existing SFT allows the computations to "read" what is written on the existing configurations, and eliminate any configurations which the algorithm deems unsatisfactory. The effect is to forbid more (even infinitely many) patterns from the original SFT, at the cost of covering it with computation graffiti. It is not well-understood which classes of additional patterns can be forbidden in this way; Durand, Levin and Shen [5] have found complexity-related restrictions. Also, the computation infrastructure has the potential to modify more properties of the original SFT beyond forbidding words. For example, in [9], since they wanted to control entropy, they needed the computation to contribute zero entropy to the SFT. Thus while $\mathbb{Z}^2$-SFTs can demonstrate universal behavior in some cases, it is not at all obvious when they will do so.

In a recent series of papers, Pavlov [12, 13] and Barbieri and García-Ramos [1] explored the property of topological completely positive entropy (TCPE) in $\mathbb{Z}^d$-SFTs. Defined by Blanchard [3] as a topological analog of the K-property for measurable dynamical systems, TCPE seems rather poorly behaved compared to the K-property. However, Pavlov was able to give a simple characterization of TCPE for $\mathbb{Z}$-SFTs. The question then remained: will there also be a simple

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characterization for $\mathbb{Z}^d$-SFTs? In this paper we give a negative answer; TCPE is just as complex in $\mathbb{Z}^2$-SFTs as it is in general topological dynamical systems.

By definition, a topological dynamical system has TCPE if all of its nontrivial topological factors have positive entropy. Although we have not formally introduced these terms, it should be clear that this definition involves a quantification over infinite objects (the nontrivial topological factors). Contrast this with, for example, the definition of a Cauchy sequence of real numbers: (x_n) is Cauchy if for every rational $\varepsilon > 0$ there is a natural number N such that etc. Here all the quantifications involve finite objects. A property that can be expressed using only quantifications over finite objects is called *arithmetic* and such properties are typically easier to work with than properties which require a quantification over infinite objects in their definition.

Roughly speaking (see the Preliminaries for precise definitions), a property is coanalytic, or Π^1_1, if it can be expressed using a single universal quantification over infinite objects[1] and any amount of quantification over finite objects. A property is Π^1_1-complete if it is universal among Π^1_1-properties. If a property is Π^1_1-complete, it has no arithmetic equivalent description.

In [12] and [13], Pavlov introduced two arithmetic properties that a $\mathbb{Z}^d$-SFT X could have. One of them, ZTCPE, he showed was strictly weaker than TCPE. The other property implied TCPE, and he asked if it provided a characterization. This question was answered in the negative by Barbieri and García-Ramos [1], who constructed an explicit counterexample with $d = 3$. We generalize both of these results by showing that there is no arithmetic property that characterizes TCPE in the $\mathbb{Z}^2$-SFTs.

Theorem 1. *The property of TCPE is Π^1_1-complete in the set of $\mathbb{Z}^2$-SFTs.*

In the course of proving their result, Barbieri and García-Ramos defined an ω_1-length hierarchy within TCPE which stratified TCPE into subclasses. Their explicit counterexample was a $\mathbb{Z}^3$-SFT at level 3 of this hierarchy, and asked whether there could be a $\mathbb{Z}^2$-SFT at level 3. We answer this question in a quite general way. Standard methods of effective descriptive set theory imply that the TCPE rank of any $\mathbb{Z}^2$-SFT must be a *computable ordinal*, that is, a countable ordinal α for which there is a computable linear ordering $R \subseteq \omega \times \omega$ whose order type is α. We show that this is the only restriction. Below, ω_1^{ck} denotes the supremum of the computable ordinals.

Theorem 2. *For any ordinal $\alpha < \omega_1^{ck}$, there is a $\mathbb{Z}^2$-SFT with TCPE rank α.*

In fact, these two main theorems are closely related; a Π^1_1 set can always be decomposed into an ordinal hierarchy of simpler subclasses. Such hierarchy is called a Π^1_1 rank and a standard reference on the topic is [10]. Frequently, when all subclasses are populated, the same methods used to populate the hierarchy yield a proof of Π^1_1-completeness. That has happened in this case.

Silvere Gangloff has kindly let us know of his progress on this problem: he has independently constructed $\mathbb{Z}^2$-SFTs of rank α for each $\alpha < \omega^2$ by a different method [8]. We have also learned Ville Salo has recently constructed $\mathbb{Z}$-subshifts of all TCPE ranks [16], and conjectured our Theorem 2. The author would also like to thank Sebastián Barbieri for interesting discussions on this topic.

[1]technically, quantification over elements of a Polish space

2. Preliminaries

2.1. **Subshifts and TCPE.** Let Δ denote a finite alphabet. A $\mathbb{Z}^d$-*subshift* is a subset of $\Delta^{\mathbb{Z}^d}$ that is topologically closed (in the product topology, where Δ has the discrete topology) and closed under the d-many shift operations and their inverses. An element of Δ^S is also called a *configuration.* A *pattern* is an element of Δ^S where $S \subseteq \mathbb{Z}^d$ is a finite subset. A pattern w *appears in* a configuration x if there is some $g \in \mathbb{Z}^d$ such that $w = x \restriction g^{-1}(S)$. A pair of patterns w and v *coexist* in x if they both appear in x. If $g \in \mathbb{Z}^d$, and w is a pattern or x a configuration, let gw and gx denote the corresponding shifted versions of w and x, that is, $gx(h) = x(g^{-1}h)$ and $gw(h) = w(g^{-1}h)$.

A subshift $X \subseteq \Delta^{\mathbb{Z}^d}$ is completely characterized by the set of patterns which do not appear in any configuration of X. Conversely, for any set F of patterns, the set

$$W_F := \{x \in \Delta^{\mathbb{Z}^d} : \text{ for all } w \in F, w \text{ does not appear in } x\}$$

is a subshift. A subshift X is called a *shift of finite type* if $X = W_F$ for some finite set F of forbidden patterns.

More generally, a $\mathbb{Z}^d$-*topological dynamical system* (TDS) is a pair (X, T) where X is compact separable metric space and T is an action of $\mathbb{Z}^d$ on X by homeomorphisms. A $\mathbb{Z}^d$-subshift is a special case of this. If (X, T) and (Y, S) are two $\mathbb{Z}^d$-TDS, we say that (Y, S) is a *factor* of (X, T) if there is a continuous onto function $f : X \to Y$ such that $Sf = fT$. A *sofic* $\mathbb{Z}^d$-subshift is a $\mathbb{Z}^d$-subshift which is a factor of a $\mathbb{Z}^d$-SFT.

A TDS has *topological completely positive entropy* if all of its nontrivial factors have positive entropy. The unavoidable trivial factor is one where Y consists of a single element only.

For the purposes of this paper, we work almost entirely with an equivalent characterization of TCPE due to Blanchard [4]. This characterization makes use of Blanchard's local entropy theory and so a few definitions will be required.

If $\mathcal{U}$ is an open cover of a subshift X, let $\mathcal{U}_n$ denote the open cover of X which is the common refinement of the shifted covers $g^{-1}\mathcal{U}$ for $g \in [0, n)^d$,

$$U_n = \bigvee_{g \in [0,n)^d} g^{-1}\mathcal{U}.$$

Let $\mathcal{N}(\mathcal{U}_n)$ denote the smallest cardinality of a subcover of $\mathcal{U}_n$. Then $\log \mathcal{N}(\mathcal{U}_n)$ can be thought of as the minimum number of bits needed to communicate, for each $x \in X$ and $g \in [0, n)^d$, an element of $\mathcal{U}$ containing $g^{-1}x$. The *topological entropy* of X relative to $\mathcal{U}$ is

$$h(X, \mathcal{U}) = \lim_{n \to \infty} \frac{\log \mathcal{N}(\mathcal{U}_n)}{n^d}$$

A pair of elements $x, y \in X$ are an *entropy pair* if $h(X, \{K_x^c, K_y^c\}) > 0$ for every disjoint pair of closed sets K_x, K_y containing x and y respectively, where K^c denotes the complement of K. Blanchard [4] proved the following theorem for $\mathbb{Z}$-topological dynamical systems. (It also holds in the more general context of a G-topological dynamical system, where G is a countable amenable group, but we do not need the more abstract formulation; a reference is [**?**].) Here is the version we need.

Theorem 3 (Blanchard)**.** *A subshift X has topological completely positive entropy if and only if the smallest closed equivalence relation on X containing the entropy pairs is all of X^2.*

A useful sufficient condition for the entropy pairhood of x and y, which comes up in all works on this topic, is the following. Two patterns $w, v \in \Delta^S$ are *independent* if there is a positive density subset $J \subseteq \mathbb{Z}^d$ such that for all $I \subseteq J$, there is some configuration $x \in X$ such that $x \upharpoonright g^{-1}S = w$ for all $g \in I$ and $x \upharpoonright g^{-1} = v$ for all $g \in J \setminus I$. In English, w and v are independent if there is a positive density of locations such if we place w or v at each of those locations (free choice), regardless of our choices it is always possible to fill in the remaining symbols to get a valid configuration $x \in X$. Observe that if $x \upharpoonright S$ and $y \upharpoonright S$ are independent patterns for all finite $S \subseteq \mathbb{Z}^d$, then x and y are an entropy-or-equal pair.

Barbieri and García-Ramos [1] defined the following hierarchy of closed relations and equivalence relations on X. They first define the set of *entropy-or-equal pairs*,

$$E_1 = \{(x, y) \in X^2 : x = y \text{ or } (x, y) \text{ is an entropy pair}\},$$

and note that this set is closed. At successor stages, define

$$E_{\alpha+1} = \begin{cases} \text{the topological closure of } E_\alpha & \text{if } E_\alpha \text{ is not closed} \\ \text{the transitive, symmetric closure of } E_\alpha & \text{if } E_\alpha \text{ is not an equiv. rel'n} \\ E_\alpha & \text{if } E_\alpha \text{ is a closed equiv. rel'n} \end{cases}$$

At limit stages, $E_\lambda = \cup_{\alpha<\lambda} E_\alpha$. They show that X has TCPE if and only if $E_\alpha = X^2$ for some α, and in this case they define the *TCPE rank* of X to be the least α at which this occurs. They construct a $\mathbb{Z}^3$-SFT of TCPE rank 3, and they ask whether this can be improved to a $\mathbb{Z}^2$-SFT.

Question 1 (Barbieri & García-Ramos)**.** *Is there a $\mathbb{Z}^2$-SFT of TCPE rank 3?*

Our Theorem 2 answers this question positively and then characterizes those TCPE ranks obtainable by $\mathbb{Z}^2$-SFTs to be exactly the computable ordinals.

2.2. **Effective descriptive set theory.** With the exception of Proposition 1 below, we have attempted to make the paper self-contained with respect to effective descriptive set theory. However, we have surely not succeeded completely in this, so we also refer the reader to the books [**?**] on descriptive set theory and [15] on its computable (effective) aspects.

Let ω^ω denote the space of all infinite sequences of natural numbers, with the product topology. Most mathematical objects can be described or encoded in a natural way by elements of ω^ω. A set $A \subseteq \omega^\omega$ is Π^0_n if there is a computable predicate P such that for all $x \in \omega^\omega$,

$$x \in A \iff \forall m_1 \exists m_2 \dots Q m_n P(x, m_1, \dots, m_n)$$

where each $m_i \in \omega$ (or in a set whose members are coded by elements of ω) and Q is either $\forall$ or $\exists$, depending on the parity of n. For example, the set of all convergent sequences of rational numbers is Π^0_3

$$(q_n)_{n\in\omega} \text{ converges } \iff (\forall \epsilon \in \mathbb{Q})(\exists N)(\forall n, m)[n, m > N \implies |q_n - q_m| \leq \epsilon].$$

A set $A \subseteq \omega^\omega$ is *arithmetic* if it is Π^0_n for some n. A set $A \subseteq \omega^\omega$ is *coanalytic*, or Π^1_1, if there is an arithmetic predicate P such that for all $x \in \omega^\omega$,

$$x \in A \iff (\forall y \in \omega^\omega) P(x, y)$$

For example the property of TCPE is Π^1_1. Let $K(X^2)$ denote the closed subsets of X^2 with the Hausdorff metric (appropriately encoded as a subset of ω^ω). Then

$$X \text{ has TCPE } \iff (\forall E \in K(X^2))[(E \text{ is an equivalence relation and } E \text{ contains the entropy-or-equal pairs}) \implies E = X^2]$$

A *tree* $T \subseteq \omega^{<\omega}$ is any set closed under taking initial segments. For $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma \prec \tau$ to indicate that σ is a strict initial segment of τ. A string $\sigma \in T$ is called a *leaf* if there is no $\tau \in T$ with $\sigma \prec \tau$. The empty string is denoted λ. A *path* through a tree T is an infinite sequence $\rho \in \omega^\omega$, all of whose initial segments are in T. The set of paths through T is denoted $[T]$. A tree T is *well-founded* if $[T] = \emptyset$. Let WF denote the set of well-founded trees, which is Π^1_1.

$$T \in WF \iff \forall \rho \in \omega^\omega[\rho \text{ has some initial segment not in } T]$$

A Π^1_1 set A is called Π^1_1-*complete* if for every other Π^1_1 set B, there is a computable function f such that for all $x \in \omega^\omega$,

$$x \in B \iff f(x) \in A$$

The set WF is Π^1_1-complete. No Π^1_1-complete set is arithmetic.

So far we have discussed only the descriptive complexity of subsets of ω^ω. There is a miniature version of this theory for subsets of ω (and by extension, subsets of any collection of finitely-describable objects, such as SFTs). A set $A \subseteq \omega$ is Π^0_n, arithmetic, Π^1_1, or Π^1_1-complete exactly when the same definitions written above are satisfied, with the only change being that the elements x whose A-membership is being considered are now drawn from ω rather than ω^ω, and also $B \subseteq \omega$ in the definition of Π^1_1-complete. No Π^1_1-complete subset of ω can be arithmetic either.

A tree $T \subseteq \omega^{<\omega}$ is computable if there is an algorithm which, given input $\sigma \in \omega^{<\omega}$, outputs 1 if $\sigma \in T$ and 0 otherwise. An *index* for a computable tree T is a number $e \in \omega$ such that the eth algorithm in some canonical list computes T in the sense described above. The set of indices of computable well-founded trees is a canonical Π^1_1-complete subset of ω.

On every Π^1_1 set, it is possible to define a Π^1_1 *rank*, a function which maps elements of the set to an ordinal rank $< \omega_1$ in a uniform manner (for details see [10]). A natural Π^1_1 rank on well-founded treess T is defined by induction as follows. The rank of a leaf $\sigma \in T$ is $r_T(\sigma) = 1$. For any non-leaf $\sigma \in T$, the rank of σ is $r_T(\sigma) = \sup_{\tau \in T : \sigma \prec \tau}(r_T(\tau) + 1)$. The rank of T is $r(T) = r_T(\lambda)$. Colloquially, the rank of a well-founded tree is the ordinal number of leaf-removal operations needed to remove the entire tree.

If $A \subseteq \omega^\omega$ is Π^1_1-complete, then for any Π^1_1 rank on A, the ranks of elements of A are cofinal (unboundedly large) below ω_1. If $A \subseteq \omega$ is Π^1_1-complete, then since A is countable, there must be some countable upper limit on the ranks of elements of A. A countable ordinal α is *computable* if there is a computable linear ordering $R \subseteq \omega \times \omega$ whose order type is α. The computable ordinals are also exactly those ordinals which can be the rank of a computable well-founded tree. The supremum of all computable ordinals is denoted ω_1^{ck}. If $A \subseteq \omega$ is Π^1_1-complete, then for any Π^1_1 rank on A, the ranks of elements of A are cofinal in ω_1^{ck}.

Heuristically, a sort of converse holds. If one can show that all countable (resp. computable) levels of a Π^1_1 hierarchy on a subset of ω^ω (resp. ω) are populated, typically one also has the tools to show that the set in question is Π^1_1-complete.

Barbieri and García-Ramos found topological dynamical systems at every level of the TCPE hierarchy, giving strong evidence for the following theorem (which will also be a side consequence of our methods).

Theorem 4. *The set of $\mathbb{Z}^2$-TDS with TCPE is Π^1_1-complete, and the TCPE rank is a Π^1_1 rank on this set.*

Here the arbitrary $\mathbb{Z}^d$-TDS are appropriately encoded using elements of ω^ω.

Our main goal is to show that the situation is no simpler in $\mathbb{Z}^d$-SFTs, which are appropriately encoded using elements of ω.

Theorem 5. *The set of $\mathbb{Z}^2$-SFTs with TCPE is Π^1_1-complete, and the TCPE rank is a Π^1_1 rank on this set.*

The first step to proving that the TCPE rank is a Π^1_1 rank is to show that every $\mathbb{Z}^2$-SFT which has TCPE has a computable ordinal rank. This proof is standard but does assume more familiarity with effective descriptive set theory than what was outlined in this introduction. The standard reference is [15].

Proposition 1. *If a $\mathbb{Z}^d$-TDS (X, T) has TCPE, its TCPE rank is less than $\omega_1^{(X,T)}$.*

Proof. To reduce clutter we prove the theorem for computable (X, T); the reader can check that the proof relativizes. Recall that E_α is closed whenever α is odd. If L is a computable well-order on $\omega \times \omega$ with least element 1, we say that Y is an E-hierarchy on L if

- $Y^{[1]}$ codes E_1 as a closed set,
- If $b <_L c$ are successors in L, then $Y^{[c]}$ encodes the topological closure of the transitive/symmetric closure of $Y^{[b]}$ and
- If c is a limit in L then $Y^{[c]}$ encodes the topological closure of the union of the sets coded by $Y^{[b]}$ for all $b <_L c$.

The definition of $Y^{[c]}$ from $\{Y^{[b]} : b <_L c\}$ is arithmetic and the definition of an E-hierarchy overflows to computable pseudo-wellorders.

Suppose for the sake of contradiction that the TCPE rank of X is at least ω_1^{ck}. If $E_{\omega_1^{ck}+1} = X^2$, we would have the following Σ^1_1 definition of $\mathcal{O}$.

$$a \in \mathcal{O} \iff a \in \mathcal{O}^* \text{ and}$$
$$\exists Y (Y \text{ is an } E\text{-hierarchy on } \{b : b \leq_{\mathcal{O}} a\} \text{ and } Y^{[a]} \neq X^2)$$

This is a contradiction since $\mathcal{O}$ is Π^1_1-complete. Similarly, if $E_{\omega_1^{ck}+1} \neq X^2$, then since X has TCPE, the next closed set $E_{\omega_1^{ck}+3}$ is strictly larger than $E_{\omega_1^{ck}+1}$. Let $U \subseteq X^2$ be a basic open set such that $E_{\omega_1^{ck}+3} \cap U \neq \emptyset$ but $E_{\omega_1^{ck}+1} \cap U = \emptyset$. In this case we could also define $\mathcal{O}$ by

$$a \in \mathcal{O} \iff a \in \mathcal{O}^* \text{ and}$$
$$\exists Y (Y \text{ is an } E\text{-hierarchy on } \{b : b \leq_{\mathcal{O}} a\} \text{ and } Y^{[a]} \cap U = \emptyset)$$

This provides a Σ^1_1 definition of $\mathcal{O}$, for if $a^* \in \mathcal{O}^* \setminus \mathcal{O}$, there is some $b^* \in \mathcal{O}^* \setminus \mathcal{O}$ with $b^* <_{\mathcal{O}} a^*$. Then $E_{\omega_1^{ck}+1}$ is a subset of $Y^{[b^*]}$, so $E_{\omega_1^{ck}+3}$ is a subset of $Y^{[a^*]}$, and thus $Y^{[a^*]} \cap U \neq \emptyset$. Again, contradiction. Therefore, the TCPE rank of X is less than ω_1^{ck}. □

Corollary 1. *If a $\mathbb{Z}^d$-SFT has TCPE, then its TCPE rank is a computable ordinal.*

Corollary 2. *The TCPE rank is a Π^1_1-rank on the set of $\mathbb{Z}^d$-TDS and the set of $\mathbb{Z}^d$-SFTs.*

Proof. If X_1 and X_2 are $\mathbb{Z}^d$-TDS or $\mathbb{Z}^d$-SFTs and X_2 has TCPE, then the following are equivalent:

(1) The TCPE rank of X_1 is less than or equal to the TCPE rank of X_2.
(2) There is an $a \in (\mathcal{O}^*)^{X_2}$ and E-hierarchies Y_1 and Y_2 (for X_1 and X_2 respectively) on a such that $Y_1^{[a]} = X_1^2$ and $Y_2^{[b]} \neq X_2^2$ for any $b <_{\mathcal{O}}^{X_2} a$.
(3) For all $a \in (\mathcal{O}^*)^{X_2}$ and all E-hierarchies Y_1 and Y_2 on a, if $Y_1^{[a]} = X_1^2$ then $Y_2^{[a]} = X_2^2$.

Proposition 1 guarantees that it is safe to use $\mathcal{O}^*$ in two places where we wanted to use $\mathcal{O}$, but could not. □

2.3. **SFT computation.** This section introduces the main technical tool used in this paper, the tiling-based SFT computation framework of Durand, Romashchenko and Shen [7]. A more motivated and detailed description of that framework can be found in their original paper. We also mention that [**?**, Chapter 3, Chapter 7.1-2] contain a good technical introduction to Turing machines and polynomial time complexity, and we assume the reader is fluent in this topic. Here we give a general overview of the ideas and terminology of the DRS construction, followed by a more technical description of their basic module, which will serve as the basis for our constructions.

A *Wang tile* is a square with colored sides. Two Wang tiles may be placed next to each other if they have the same color on the side that they share. We do not rotate the tiles. A *tileset* is a finite collection of Wang tiles. Given a finite tileset, the collection of infinite tilings of the plane which can be made with that tileset is a $\mathbb{Z}^2$-SFT. From here on we refer to infinite tilings of the plane as configurations. Wang [17] described a method for turning any Turing machine into a tileset such that any configuration which contains a special *anchor tile* is also forced to contain a literal picture of the space-time diagram of an infinite run of the Turing machine. If the Turing machine runs forever, the tiling can go on forever; if the Turing machine halts, there is no configuration because there is no way to continue the tiling.

The anchor tile contains the head of the Turing machine and the start of the tape. If we would like to force computations to appear in every configuration, we must require the anchor tile to appear in every configuration. By compactness, the only way to do this in a subshift is to require the anchor tile to appear with positive density. This means that many computations go on simultaneously. It is a technical challenge to organize the infinitely many computations so they do not interfere with each other, and to guarantee that the algorithm gets enough time to run. This challenge was first solved by Berger [2] with an intricate fractal construction that was subsequently simplified by Robinson [14]. Several other solutions have occurred over the years, including the one in [7] which is used in this paper.

In the DRS system, tiles use a *location part* of their colors to arrange themselves into an $N \times N$ grid pattern for some large N. Central to each $N \times N$ region, there is a *computation zone*; tiles in this zone must participate in building a space-time diagram (and one in particular must host the anchor tile). Simultaneously, the entire $N \times N$ region could itself be considered as a huge tile, or *macrotile*. The *macrocolors* of the macrotile are whatever color combinations appear on the boundary of the $N \times N$ region. To control what kind of tileset is realized by the macrotiles, tiles use a *wire part* of their colors to transport the bits displayed on the outside of the macrotile onto the input tape of the computation zone. The algorithm reads the color combination and makes the determination whether this kind of macrotile will be allowed (halting if the color combination is unsatisfactory). By design, the algorithmic winnowing forces the macrotiles to belong to a tileset that is very similar to the original tileset, but with one change: N is increased so that the algorithm at the next level gets more time to run. In essence, the algorithm copies its own source code (with the one change in N) up to the next level. Then with any time left over after checking the color combinations, the algorithm can use to do arbitrary other computations (possibly halting for other reasons).

So the tiles organize into macrotiles, the macrotiles organize into macromacrotiles, and so on. When talking about adjacent layers of macrotile, we refer to the smaller macrotiles as the *children*. An $N \times N$ group of child tiles make up a *parent* tile. Two adjacent tiles at the same level are *neighbors*. If two child tiles belong to the same parent tile we call the child tiles *siblings*. The smallest macrotiles (the original tileset) are called *pixel* tiles.

The computations which we cause the SFTs to perform in this paper are best understood by starting with a simple module, to which we add more and more features to obtain more general results. We begin here with the most basic module, which simply demonstrates the undecidability of the SFT emptiness problem using the DRS framework.

2.3.1. *Basic DRS module.* Given an index e_0, we produce a $\mathbb{Z}^2$-SFT that is empty if and only if the e_0th Turing machine halts on empty input.

Fix the zoom sequence $N_i = 2^i$.

Fix a universal Turing machine with binary tape alphabet. We can assume that the universal machine operates as follows. It reads the tape up until the first 1 and interprets the number of 0's it saw until that point as the program number. Then it applies that program to the entire original contents of the tape. In this way the program number itself is accessible to the computation.

In the basic module, a *level i macrocolor* is a binary string of length $s_i = 2 \log N_i + 2 + \log k$, where k is the number of colors needed to implement a tileset whose configurations simulate computations from the universal Turing machine, provided they contain the anchor tile. Here there will be $2 \log N_i$ bits for the location part, 2 bits for the wire part, and $\log k$ bits for the computation part.

Let $A(t)$ be the following algorithm, where t is a binary string input:

(1) (Parsing)
   (a) Start reading $t = 0^e 1 0^i 1 \ldots$, and check that $e < i$.
   (b) Compute s_i and check that the rest of t has length $4s_i$; interpret the rest of t as colors $c_1, \ldots, c_4$.
(2) (Consistency)
   (a) Check all four colors have compatible location (x, y) (see Figure [**?**])
   (b) Based on the location, check that wires and/or computation parts either appear or not, as appropriate. Set up the s_{i+1}-width wires so that they deliver the colors to the computation zone starting at location $(e + i + 2)$ of the parent tape. (see Figure [**?**])
   (c) If wire and/or computation bits are used, check they are a valid combination. (see Figure [**?**])
(3) (Sync Levels) If (x, y) can view the nth bit of the parent's tape, check:

(a) If $n < e + i + 2$, check the tape has 1 on it if $n = e$ or $n = e + i + 2$; otherwise check the tape has 0 on it. (This makes the parent computation have $0^e 1 0^{i+1} 1 \ldots$, so the parent knows it is at the next level, which is $i + 1$)
(b) Otherwise, if $n \geq e + i + 2 + 4s_{i+1}$, check the parent tape is blank.

(4) Halt if any checks above fail.
(5) Simulate Turing machine e_0 on empty input.

All steps except the last one take time at most $\mathrm{poly}(\log N_i)$, equivalently in our case to $\mathrm{poly}(i)$. Also, the universal machine simulation process only adds polynomial overhead. Therefore, for any $e < i$ and any t that begins with $0^e 1 0^i 1 \ldots$, the universal machine completes the first four steps within $\mathrm{poly}(i)$ time.

Fix e to be the Turing machine index of the algorithm A above. Let $i_0 > e$ be large enough that for all $i \geq i_0$ and all inputs t beginning with $0^e 1 0^i 1 \ldots$, steps 1-4 are completed within $N_{i-1}/4$ steps on the universal machine. (Here $N_{i-1}/2$ is the size of the computation zone in the macrotile that will be running this computation.)

For $i \geq i_0$, let T_i be the tileset

$$T_i = \{(c_1, c_2, c_3, c_4) : A(0^e 1 0^i 1 c_1 c_2 c_3 c_4) \text{ runs at least } N_{i-1}/2 \text{ steps}\}$$

Observe that i is large enough to permit the deterministic construction of a valid layout of s_{i+1}-thickness wires in each $N_i \times N_i$ parent tile, and large enough that if any of the checks fail, they fail within $N_{i-1}/4$ steps. Therefore, provided Turing machine e_0 runs forever, for each possible location in $N_i \times N_i$, there is a $\bar{c} \in T_i$ which has that location. If the location should contain a wire alone then there are exactly two $\bar{c} \in T_i$ with that location (one for each bit the wire could carry) and if the location should contain computation bits then exactly the valid computation fragments can appear there. On the other hand, if machine e_0 halts, then eventually the tilesets T_i become empty.

Recall from the DRS construction the following definition: a tileset T *simulates* a tileset S at zoom level N if there is an injective map $\phi : S \to T^{N^2}$ which takes each tile from S to a valid $N \times N$ array of tiles from T, such that

- For any S-tiling U, $\phi(U)$ is a T-tiling.
- Any T-tiling W can be uniquely divided into an infinite array of $N \times N$ macrotiles from the image of S.
- For any T-tiling W, $\phi^{-1}(W)$ is an S-tiling.

Here we have abused notation to let ϕ map tilings to tilings in the obvious way.

Proposition 2. *For each $i \geq i_0$, T_i simulates T_{i+1} at zoom level N_i.*

Proof. Given $S = \bar{d} \in T_{i+1}$, map it to the unique $N_i \times N_i$ pattern of T_i-tiles whose wires carry data $\bar{d}$. The content of the wires uniquely determines the content of the computation zone because the computation is deterministic, the starting tape contents are uniquely determined by $\bar{d}$, and T_i only includes tiles which contain valid computation fragments. Since $A(0^e 1 0^{i+1} 1 \bar{d})$ is still running after $N_i/2$-many steps, the whole computation zone can be filled with T_i-tiles. □

Then by induction, for each $i \geq i_0$, T_{i_0} simulates T_i at zoom level $\prod_{i_0 \leq k < i} N_k$.

Proposition 3. *There is a T_{i_0}-tiling if and only if Turing machine e_0 runs forever.*

Proof. By compactness, there is T_{i_0}-tiling if and only if T_i is non-empty for all $i > i_0$. This occurs if and only if Turing machine e_0 runs forever. □

This completes the exposition of the basic module. For further details we refer the reader to [7].

2.4. **Overview of the paper.** The rest of the paper is divided into two parts. In Secton 3 we construct a $\mathbb{Z}^2$-SFT of TCPE rank 3, answering Barbieri and García-Ramos and laying the foundation for the more general results.

The construction proceeds in stages. First we define an effectively closed subshift X which has TCPE rank 3. Next we describe the computational overlay which allows us to replace the infinitely many restrictions defining X with an algorithm that will simulate those restrictions. Finally, we tweak the computation framework so that it provides no interference to the local entropy properties of X.

In slightly more details, the configurations of X consist of *pure types*, which are seas of squares, tightly packed together, all the same size; and *chimera types*, which contain up to two sizes of squares, where the sizes must be adjacent integers. Infinite, degenerate squares also inevitably result; they cannot coexist with finite squares. Because of the chimera types, a configuration of pure type n will form an entropy pair with a configuration of pure type $n+1$, but the entropy pairhood relation cannot extend to larger gaps. The infinite, degenerate types are connected to the finite types by topological closure only. However, X is topologically connected enough that the TCPE process finishes.

To show that such a shift is sofic is a straightforward application of the DRS framework, simpler than the related square-counting construction in [18]. However, no shift with TCPE can contain a rigid grid (erasing everything but the grid would yield a non-trivial zero entropy factor). This apparent problem is solved by imagining the entire subshift is printed on a piece of fabric, which we then pinch and stretch so that the deformed grid itself bears entropy. The same idea was used by Pavlov [13] and we build on his construction.

Finally, a technical problem arises involving the interaction of rare computation steps with the need for a fully supported measure. This problem is solved with the notion of a *trap zone*, an idea which originated in [6]. The problem is not of any fundamental importance and the solution is technical, so it could be skipped on a first reading.

The second part of the paper, Section 4, builds heavily on the first part and contains all the main results. Again we build an effectively closed subshift, this time with a high TCPE rank, superimpose a computation to show it is sofic, and put it on fabric to make the computation transparent to local entropy.

The TCPE process in X finished quickly because all the pure types were transitively chained together. We can make the process finish more slowly by putting topological speed bumps between the pure types. Forbid most of the chimera types of X, leaving only *regular* chimera types – those in which the squares occur in a regular grid pattern only. This kind of regular grid pattern is no good for connecting pure types, so the entropy pair connections are broken. Since two sizes of square now occur in a regular grid, a configuration of chimera type can be parsed as a configuration on a macroalphabet where the two macrosymbols are the two different squares. Apply the exact same restrictions that defined X to the configurations on this macroalphabet. Now the pure types will still get connected, but instead of being connected immediately as an entropy pair, they have to wait until the TCPE process on the chimera types finishes. Topological speed bumps can be introduced into the chimera types of the chimera types, further lengthening the process.

Too many speed bumps, and the TCPE process will not finish. But if the speed bumps are organized using a well-founded tree, they will not hold things up forever. Some ill-founded trees even produce subshifts with TCPE. We show that the length of the TCPE process is controlled by the Hausdorff rank of the lexicographical ordering on $T \cup [T]$.

Showing that the resulting subshifts are sofic, and superimposing the computations transparently to local entropy, requires more technical work but contains no surprises.

Finally, all constructions are completely uniform, so in the end we can produce a procedure which maps a tree T to a $\mathbb{Z}^2$-SFT Y in such a way that Y has TCPE if and only if T is well-founded, and when this happens the well-founded rank of T and the TCPE rank of Y are related in a predictable way. This simultaneously gives both the Π^1_1-completeness of TCPE and the population of the computable ordinal part of the hierarchy.

3. A $\mathbb{Z}^2$-SFT with TCPE rank 3

We begin with constructing a $\mathbb{Z}^2$-SFT with TCPE rank 3. In this construction, many of the features of the general construction already appear.

First we will define an effectively closed $\mathbb{Z}^2$ subshift with TCPE rank 3. Next we will argue this subshift is sofic. Finally, we will show how to modify it to obtain a SFT with the same properties.

3.1. An effectively closed $\mathbb{Z}^2$ subshift of TCPE rank 3.

Definition 1. *An n-square on alphabet $\{A, B\}$ is an $n \times n$ square of B's, surround by an $(n+2) \times (n+2)$ border of A's.*

Definition 2. *For any alphabet $\{A, B\}$, let $F_{A,B}$ denote a computably enumerable set of forbidden patterns which achieves the following restrictions*

1. *Every 2×2 block of B's is in the interior of an n-square.*
2. *Every A is part of the border of a unique n-square. (Infinite, degenerate squares are possible.) Two such squares may be adjacent (see Figure 1), but boundaries may not be shared.*
3. *If a configuration contains an n-square and an m-square, then $|m-n| \leq 1$.*

We will show that the subshift $X \subseteq 2^{\mathbb{Z}^2}$ defined by forbidden word set $F_{0,1}$ has TCPE rank 3. In order to do this, we partition X into countably many pieces, or types, as follows. The possible types are $\omega \cup \{(n, n+1) : n \in \omega\} \cup \{\infty\}$. To determine the type of some configuration $x \in X$, examine what n-squares appear in x.

Definition 3. *If $x \in X$, the* type *of x is*

$$\operatorname{type}(x) = \begin{cases} n & \text{if } x \text{ contains only } n\text{-squares} \\ (n, n+1) & \text{if } x \text{ contains } n\text{-squares and } (n+1)\text{-squares} \\ \infty & \text{if } x \text{ contains no finite } n\text{-squares for any } n \end{cases}$$

Observe that if $\operatorname{type}(x) = \infty$, then either $x = 1^{\mathbb{Z}^2}$, or the 0's which do appear in x appear as the boundaries of up to four infinite n-squares. Otherwise, if a finite n-square appears, no infinite n-square may appear, because by part (3), it is forbidden to have both an n-square and another square that appears to be very large. And if a square of any other size appears, then again by part (3), the sizes can differ by only one, so if any finite square appears, then x must have type n or $(n, n+1)$ for some n. The restriction in part (1) guarantees that the squares are close together; either touching, or separated by just a small space.

Now let us identify the entropy pairs. The fact that squares may be either touching or separated by one unit provides freedom for gluing blocks. More precisely, if $i, j \in \mathbb{Z}$, we let $[i, j)$ denote the set $\{i, i+1, \ldots, j-1\} \in \mathbb{Z}$. If v is a pattern in $\mathcal{L}(Y_1)$, define the *type* of v as

$$\operatorname{type} v = \{\operatorname{type}(x) : x \in X \text{ and } h(v) \in x\},$$

where $h : Y_1 \to 2^{\mathbb{Z}^2}$ is the obvious factor map. We have the following lemma, which shows that any pattern consistent with a configuration of a finite type can be extended to a rectangular block of completed squares. The point is that blocks of the kind guaranteed below can be placed adjacent to each other freely without breaking any rules of $F_{0,1}$. We state the lemma here but leave the proof for the end of the subsection.

Lemma 1. *For any type $t \in \omega \cup \{(n, n+1) : n \in \omega\}$, and any $k \in \omega$, there is an N large enough so that for all $v \in \Lambda^{[0,k)^2}$, if $t \in \operatorname{type}(v)$, then for any rectangular region R which contains $[-N, k+N)^2$, we can extend v to $v' \in \Lambda^R$ such that*

(1) $t \in \operatorname{type}(v')$
(2) v' contains only completed squares (every boundary arrow in v' is part of an n-square fully contained in v'.)
(3) The boundary of v' does not contain any two adjacent 1's, nor any 1's at a corner.

Now we can describe the entropy pairhood facts, which depend only on type.

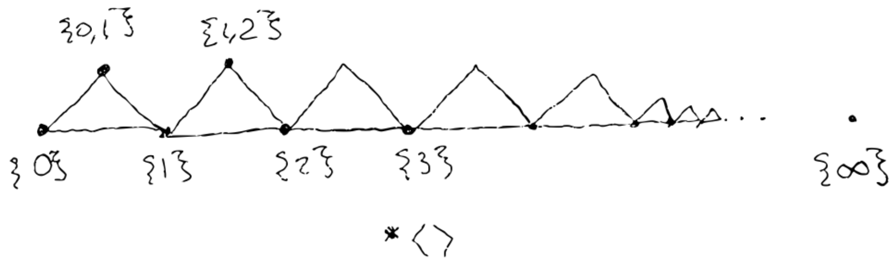
Lemma 2. *If $x, y \in X$, then the following table summarizes exactly when x and y are an entropy-or-equal pair (redundant boxes are left blank).*

type(x) \ type(y)	n	$(n, n+1)$	∞
m	*iff* $\lvert n-m\rvert \le 1$	*iff* $m \in \{n, n+1\}$	*never*
$(m, m+1)$		*iff* $m = n$	*never*
∞			*always*

Proof. First we prove all the "if" directions. Suppose x and y have types which the table indicates should be entroy-or-equal pair types. In each of the finite cases, there exists a finite type t such that arbitrarily large patterns of x and y are each consistent with type t. In the (∞, ∞) case, for any pair of patterns from x and y, there is also always a finite type t that is consistent with both patterns (any sufficiently large finite type will do). Given v and w $k \times k$ patterns from x and y, let t be such a finite type, and let N be the number guaranteed by Lemma 1. Partition the $\mathbb{Z}^2$ into square plots of side length $2N + k$. In the center of each plot, place a copy of either v or w (independent choice). By Lemma 1, fill in the rest of each plot in a way consistent with t. The result obeys all the rules of $F_{0,1}$, so x and y are an entropy pair.

In the other direction, if x and y have types which the table indicates should not be entropy-or-equal pairs, that means that x and y have squares of size difference more than one (in some cases we are looking at a finite square and an infinite, degenerate square, which is also a size difference more than one). If w and v are patterns of x and y which are large enough to show these too-different squares, then w and v are forbidden from appearing in the same configuration, so x and y are not an entropy pair. □

Observe that if $\text{type}(x) = \text{type}(y)$, then x and y are always an entropy pair. Therefore, we may represent the entropy pairhood relation E_1 by the following graph.



Next, E_2 is obtained by taking the transitive closure of E_1. We see that E_2 is an equivalence relation with two classes: $\{x : \text{type}(x) \text{ is finite}\}$ and $\{x : \text{type}(x) = \infty\}$. Finally, E_3 is obtained as the topological closure of E_2. Observe that $E_3 = X^2$ (every x with infinite type is the limit of a sequence of y of increasing finite type). Therefore, the TCPE rank of X is 3.

We conclude this subsection with the proof of Lemma 1.

Proof of Lemma 1. We first describe a way to construct v' which works if v is nice. Then we explain how to expand any v to a nice v.

First, complete any partial squares to produce a pattern consistent with type t. Call a square s a "top square" if there are no squares intersecting the space directly above s. Similarly, we have left squares, right squares, and bottom squares. The top squares are strictly ordered left to right. Let us say that the row of top squares is *nice* if for every adjacent pair (s, s') of top squares, there is at most one pixel gap between the columns that intersect s and the columns that intersect s'. We can make a similar definition for the left, right and bottom row of squares. Call v nice if all four of its rows are nice.

If v is nice, we can extend v to such v' by the following algorithm. Directly above each top square, place another square of the same size, either with boundaries touching, or with a one pixel gap. Keep adding squares directly above existing ones until the top of R is reached. Use the freedom of choice in spacing to make sure the topmost square has its top row flush with the boundary of R. This is possible if the boundary is at least distance $O(n^2)$ away, where n is the size of the top square we started with. Do the same for the bottom squares, but going down. Since v is nice, there is no more than one pixel wide gap between these towers of squares, so the restrictions $F_{0,1}$ are so far satisfied.

Turn our attention now to the left row. Let s be the left-most top square. Then s is also a left square. (If it were not a left square, there would be a square that lies completely in the half-plane to the left of it; a top-most such square would be a top square further left than s.) The tower of squares which we placed above s, together with the tower of squares which we placed below the left-most bottom square, form a natural extension of the left row. This extended left row is nice, and reaches from the top to the bottom of R. Directly to the left of each square in the extended left row, place another square of the same size, either with boundaries touching, or with a one-pixel gap, strategically chosen. Keep adding squares until the left boundary of R is reached. Again we can ensure that the left-most added squares have boundary flush with the left edge of R. Doing the same on the right side completes the construction in case v is nice.

Now we deal with the case where v is not nice. Let x be a configuration of type t in which v appears. We are going to take a larger pattern from x which is nice and includes v. To find the top row of this pattern, start with a square s_0 in x that is located directly above v and is at least k distance away from the top of v. Now consider the space directly to the right of s_0. There must be a square s_1, intersecting this space, such that the the gap between the right edge of s_0 and the left edge of s_1 is no more than one pixel. Going in both directions from s_0, fix a bi-infinite sequence $(s_i)_{i\in\mathbb{Z}}$ of squares such that for each i, the square s_{i+1} intersects the space directly to the right of s_i, and there is at most one pixel gap between the columns that intersect s_i and the columns that intersect s_{i+1}. Call this sequence the "top line", although it is not a line, as it may be rather wiggly. However, it is roughly horizontal; due to the fact that the square sizes are uniformly bounded, s_{i+1} intersects the space to the right of s_i by a definite fraction of its height. Therefore, the slope of the line which connects the center of s_i and the center of s_{i+1} has magnitude less than $1-\varepsilon$ for some ε depending only on t. Since s_0 is located at least k above v, an intersection would require a secant slope of magnitude at least 1 in the top line, so it follows that the top line cannot intersect v. Similarly, make a left line, a right line and a bottom line. The left and right lines always have secant slopes at least $1+\varepsilon$ in magnitude. Due to these approximate slopes, the top line and the bottom line must each intersect the right line and the left line, and they do so within a bounded distance. Take the pattern which consists of the loop made by the four lines and all the squares inside that loop. This pattern is nice. Apply the argument above. ☐

3.2. **Enforcing shape restrictions by a SFT on an expanded alphabet.** The shift X from the previous subsection was only effectively closed, while our goal is to create a SFT with the same properties. First we show how to realize restrictions (1) and (2) of $F_{A,B}$ using local restrictions on an expanded alphabet. Let

$$\Lambda = \{1, \circ, \leftarrow, \rightarrow, \uparrow, \downarrow, \ulcorner, \urcorner, \llcorner, \lrcorner, \colorbox{lightgray}{$\leftarrow$}, \colorbox{lightgray}{$\rightarrow$}, \colorbox{lightgray}{$\uparrow$}, \colorbox{lightgray}{$\downarrow$}, \colorbox{lightgray}{$\ulcorner$}, \colorbox{lightgray}{$\urcorner$}, \colorbox{lightgray}{$\llcorner$}, \colorbox{lightgray}{$\lrcorner$}\}.$$

We consider the corner pieces to also be arrows; corner arrows always point counterclockwise.

Definition 4. *Let Y_1 be the subshift whose configurations consist of square blocks of arrows on a background of 1's, where*

- *Each block of arrows consists of of nested counterclockwise squares of arrows, the outermost of which are colored gray, while the inner are colored white.*
- *The 2×2 block of 1's is forbidden.*

See Figure 1.

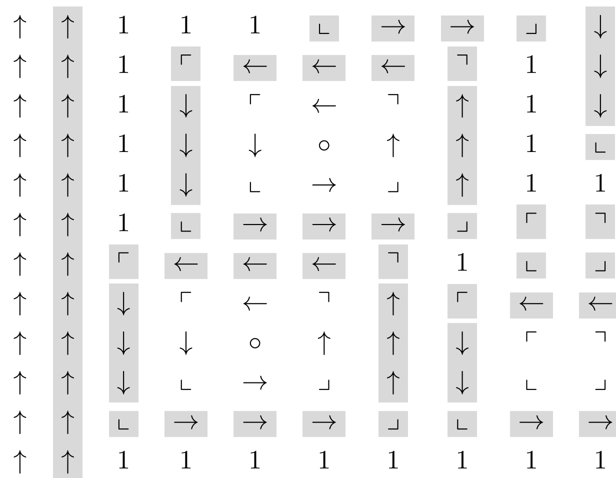


FIGURE 1. A permitted pattern from an element of Y_1

Proposition 4. *The set of 2×2 patterns from Λ which never appear in Y_1, when taken as a set of forbidden patterns, define Y_1 as a SFT.*

Proof. This is essentially identical to the proof of Lemma 1 in [**?**]. Note that the 2×2 restrictions suffice to guarantee that each arrow is continued by another arrow of the same color, forming oriented paths which make only left turns. The 2×2 restrictions also suffice to guarantee that on the inner edge of each oriented path, there is a tightly nested white path of the same orientation; that on the outer edge of each oriented white path there is a tightly nested path of the same orientation; and that on the outer edge of each oriented gray path we find only 1's and oppositely-oriented gray paths. Starting from a finite gray path and following its nested paths inward, we must terminate at nothing, a $\circ$, or a $\begin{array}{cc} \ulcorner & \urcorner \\ \llcorner & \lrcorner \end{array}$, which guarantees that the gray path was a square. Infinite gray paths must be either one-sided or two-sided; the three-sided case is ruled out by following nested white paths inward until one of the forbidden configurations $\downarrow \ \uparrow$ or $\begin{array}{c} \leftarrow \\ \rightarrow \end{array}$ is found. □

Observe that if we let h be the factor map which takes white symbols to 1 and gray symbols to 0, then $h(Y_1)$ is precisely the shift given by the restrictions (1) and (2) of $F_{0,1}$ from Definition 2. In the next subsection we realize restriction (3) with SFT computation.

3.3. **Enforcing size restrictions with SFT computation.** We now show that the subshift X defined by restrictions $F_{0,1}$ is sofic, using a simple application of the Durand-Romashchenko-Shen SFT computation framework. This framework is likely overpowered for this application, because there is no arbitrary algorithm appearing in the definition of X. However, in the general case we must have complete computational freedom, so we introduce it now in the simpler setting. We begin with the SFT Y_1 on alphabet Λ defined above, with all 2×2 restrictions. We will use a superimposed computation to realize the square size restrictions.

We now describe a modification of the basic DRS module. As in the introduction, we will ultimately choose some large i_0, form the tileset T_{i_0}, and superimpose a tile from T_{i_0} onto each symbol of Λ (subject to some restrictions). Note that the side-length of a macrotile at level i_0 is one pixel, while the side-length of a macrotile at level $i > i_0$ is $L_i := \prod_{i_0 \leq j < i} N_j$ pixels.

Definition 5. *We define the* responsibility zone *of a level i macrotile M as follows*

(1) If $i = i_0$, the responsibility zone is the pixel tile M itself.
(2) If $i > i_0$, consider the $(N_{i-1} + 2)$-side-length square of level $i - 1$ macrotiles which concentrically contains M. The responsibility zone of M is the union of the responsibility zones of these level $i - 1$ macrotiles.

Equivalently, the responsibility zone of a level i macrotile M is the square of pixel-side-length $R_i := (L_i + \sum_{j=i_0}^{i-1} 2L_j)$ which concentrically contains M.

Macrotiles will need to "know" (have encoded somehow in their colors) what is going on inside their responsibility zone. Recall that the macrotiles will be superimposed on Y_1-configurations. Therefore, it makes sense to discuss Y_1-squares and their relative location to the macrotiles.

Definition 6. *A macrotile M is* responsible for *a Y_1-square if there are macrotiles M_1 and M_2, both at the same level, such that*

(1) M_1 and M_2 are fully contained in the responsibility zone of M,
(2) M_1 and M_2 are either same tile or neighbors, and
(3) the square has one corner in M_1 and another corner in M_2.

Proposition 5. *If M is a macrotile of level i, and S is a Y_1-square, then M is responsible for S if and only if one of the following holds.*

(1) There is a level $i-1$ macrotile M_1 within the responsibility zone of M, and M_1 is responsible for S.
(2) There are two adjacent level $i-1$ macrotiles M_1, M_2 within the responsibility zone of M, and S has one corner in each of M_1 and M_2.
(3) There are two (possibly non-adjacent) level $i - 1$ macrotiles M_1, M_2 which are both children of M, and S has one corner in each of M_1 and M_2.

Proof. It is clear that in each of the three cases above, M is responsible for S. Suppose M is responsible for S via level j macrotiles M_1 and M_2, where j is chosen as small as possible. If $j = i$, then $M_1 = M_2 = M$ and since no smaller j works, it must be that S has one corner in each of two non-adjacent children of M, so case (3) above applies. If $j = i - 1$ and $M_1 = M_2$, then M_1 is responsible for S, so case (1) above applies. If $j = i - 1$ and $M_1 \neq M_2$, then case (2) above applies. Finally, if $j < i - 1$ then M_1 and M_2 both lie in the responsibility zone of a single level $i - 1$ macrotile, so case (1) above applies. □

We need a macrotile M to "know about" all the sizes of square that it is responsible for. We also need M to know about the location of all partial corners or partial sides of squares that intersect M, but are too big for M to be responsible for them. Each macrotile will tell its neighbors what it knows. Some of what it knows will be told to all neighbors, other things it knows will be shared only with its siblings.

In the basic construction, all the information displayed in the colors could be thought of as information that is shared equally between the two macrotiles whose adjacent sides have that color. Going forward, it becomes convenient to think of some parts of the macrocolor as having a direction of information flow. For example, the first bits of a color could be for information that is flowing rightward (for a vertical color) or upward (for a horizontal color), while the later bits could be for information that is flowing leftward or downward. In this way, a macrotile can tell its neighbors what it knows by using the first part of the color on its right and top edges, and the second part of the color on its left and bottom edges. It uses the second part of its right and top colors, and the first part of its left and bottom colors, to receive messages sent by its neighbors.

With this information flow idea in mind, a level i macrocolor for a tile M is a binary string which contains the following information:

(1) Location, wire and computation bits as in the basic module (synchronized, space $O(\log N_i)$)).
(2) Self-knowledge bits (sending):
  (a) A *type*, which is a list of up to two numbers less than or equal to L_i, representing all sizes for squares that M is responsible for. (space: $O(\log L_i)$)
  (b) If $i > i_0$, a *justification* for each number above. The justification is the location of the lexicographically least level $i-1$ macrotile from M's responsibility zone which reported this information to M. The method of reporting is described later. (space: $O(\log N_i)$)
  (c) A list of up to 4 pixel-locations, measured relative to M, of corners of large squares, together with orientation information for the corner (space: $O(\log L_i)$)
  (d) A list of up to 2 pixel-locations, measured relative to M, of sides of large squares that pass through M, together with orientation information. Since the side passes all the way through, the location is just a single x-pixel-coordinate or y-pixel-coordinate. (space: $O(\log L_i)$)
(3) Neighbor-knowledge bits (receiving): The tile M receives self-knowledge information as above from its neighbor.
(4) Diagonal-neighbor-knowledge bits (sending): The tile M will tell its neighbor M' about the self-information of the two tiles which are adjacent to M and diagonal to M'.
(5) Diagonal-neighbor-knowledge bits (receiving): The tile M will receive from its neighbor M' the self-information of the two tiles which are adjacent to M' and diagonal to M.
(6) Parent-knowledge bits (synchronized, shared only with siblings).
(7) Corner message passing bits (outgoing, shared only with siblings). If M's self-information includes a partial corner, it sends out the pixel location, relative to its parent, of that corner in the directions of the two arms of the corner. Or, if M receives a corner message on one side, it sends the same message out on the opposite side.
(8) Corner message receiving bits (incoming, shared only with siblings).

The combined self-knowledge bits make it so that the computation going on inside M will be aware of what is happening in the responsibility zone of M's eight neighbors.

The parent-knowledge bits allow M and all of its siblings to be aware of what is happening in the responsibility zone of their shared parent.

The corner message passing bits allow M to use its colors to communicate long distances with its siblings, so they can together ascertain the sizes of any square which has two corners in their shared parent. This is done by the same method as in [18]. Any macrotile with a partial corner must send out a message containing the deep coordinate of that side. To send a message in one particular direction (north, south, east or west), the macrotile displays the message in its macrocolor on just one side. Any sibling macrotile receiving the message must pass the message on (unless the parent boundary is reached, in which case the message stops); eventually the message may reach a sibling macrotile with a matching corner. Since the recipient macrotile also knows the deep coordinates of its own corner, it can calculate the distance the message traveled, which is the side length of the square.

We now informally describe the rest of what the algorithm running inside M does with all the color information above, followed by a step-by-step summary of the algorithm.

Based on the color information above, M has many ways to learn about sizes of squares that could be in the responsibility zone of M's parent. Macrotile M makes sure that its parent-knowledge includes:

(1) Any size that appears in M's self-knowledge or any of M's neighbors self-knowledge (this takes care of all squares that M and each of its neighbors are responsible for).
(2) The combined self-knowledge of M and its eight neighbors is also enough to deduce whether there are neighbor squares M_1, M_2 among these nine, and a Y_1-square S, such that S has one corner in M_1 and another corner in M_2. If this occurs, M must also make sure that its parent has recorded the size of S.
(3) Based on the corner message passing, M may learn about a square which has one corner in M and another corner in a sibling of M. If this happens M must make sure its parents knows the size of that square.
(4) On the other hand, if M sends corner messages out and gets none back, it should make sure that its parent-knowledge includes knowledge of that corner.
(5) If the parent cites M as a source of any of its knowledge, M must check that this is true. If the parent cites another source for knowledge provided by M, then M must check this other source has a location lexicographically less than M's own location.

Proposition 5 implies that steps (1)-(3) above are enough to ensure that if M's parent is responsible for a square S, and if all macrotiles at the same level as M have accurate self-information, then M's parent will have accurate self-information.

Now we summarize the algorithm run by macrotiles at all levels. Let $A(t)$ be the algorithm which does the following:

(1) (Parsing)
    (a) Start reading $t = 0^e 1 0^{i_0} 1 0^i 1 \dots$ and check that $e < i_0 \leq i$.
    (b) Check the rest of t has the right length for four level-i macrocolors.
(2) (Consistency)
    (a) Check location, wire, and computation parts are consistent (same as basic module).

(b) Check that the same self-knowledge appears in all four colors, that all outgoing diagonal-neighbor-knowledge agrees with corresponding incoming neighbor-knowledge, and that the incoming diagonal-neighbor-knowledge is consistent.

(c) Check the parent-knowledge is the same on all colors shared with siblings, and blank on all colors shared with non-sibling neighbors.

(d) If self-knowledge indicates a corner, use i_0, i, and the location bits to compute the pixel location of the corner relative to the parent. Check that messages are sent out along the corner arms; the content of the message is the pixel location of the corner relative to the parent. If corner messages are incoming, make sure they either match a corner from the self-knowledge, or are sent outgoing on the opposite side (exception: do not send corner messages to non-siblings).

(e) If any self-knowledge, neighbor-knowledge, or diagonal-neighbor-knowledge includes n in its type, make sure the parent-knowledge includes n in its type.

(f) If the combined self-, neighbor-, and diagonal-neighbor-knowledge reveals that among these 9 macrotiles there is a square and a pair of neighbors which each contain a corner of that square, compute the size of the square, and make sure the parent-knowledge includes that size in its type.

(g) If the self-knowledge has a corner and receives a matching corner-message, compute the size of the associated square and make sure the parent-knowledge includes that size in its type, and that the parent-knowledge does not include that corner. If the self-knowledge has a corner and does not receive a matching message, make sure the parent-knowledge includes that corner. If the self-knowledge has a partial side, make sure the parent-knowledge has that partial side if and only if no messages came along it in either direction.

(h) If the parent cites the location of this macrotile as justification for a size, make sure that is true.

(i) If the parent cites the location of another macrotile for information that this macrotile provided, check that the cited location is lexicographically less than the location of this macrotile.

(j) If the parent claims a partial corner or side that would intersect this macrotile, make sure that is true.

(k) Check that the type of the parent-knowledge is either empty, contains a single number, or contains two adjacent numbers $n, n+1$. If there is a number n and the parent-knowledge also contains a partial corner or side, halt if the partial corner or side is already too long compared to n.

(3) (Sync Levels)

(a) If the location of this tile is on the parent tape, check the parent tape begins with $0^e10^{i_0}10^{i+1}1\dots$ and ends with blanks.

(b) If the location of this tile is on the parent tape and can view the parent's self-knowledge part on the parent tape, check that what is seen agrees with the parent-knowledge recorded in the colors c_i.

(4) If any of the above steps do not check out, halt. Otherwise, run forever.

When $e \leq i_0 \leq i$, the steps described take poly($\log L_{i+1}$) time to run, equivalently in our case to poly(i). Fix e to be the index of the algorithm described above; let $i_0 \geq e$ be large enough that for all $i \geq i_0$, if t begins with $0^e 1 0^{i_0} 1 0^i 1 \ldots$, then $A(t)$ finishes all the checks within $N_{i-1}/2$ steps. For $i \geq i_0$, define

$$\hat{T}_i = \{(c_1, c_2, c_3, c_4) : A(0^e 1 0^{i_0} 1 0^i 1 c_1 c_2 c_3 c_4) \text{ runs forever}\}$$

Let T denote the set of all valid T_{i_0}-tilings.

Next we are going to define a SFT subset of $T \times Y_1$ which has $X_{0,1}$ as a factor. To specify which subset, we define a notion of accuracy that can hold between the T-part and the Y_1-part of our SFT. Recall that R_i denotes the pixel-side-length of the responsibility zone for a macrotile of level i.

Definition 7. *Given the self-knowledge s of an i-level macrocolor, and $\bar{a}$ a permitted pattern from Y_1 of size $R_i \times R_i$, we say that s is* accurate *for $\bar{a}$ if, considering a level-i macrotile M superimposed centrally in $\bar{a}$,*

(1) The type of s equals the set of sizes of squares for which M is responsible,

(2) Justification of s are accurate: if n is justified by giving the location of a level $i-1$ macrotile M', then M' must be the lexicographically least child of M which either is
- *itself responsible for an n-square, or*
- *has one corner of an n-square, the other corner of which is contained in another child of M, or*
- *there is an n-square with one corner in each of two adjacent level $(i-1)$ macrotiles from the 3×3 block of macrotiles centered at M'.*

(3) Corner and partial side locations of s agree exactly with the actual locations of partial corners and sides in the central $L_i \times L_i$ region of $\bar{a}$.

We will also say that a the self-information of a macrotile $\bar{c} \in \hat{T}_i$ is accurate for $\bar{a}$ in cases where $\bar{a}$ is larger than the responsibility zone of a level-i macrotile, in cases where it is clear precisely where we intend to superimpose $\bar{c}$ on $\bar{a}$.

Definition 8. *Let $y \in Y_1$, let u be a $L_i \times L_i$ subset of $\mathbb{Z}^2$, and let $\bar{c}$ be a level-i macrocolor. We say that*

(1) $\bar{c}$ is self-accurate *at u in y if the self-information of $\bar{c}$ is accurate for y when $\bar{c}$ is superimposed on u.*

(2) $\bar{c}$ is accurate *at u in y if $\bar{c}$ is self-accurate and additionally all 8 neighbor-informations of $\bar{c}$ are accurate for y when superimposed on the corresponding eight $L_i \times L_i$ regions concentrically surrounding u.*

Definition 9. *Let $y \in Y_1$, let u be a $L_i \times L_i$ subset of $\mathbb{Z}^2$, and let $\bar{t}$ a permitted pattern from T consisting of a single a level-i macrotile. We say that*

(1) $\bar{t}$ is self-accurate *at u in y if the top-level macrocolor of $\bar{t}$ is self-accurate at u in y, and for each $j < i$ and each level-j macrotile appearing in $\bar{t}$, that macrotile is accurate at its location in y.*

(2) $\bar{t}$ is accurate *at u in y if it is self-accurate, and additionally its top-level macrocolor is accurate.*

Proposition 6. *Given $y \in Y_1$, and $x \in T$, if each T_{i_0} tile of x is self-accurate at its location in y, then whenever $x \upharpoonright u$ is a macrotile, it is accurate at u in y.*

Proof. By induction. Assume that $i_0 \leq j < i$ and all level-j macrotiles in x have self-information that is accurate at their location in y. Then each level-j tile is accurate, because its neighbor-information is copied from its self-accurate neighbor tiles. Let M be a level-$(j+1)$ macrotile. The children of M which live on the tape of M have made sure M's self-knowledge is in the parent information part of M's children.

Any child of M whose self-information includes n in its type, has forced M to include n in its type. Any child of M who has a square of length n straddling two level-j tiles in its 3×3 neighborhood, has discovered this and forced M to include n in its type. Any child of M who has a corner of a square of length n, the other corner of which is in another sibling, has found it via corner message passing, and forced M to include n in its type. Any child of M who has a corner or partial side which did not collect matching messages, has forced M to include that corner or partial side.

Conversely, M must include justifications for its type, and each child does check to make sure that if M uses that child as justification, that this is accurate. Children located on partial corners/sides claimed by M also check to make sure M has that information correct. □

The preceding proposition shows that the requirement to self-accurately superimpose $\hat{T}_{i_0}$-tiles onto element of Λ suffices to ensure that if any tiling is created, then the macrotiles at all levels have accurate information about the squares upon which they are superimposed. In other words, accuracy at the bottom level flows up. This motivates the following definition.

Definition 10. *Let $\Lambda_2 = \{(\bar{c}, a) \in \hat{T}_{i_0} \times \Lambda : \bar{c}$ is self-accurate for $a\}$. Let Y_2 be the SFT whose restrictions are the color-matching restrictions from $\hat{T}_{i_0}$, and the 2×2 restrictions from Y_1 on the alphabet Λ.*

Towards showing that $X_{0,1}$ is indeed a factor of Y_2 via the natural factor map, we need the following lemmas.

Lemma 3. *Given $y \in Y_1$, suppose that $\bar{t}_1$ and $\bar{t}_2$ are level-i macrotiles from T which are self-accurate at locations u_1 and u_2 in y. Then if u_1 and u_2 are adjacent and the top-level colors of $\bar{t}_1$ and $\bar{t}_2$ are permitted adjacent, then $\bar{t}_1$ and $\bar{t}_2$ are permitted adjacent in T.*

Proof. By induction. The $i = i_0$ case is trivial. Suppose the lemma holds for each macrotile of level $i-1$. Without loss of generality suppose that $\bar{t}_1$ is left of $\bar{t}_2$. It suffices to show that $\bar{s}_1$ is permitted adjacent to $\bar{s}_2$ whenever s_1 is a child macrotile of $\bar{t}_1$ on its right edge and $\bar{s}_2$ is the corresponding child macrotile of $\bar{t}_2$ on its left edge. By definition of self-accuracy, all child macrotiles found in $\bar{t}_1$ and $\bar{t}_2$ are accurate at their location in y. So by induction it suffices to check that $\bar{s}_1$ and $\bar{s}_2$ have matching top-level colors. Their locations match by the way they were chosen. Their wire bits match, if they have wire bits, because the top-level colors of $\bar{t}_1$ and $\bar{t}_2$ match. They do not display computation bits. Their self-knowledge and neighbor-knowledge bits match because all these fields are set to their unique accurate values. Finally, no parent-information nor corner-message passing bits are displayed across the parent boundary, so they match by both displaying nothing. □

Lemma 4. *Suppose we have an element $y \in Y_1$ whose squares have not-too-different sizes, and an $L_i \times L_i$ region $u \subseteq \mathbb{Z}^2$. Then for any level-i macrocolor $\bar{c}$ that is self-accurate at location u in y, there is a level-i macrotile $\bar{t}$ from T which is self-accurate at u in y and has top-level macrocolors $\bar{c}$.*

Proof. The proof is by induction on i, and the i_0 case is trivial. Given the level i-macrocolor $\bar{c}$, consider u as a union of $L_{i-1} \times L_{i-1}$ regions corresponding to child macrotiles of level $i-1$. For each such child macrotile M, we assign an accurate macrocolor as follows.

(1) Set the all the self-knowledge, neighbor-knowledge and diagonal-neighbor-knowledge of M to be accurate in y. All these knowledge fields are completely deterministic.
(2) Each M copies its parent-knowledge field from $\bar{c}$.
(3) Set the corner-messages also to reflect the reality of y: if M contains a partial corner or side which has a matching corner in the same parent, M should receive a message; if M receives a message or has a corner, M should send a message.
(4) The location part of M's colors is uniquely determined by its location in u.
(5) The wire parts and computation parts of M's colors are uniquely determined by $\bar{c}$.

Observe these macrocolors are designed so that they are accurate and so that any two child macrotiles with adjacent locations have matching macrocolors on the side they share. By induction assume we have a self-accurate macrotile with the chosen macrocolor at each child location. By Lemma 3, these child macrotiles are permitted adjacent, so together they form a level-i macrotile $\bar{t}$ from T. The wire parts of the child macrocolors were chosen to ensure that $\bar{t}$'s top macrocolor is $\bar{c}$. □

Definition 11. *Let $h_2 : Y_2 \to 2^{\mathbb{Z}^2}$ be the factor map which maps $(\bar{c}, a)$ to 0 if a is a boundary symbol and 1 otherwise.*

That is, h_2 recovers the square boundaries from the Y_1 part of $x \in X_2$.

Proposition 7. *The subshift $X_{0,1}$ is sofic, and furthermore $h_2(Y_2) = X_{0,1}$.*

Proof. Given $y \in Y_1$ with squares of not-too-different sizes, by Lemma 4, arbitrarily large accurate macrotiles from T can superimposed on a central region of y. It follows by compactness that there is an infinite T_{i_0}-tiling which can be superimposed accurately on y. In particular, the superposition is self-accurate at the pixel level. Therefore there is $x \in Y_2$ such that $h_2(x)$ gives the square outlines of y.

On the other hand, consider an element $y \in Y_1$ with squares of size difference more than 1 (the case of a finite square and an infinite square is included in this possibility). Suppose for contradiction that there is $x \in T$ such that each pixel of x is self-accurate at its location in y. Then there is a macrotile M in x large enough to be responsible for two squares of too-different sizes (or large enough to be responsible for a finite square and a very long partial corner or side of an infinite square). By Proposition 6, the part of x which corresponds to the responsibility zone of M is accurate. Therefore, the self-knowledge of M includes the two squares. Therefore the children of M see the two squares in their parent-knowledge. Therefore the children of M halt at the last consistency step of the algorithm. Contradiction. □

If there is a macrotile that has n written on its parameter tape, then for every pixel location $t \in \mathbb{Z}^2$, every sufficiently large macrotile containing t also has n written on its parameter tape. Therefore, an element of Y_2 could be said to have a limiting *information type*, which is by definition equal to the collection of all sizes recorded on any parameter tape, or ∞ if no finite size is ever recorded. Observe that the information type of an element $x \in Y_2$ is equal to $\mathrm{type}(h_2(x))$.

3.4. **Preserving TCPE rank.** We have seen that $X_{0,1}$ is sofic, but so far only via an SFT extension Y_2 that does not have TCPE. The DRS-type construction is tiling-based, so any subshift that uses it factors onto a zero-entropy subshift that retains only the tiling structure. Below, we describe how to modify Y_2 to fix this problem.

The idea is to imagine the tiling structure is printed on a piece of fabric, which can be pinched and stretched so that the alignment of tiles in one region has no bearing on the alignment of tiles far away. Now the tiling structure itself bears entropy (information about pinching and stretching), so any factor map which retains any part of the computation also retains the entropy of the tiling structure on which the computation lives. This idea is made precise below with an analysis of a construction by Pavlov [13]. He shows the following.

Theorem 6 (Pavlov [13, Theorem 3.5])**.** *For any alphabet A, there is an alphabet B and a map taking any orbit of a point in $A^{\mathbb{Z}^2}$ to a union of orbits of points in $B^{\mathbb{Z}^2}$ with the following properties:*

(1) $O(x) \neq O(x') \implies f(O(x)) \cap f(O(x')) = \emptyset$.
(2) If $W \subseteq A^{\mathbb{Z}^2}$ is a subshift (resp. SFT), then $f(W)$ is a subshift (resp. SFT).
(3) If W is a $\mathbb{Z}^2$-subshift with a fully supported measure, and there exists an N such that for every $w, w' \in L(W)$, there are patterns $w = w_1, w_2, \ldots, w_N = w$ such that for all $i \in [1, N)$, w_i and w_{i+1} coexist in some point of W, then $f(X)$ has TCPE.

However, in light of the subsequent work by Barbieri and García-Ramos [1], stronger claims should be made about Pavlov's construction. We make the following definition.

Definition 12. *Let W be a $\mathbb{Z}^2$-subshift. We say $(x, x') \in W$ is a* transitivity pair *if for every pair of patterns v, v' that appear in x and x' respectively, v and v' coexist in some point of W.*

Examination of Pavlov's proof shows he also proved the following. The point is that this transformation preserves many things about an SFT with a fully supported measure, while upgrading all transitivity pairs into entropy-or-equal pairs.

Theorem 7 (essentially Pavlov [13, Theorem 3.5])**.** *For any alphabet A, there is an alphabet B and a map taking any orbit of a point in $A^{\mathbb{Z}^2}$ to a union of orbits of points in $B^{\mathbb{Z}^2}$ with the following properties:*

(1) $O(x) \neq O(x') \implies f(O(x)) \cap f(O(x')) = \emptyset$.
(2) If W is a subshift (resp. SFT), then $f(W)$ is a subshift (resp. SFT).
(3) If W is a subshift with a fully supported measure, then $f(W)$ has a fully supported measure.
(4) If W is a subshift with a fully supported measure, and $x, x' \in W$, the following are equivalent:
  (i) (x, x') is a transitivity pair in W.
  (ii) (y, y') is an entropy-or-equal pair in $f(W)$ for some $(y, y') \in f(x) \times f(x')$
  (iii) (y, y') is an entropy-or-equal pair in $f(W)$ for every $(y, y') \in f(x) \times f(x')$
(5) If W is a subshift, $A \subseteq W$ is a shift-invariant set, and $x \in W$, then $x \in \overline{A}$ if and only if $f(x) \subseteq \overline{f(A)}$.

Proof. The proofs of (1)-(3) are exactly as in the original. The proof of (4) is also essentially there, if a bit roundabout. Here we sketch a direct route to (4), using the same language as in the original proof.

$(i) \Rightarrow (iii)$. First, suppose (x, x') is a transitivity pair. Let $(y, y') \in f(x) \times f(x')$, with $y \neq y'$, and fix S and $w = y \upharpoonright S$ and $w' = y' \upharpoonright S$ such that $w \neq w'$. Let v, v' be patterns in W such that v and v' induce w and w'. Then v appears in x and v' appears in x', so there is a point $x'' \in W$ in which v and v' both appear. Because W has a fully supported measure, we may assume that a pattern v''' containing both v and v' appears with positive density in x''. Let $y'' \in f(x'')$ be chosen so that all ribbons of y are perfectly horizontal or vertical and spaced 3 apart. Pick an infinite, positive density subset of patterns in y'' which are induced by v''' and which are far enough apart. At each of these locations, finitely perturb the ribbons to wiggle a copy of w or w' (independent choice) into those locations. The independent choice is possible because the locations are far enough apart.

$(ii) \Rightarrow (i)$. If $(y, y') \in f(x) \times f(x')$ is an entropy-or-equal pair, it is a transitivity pair. So if v and v' are patterns that appear in x and x', they induce w and w' in y and y', and there is some $y'' \in f(W)$ in which w and w' coexist. Then v and v' coexist in each element of $f^{-1}(y'')$.

For (5), the forward direction follows for any given $y \in f(x)$ by considering elements of $f(A)$ that have the same ribbon structure[2] as y, and the reverse direction also follows by restricting attention to a single fixed ribbon structure. □

For any subshift W, its transitivity pair relationship is closed. So we may define a *generalized transitivity* hierarchy similar to the TCPE hierarchy as follows.

Definition 13. *If W is a subshift, define $T_1 \subseteq W^2$ to be its set of transitivity pairs. Then for each $\alpha < \omega_1$ define $T_{\alpha+1}$ to be the transitive closure of T_α if T_α is closed, and to be the closure of T_α otherwise. For λ a limit, define $T_\lambda = \cup_{\alpha<\lambda} T_\alpha$. We say that W is* generalized transitive *if there is some α such that $T_\alpha = W^2$, in which case the least such α is called the* generalized transitivity rank *of W.*

Observe that the shifts with transitivity rank 1 are exactly the transitive ones. We have the following relationship between transitivity rank and TCPE rank.

Theorem 8. *Suppose that W is a subshift with a fully supported measure. Let f be the operation of Theorem 7. Then $f(W)$ has TCPE if and only if W is generalized transitive, and in this case the TCPE rank of $f(W)$ and the transitivity rank of W coincide.*

Proof. For any $T \subseteq W^2$, let $f(T)$ denote $\bigcup_{(x,x')\in T} f(x) \times f(x')$. Let E_α denote the sets of the TCPE hierarchy on $f(W)$. We claim that for all $\alpha \in [1, \omega_1)$, we have $f(T_\alpha) = E_\alpha$. The fact that $f(T_1) = E_1$ follows from Theorem 7 part (4). The limit step cannot cause any discrepancy. In consideration of the successor step, suppose that $f(T_\alpha) = E_\alpha$ for some α.

We claim T_α is closed if and only if E_α is closed. Observe T_1 is shift-invariant in the sense that $(x, x') \in T_1$ implies $O(x) \times O(x') \subseteq T_1$, and therefore each T_α is also shift-invariant in that sense (this shift-invariance is not destroyed by topological or transitive closures, or by limits). Therefore, if T_α is closed, its complement is a union of sets of the form $U_v \times U_{v'}$, where v and v' are patterns and

$$U_v := \{x : \ v \text{ appears in } x\}.$$

[2]In [13], "the first two coordinates of y".

Therefore the complement of E_α is a union of sets of the form $f(U_v) \times f(U_{v'})$. Theorem 7 parts (1) and (5) imply that f maps open shift-invariant sets to open shift-invariant sets. Since the U_v are shift-invariant, it follows that if T_α is closed, then so is E_α. On the other hand, if T_α is not closed, there are $(x, x') \notin T_\alpha$ but $(x_n, x'_n) \in T_\alpha$ with $(x_n, x'_n) \to (x, x')$. Mapping these points all over to $f(W)$ using the same ribbon structure and same origin yields the analogous situation in E_α. This completes the proof of the claim that T_α is closed if and only if E_α is closed.

If T_α is closed, then it is easy to check that $(x, x') \in T_{\alpha+1}$ if and only if $f(x) \times f(x') \subseteq E_{\alpha+1}$. On the other hand, if T_α is not closed, then an argument as above shows that $(x, x') \in T_{\alpha+1}$ implies $f(x) \times f(x') \subseteq E_{\alpha+1}$ and vice versa. $\square$

Therefore it would suffice to show that the SFT Y_2 defined in the previous section has a fully supported measure and has generalized transitivity rank 3. Unfortunately, it is possible that Y_2 does not have a fully supported measure. However, the reasons are non-essential and we can modify the tiling construction to ensure a fully supported measure.

3.5. **An SFT with a fully supported measure and generalized transitivity rank 3.** We now describe why Y_2 may not have a fully supported measure, and how to fix this. We would like to show that given an element $x \in Y_2$ and a pattern $v \in x$, that there is some $x' \in Y_2$ such that v appears with positive density in x'. For simplicity, let us consider a type ∞ element of Y_2 in which the Y_1 part uses only the symbol $\downarrow$. Consider its tiling structure. We could imagine a very unlucky pattern v with the following properties:

- v is a macrotile located in in the computation part of its parent
- The parts of the parent computation that v sees are actually a rare combination that occurs in only one of all possible parent macrotiles
- The uniquely implied parent macrotile also has these three properties.

Such unlucky v may well exist, and if it exists it could occur at most one time in any configuration.

To fix this we use a trick similar to one Durand and Romashchenko used for a similar problem in [6]. We make a "trap zone" of size 2×2 child macrotiles, located in a part of the parent macrotile which is out of the computation zone and out of the way of the wires. Where exactly to put the trap zone can be efficiently computed in the same way as the wire layout is computed. The idea is that any 2×2 block of level-i macrotiles that is permitted to appear at all, in any location, should be permitted to appear in the appropriately sized trap zone. To achieve this, we modify the direction of information flow at the eight level-i macrocolors which form the boundary of the trap zone, so that all information at level i is flowing out of the trap zone only.

We relax the consistency requirements for tiles whose location is north, south east or west of one of the tiles in the trap zone. Note that if a tile has three of its location parts in agreement, the location of that tile is already determined. So if a tile knows itself (based on 3 location parts) to be a neighbor of the trap zone, then it will require only the consistency of the three parts of its color not touching the trap tile, and it will allow its trap-adjacent macrocolor to be unrestricted. Thus a trap neighbor M will just observe whatever the adjacent trapped tile M' chooses to hallucinate in all fields, including hallucinated "self-information of M", "corner messages sent by M", "diagonal neighbor information communicated by M" and so on. By reversing information flow in this way, any locally admissible 2×2 block of tiles is permitted to appear in the trap zone.

Now, the trapped tiles will display accurate self-information because they are internally consistent.[3] The trap neighbors must examine the self-knowledge of the trapped tiles to get some information:

- What sizes of square have appeared in the trapped tile? (The trap neighbors make sure the true parent has recorded the numbers from the trapped tiles parameter tapes.)
- What partial sides and corners appear in the trapped tile? (This is also readable from the parameter tape of trapped tiles.)

The eight trap neighbors and the four diagonal trap neighbors use a new *trap information part* of their colors to pass information in a twelve-tile loop. The passed information is:

- the combined self-information displayed by all four trapped tiles, and
- any side messages that trap neighbors would have wanted to pass through the trap zone

Based on this information, the trap neighbors are fully equipped to fill in all the missing size-checking functions of the trapped tiles. If they are able to compute the size of a square whose corner is in the trapped tiles, they make sure the parent records that size. If they are not able to compute the size of such a square, they make sure the parent has recorded the deep coordinates of the partial side or corner in the trap zone. If the parent cites a trapped tile as justification for a size, the trap neighbors check if this is warranted. Finally, any messages that should have passed directly through the trapped tile are routed around it, and any messages that the trapped tile should have sent out are sent out (with correct pixel coordinates, which the trapped tile would not have been able to compute by itself, since it believes itself to be in a different part of the parent).

The consistency checks for these modifications are all efficient to compute. To summarize, let Y_3 be the SFT defined just as Y_2 was defined, but using the following colors and algorithm instead.

A level-i macrocolor contains:

(1) Location, wire and computation bits
(2) Self-knowledge, neighbor-knowledge, diagonal-neighbor-knowledge, and parent-knowledge bits
(3) Corner message sending and receiving bits
(4) (NEW) Trap neighbor information circle bits
  (a) Eight self-knowledge fields, one for each side displayed by the trapped tiles. These are used to pass the self-knowledge of the trapped tiles around to all trapped neighbors.
  (b) Eight corner message sending fields, one for each trap neighbor. These are used to inform all trap neighbors of any message that a trap neighbor wanted to send through the trap zone (but was unable because information cannot flow into the trap zone).

[3]If a trapped tile thinks that it is a trap neighbor, it might display a false self-knowledge on one of its sides! But its other side will have its true self-knowledge. The information displayed by the group of trapped tiles is sufficent for the trap neighbors to correctly deduce which self-knowledge is correct if two versions are offered.

The algorithm $A(t)$ does the following (changes italicized):

(1) (Parsing)
(2) (Consistency)
  (a) Check location, wire, and computation parts are consistent, *except if three locations indicate this tile is a trap neighbor, in which case the trap-adjacent location/wire/computation is permitted to be anything.*
  (b) Check consistency of self-, neighbor-, diagonal-neighbor- and parent-knowledge across the four colors, *except if this tile is a trap tile neighbor, in which case allow anything in the trap-adjacent color, and deduce neighbor- and diagonal-neighbor-information involving trapped tiles from the trap neighbor information circle.*
  (c) *If this tile is a trap neighbor,*
    (i) check that the self-information displayed by the adjacent trapped tile also appears in the appropriate field of the information circle bits
    (ii) check that any message this tile wanted to send through the trap zone instead appears in the appropriate field of the information circle
    (iii) check that the remaining fields of the information circle agree on both sides (that is, the information that other trap neighbors are re-routing is being accurately copied around).
    (iv) Use the information circle to deduce the accurate self-information of all trapped tiles, and to learn which messages the other trapped neighbors wanted to send through the trap zone.
  (d) Perform corner-message-passing functions. *If this tile is a trap tile neighbor, do not directly exchange messages with the trapped tile. Instead, look to the information circle to see if any other trap neighbor wanted to send a message to this tile, and to see if any trapped tile should have sent a message to this tile.*
  (e) Make sure the parent knows of any sizes deducible from self-, neighbor- and diagonal-neighbor knowledge.
  (f) Make sure the parent accurately records or doesn't record any partial corner or side from this tile, as appropriate. *Trap neighbors also do this on behalf of the trapped tiles.*
  (g) If the parent cites the location of this macrotile as justification for a size, make sure that is true and that the lexicographically least macrotile was cited. *Trap neighbors also check this on behalf of the trapped tiles.*
  (h) If the parent claims a partial corner or side that would intersect this macrotile, make sure that is true. *Trap neighbors also check this on behalf of the trapped tiles.*
  (i) Check the parent-knowledge does not contain squares of too-different sizes.
(3) (Sync Levels)
(4) If any of the above steps do not check out, halt. Otherwise, run forever.

Observe that $X_{0,1}$ is still a factor of Y_3. That is because Proposition 6, Lemma 3, and Lemma 4 go through with only the following slight modifications.

(1) We say that a macrotile $\bar{t}$ from T is self-accurate at u in y if the top color of T is self-accurate, and all the child macrotiles of T are accurate *except possibly the trapped tiles, who are only required to be self-accurate.*
(2) In Proposition 6, the only change is the more convoluted process by which the self-information of trapped tiles gets to the parent.
(3) There is no modification to Lemma 3 because the trapped tiles do not occur at the edge of their parent.
(4) In Lemma 4, we build superpositions where the trapped tiles happen to hallucinate the truth, so we may continue setting all parts of all macrocolors to accurate values.

We leave the easy details to the reader.

Lemma 5. *The SFT Y_3 has a fully supported measure.*

Proof. Suppose that w is a pattern that appears in some $z = (x, y) \in Y_3$. Since every pattern is eventually contained in a 2×2 block of macrotiles at some level, without loss of generality we can assume that w is a 2×2 block of macrotiles. Let $h : Y_3 \to Y_1$ be the obvious factor map. We choose a finite number n_0 as follows. If $\text{type}(h(x)) = n$ or $\text{type}(h(x)) = (n, n+1)$, let $n_0 = n$. If $\text{type}(h(x)) = \infty$, then $h(w)$ is either part of an interior of a square, or contains some combination of partial corners and/or partial sides. Though these partial squares are infinite in x, it is consistent with $h(w)$ that they be completed into large finite squares. In this case, let n_0 be large enough that $(n_0, n_0 + 1) \in \text{type}(h(w))$, and also large enough that no square of size n_0 could have more than a partial corner or side appear in any region equal in size to the combined responsibility zones of the tiles of w.

Now, fix a non-exceptional tiling structure. The pattern w is the right shape to fit in a trap zone at some level. Trap zones the size of w appear with positive density. Let N be the number guaranteed by Lemma 1 for n_0. Let M be a number large enough that every $M \times M$ square contains a w-sized trap zone that is fairly central: the edge of the combined responsibility zones of the trapped tiles is not closer than N pixels away from the edge of the $M \times M$ square. Build $y \in Y_1$ as follows. First lay n_0-squares in horizontal stripes, so that the blank space between the stripes is M pixels tall. Then lay n_0 squares in vertical strips so that the blank space between the strips is M pixels wide. (It maybe necessary to play with the spacing of n_0-squares, adjacent vs. a one-pixel gap, in order to achieve this, but M is bigger than N, so it is possible.) Then put $h(w')$ into one fairly central trap zone in each $M \times M$ blank region, where w' is the restriction of y to the combined responsibility zones of the tiles of w. Then use Lemma 1 to fill in n_0-squares and $(n_0 + 1)$-squares through the entire remaining space in each region. The result is an element $y' \in Y_1$.

Now we argue that there is an $x' \in T$ such that $z := (x', y') \in Y_3$ contains w with positive density. Let j_0 denote the level of the four macrotiles of w. Let $u \subseteq \mathbb{Z}^2$ be an $L_i \times L_i$ region, where $i > j_0$, and where u corresponds to a single macrotile in the fixed tiling structure. We claim that for every macrocolor $\bar{c}$ that is self-accurate at u for y', there is a valid T-pattern $\bar{t}$ which is self-accurate at u for y', has top-level color $\bar{c}$, and furthermore $\bar{t}$ has w in every fairly central trap zone.

The base case occurs when $i = j_0 + 1$. If u does not contain a fairly central trap zone, we apply Lemma 4 and are done. So assume u does contain a fairly central trap zone. We are trying to construct a level-i macrotile and we will do so by selecting appropriate level-j_0 child macrotiles M and arguing that they fit together. There are three cases.

(1) If M is neither trapped nor a trap neighbor, we choose a T-pattern $\bar{t}_M$ for M exactly as in Lemma 4, choosing accurate values for all fields, and referring to $\bar{c}$ for the unique choices of wire and computation bits.
(2) If M is trapped, let $\bar{t}_M$ be the corresponding tile of w. By Proposition 6, $\bar{t}_M$ is self-accurate at its location in y. Since self-accuracy depends only on the responsibility zone, and y' copied y on the entire responsibility zone of the trapped tiles, $\bar{t}_M$ is self-accurate at its location in y'.
(3) If M is a trap neighbor, set its colors to match the corresponding trapped tile of w on its trap size. Set its standard colors on the other three sides to accurate values. Set its information circle colors to accurately reflect the self-information that all trapped tiles are displaying, and to accurately reflect the messages that should be sent through the trap zone on all sides. Since these colors are accurate, apply Lemma 4 to produce the macrotile $\bar{t}_M$.

The colors above are chosen so that all tiles are self-accurate and all adjacent tiles have top-level colors that can go next to each other. Therefore, by Lemma 3, all the child macrotiles created above can fit together to form a level-i macrotile $\bar{t}$ with the right properties.

The inductive case is now identical with the proof of Lemma 4. It now follows by compactness that there is an infinite T-tiling x' that is accurate at each pixel in y' and copies w in each fairly central trap zone. Since the fairly central trap zones occur with positive density, we are done. □

Lemma 6. *The SFT Y_3 has generalized transitivity rank 3.*

Proof. We claim that x, x' in Y_3 are a transitivity pair exactly when $h(x), h(x')$ in X are an entropy-or-equal pair. This is determined by type (see Lemma 2). When $h(x), h(x')$ are not an entropy-or-equal pair, observe it always happens for a particularly strong reason: $h(x)$ and $h(x')$ were not even a transitivity pair. Therefore, x, x' cannot be a transitivity pair either. On the other hand, if $h(x)$ and $h(x')$ have compatible type, then for any patterns $w \in x$ and $w' \in x'$, there is an n_0 such that $(n_0, n_0+1) \in \text{type}(h_1(w)) \cap \text{type}(h_1(w'))$. As in the previous lemma, it suffices to consider w and w' which are the same size and which each consist of four macrotiles in a 2×2 arrangement. As in the previous lemma, construct $y \in Y_1$ by fixing a tiling structure, laying n_0-squares in an $M \times M$ grid pattern for large enough M, placing $h_1(w)$ or $h_1(w')$ (free choice) in fairly central trap zones in each $M \times M$ region, and then filling in the rest of y and the computations just as in the previous lemma. Apply compactness and the result is an element of Y_3 in which both w and w' appear. □

Theorem 9. *There is a $\mathbb{Z}^2$-SFT with TCPE rank 3.*

Proof. Our example is $f(Y_3)$, where f is the map from Theorem 7. Apply that theorem, Lemma 5 and Lemma 6. □

4. A family of $\mathbb{Z}^2$ shifts

To generalize the previous construction to all computable ordinals, we define a transformation whose input is a tree $T \subseteq \omega^{<\omega}$ and whose output is a subshift

$X_T \subseteq 2^{\mathbb{Z}^2}$, whose TCPE status and rank are controlled by properties of T. It will be technically more convenient to use trees $T \subseteq \Omega^{<\omega}$, where $\Omega = \{(n, n+1) : n \in \omega\}$.

4.1. **Definition of the family of shifts.**

Definition 14. *Given square patterns $A, B \in 2^{[0,m)^2}$, and subpatterns $C \preceq A, D \preceq B$, let $R_{A,B,C,D}$ denote the set of restrictions which say: in every configuration in which both a C and a D appear, both an A and a B must appear. Furthermore every occurrence of A or B must have another occurrence of A or B directly north, south, east and west.*

Informally, when restrictions $R_{A,B,C,D}$ are applied, the permitted configurations fall into two categories. Configurations in which C and D do not coexist are unrestricted. But in configurations where C and D do coexist, the configuration can be understood as an (affine) configuration on $\{A, B\}^{\mathbb{Z}^2}$.

Definition 15. *For each $\sigma \in \Omega^{<\omega}$, define A_σ, B_σ to be the following patterns in $2^{[0,m)^2}$ (where m depends on σ). Let $A_\lambda = 0$ and $B_\lambda = 1$. If A_σ and B_σ have already been defined, let $B_{\sigma^\frown(n,n+1)}$ be an $(n+1)$-square on alphabet $\{A_\sigma, B_\sigma\}$, and let $A_{\sigma^\frown(n,n+1)}$ be an n-square on the same alphabet, plus a row of B_σ along the top and along the right side (to make it the same size as $B_{\sigma^\frown(n,n+1)}$.*

$$A_{\sigma^\frown(n,n+1)} = \begin{array}{cccccc} B_\sigma & \dots & \dots & \dots & \dots & B_\sigma \\ A_\sigma & \dots & \dots & \dots & A_\sigma & \vdots \\ \vdots & B_\sigma & \dots & B_\sigma & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & B_\sigma & \dots & B_\sigma & \vdots & \vdots \\ A_\sigma & \dots & \dots & \dots & A_\sigma & B_\sigma \end{array}, \quad B_{\sigma^\frown(n,n+1)} = \begin{array}{cccccc} A_\sigma & \dots & \dots & \dots & \dots & A_\sigma \\ \vdots & B_\sigma & \dots & \dots & B_\sigma & \vdots \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \vdots & B_\sigma & \dots & \dots & B_\sigma & \vdots \\ A_\sigma & \dots & \dots & \dots & \dots & A_\sigma \end{array}$$

Definition 16. *Definition of R_σ and F_σ.*

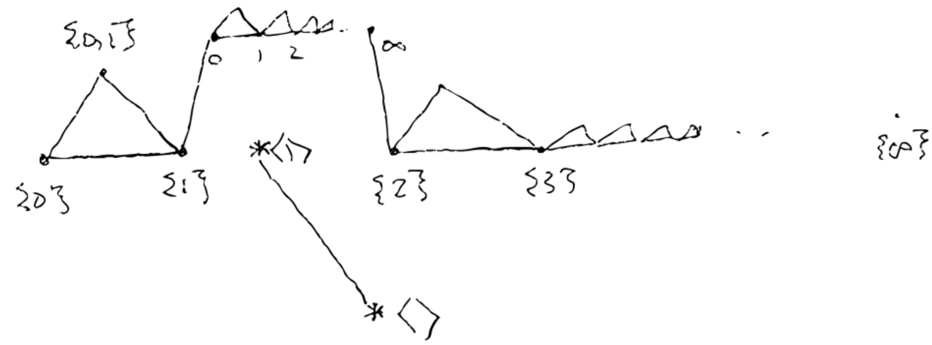
1. *No restrictions are imposed by R_λ. For $\sigma \in \Omega^{<\omega}$ of length at least 1, write $\sigma = \rho^\frown(n, n+1)$, let R_σ denote $R_{A_\rho,B_\rho,C,B_\rho}$, where C is the n-square on alphabet $\{A_\rho, B_\rho\}$.*
2. *Let F_σ denote F_{A_σ,B_σ}, with the F restrictions as in Definition 2.*

Definition 17. *For any tree $T \subseteq \Omega^{<\omega}$, define X_T to be the subshift defined by forbidding R_σ and F_σ for each $\sigma \in T$.*

Observe that if $T = \{\lambda\}$, then R_λ makes no restrictions and $F_\lambda = F_{0,1}$ ensures that X_T is equal to the subshift X considered in the previous section.

The intuition behind the definition can be understood by considering the simple case $T = \{\lambda, (1, 2)\}$. We will speak about this case informally in order to motivate the subsequent arguments. The entropy pair relations for X_T are pictured in Figure 4.1[4]. Because $\lambda \in T$, X_T is a subshift of X. The inclusion of $R_{(1,2)}$ removes many elements of X which had type $(1, 2)$, and the remaining elements of type $(1, 2)$, the pattern of 1-squares and 2-squares is very regular, these elements can be understood as configurations on the alphabet $A_{(1,2)}, B_{(1,2)}$. Therefore it makes

[4]The figure is accurate other than a technical caveat about alignment; technically there are 16 parallel widgets between type 1 and type 2.



sense to apply $F_{(1,2)}$ to these configurations. This causes a fracturing of the $(1,2)$ type into subtypes:

$$(1,2)^\frown 0, (1,2)^\frown (0,1), (1,2)^\frown 1, \dots, (1,2)^\frown \infty,$$

similar to how X was fractured. In X, an element of type 1 was an entropy pair with an element of type 2, but in X_T, no element of type 1 is an entropy pair with any element of type 2, for the same reason that $0^{\mathbb{Z}^2}$ (type 0) and $1^{\mathbb{Z}^2}$ (type ∞) were not entropy pairs in X (the connecting configurations would need to contain both 0-squares on alphabet $\{A_{(1,2)}, B_{(1,2)}\}$ and n-squares on the same alphabet, where n is large).

Note, however, that among configurations with subtype $(1,2)^\frown 0$, there is one which is is just an infinite array of $A_{(1,2)}$, i.e. an orderly array of 1-squares on the original alphabet. Therefore this configuration is also in some sense type 1. Similarly, there is a configuration with subtype $(1,2)^\frown\infty$ which is just an infinite array of $B_{(1,2)}$, i.e. consists of tightly packed 2-squares on the original alphabet. Therefore this configuration is in some sense type 2. These connections provide the reason why X_T still has TCPE (in fact, TCPE rank 4). The first step E_1 is as in Figure 4.1. The equivalence relation E_2 has three classes, which could be named "types less than $(1,2)^\frown\infty$", "finite types $\geq (1,2)^\frown\infty$" and "∞". The closed relation E_3 provides connections from the first equivalence class to the special $(1,2)^\frown\infty$ element, and from the second equivalence class to the ∞ element. And E_4 is everything.

4.2. **Types and the Hausdorff derivative.** We assign each $x \in X_T$ a *type* and an *alignment*. The type of x is a string in $(\omega \cup \Omega \cup \{\infty\})^{\leq\omega}$ which depends on x and T. The *alignment* is an element of $(\mathbb{N}^2)^{\leq\omega}$.

Given $\sigma \in \Omega^{<\omega}$, let $j_\sigma \in \mathbb{N}$ be the unique number such that $A_\sigma, B_\sigma \in 2^{[0,j_\sigma)^2}$

Definition 18. *Given $\sigma \in \Omega^{<\omega}$, we say that $x \in 2^{\mathbb{Z}^2}$ is* σ-regular *if there is some $g \in [0,j_\sigma)^2$ such that for all $h \in (j_\sigma\mathbb{Z})^2$, $x \restriction ([0,j_\sigma)^2 + h + g)$ is equal to either A_σ or B_σ. If such g exists it is unique and we say g is the* σ-alignment *of x.*

That is, x is σ-regular if x can be parsed as a tiling on alphabet $\{A_\sigma, B_\sigma\}$. Observe that if x is σ-regular, then x is τ-regular for every $\tau \preceq \sigma$, and x is not τ-regular for any τ that is incomparable with σ.

Definition 19. *For a tree $T \subseteq \Omega^{<\omega}$ and $x \in X_T$, define* $\mathrm{type}_T(x)$ *as follows. Let $\rho \in T \cup [T]$ be longest such that x is ρ-regular.*

- *If* $\rho \in [T]$*, define* $\mathrm{type}_T(x) = \rho$*.*
- *If* $\rho \in T$*, consider* x *as a configuration on alphabet* $\{A_\rho, B_\rho\}$ *and let* $t \in \omega \cup \Omega \cup \{\infty\}$ *be its unique type in the sense of Definition 3. Then define* $\mathrm{type}_T(x) = \rho^\frown t$*.*

If $\sigma \in T$ and $x \in X_T$ is σ-regular, then if x' denotes the shift of x by any amount not in $(j_\sigma\mathbb{Z})^2$, in general x and x' will not be an entropy pair, because their alignment is off. In practice this does not cause any difficulties because the alignments get progressively washed out at limit stages of the TCPE ranking process. So with the caveat that alignments are much less important than the types, but still necessary for the reader who wants all details, we define them as follows.

Definition 20. *For a tree* $T \subseteq \Omega^{<\omega}$ *and* $x \in X_T$*, define* $\mathrm{alignment}_T(x)$ *as follows. Let* $\rho \in T \cup [T]$ *be longest such that* x *is* ρ*-regular. Define* $\mathrm{alignment}_T(x)$ *to be the unique string* $\nu \in (\mathbb{N}^2)^{\leq\omega}$ *such that* $|\nu| = |\rho|$ *and for all* $k < |\nu|$*,* $\nu(k)$ *is the* $(\rho \upharpoonright k)$*-alignment of* x*.*

Every type is an element of $(\omega \cup \Omega \cup \infty)^{\leq\omega}$. Which of these elements actually appear as types?

Definition 21. *Given* $T \in \Omega^{<\omega}$*, define*

$$T^+ = \{\sigma^\frown a : \sigma^\frown a \notin T, \sigma \in T, a \in \omega \cup \Omega \cup \{\infty\}\}.$$

Proposition 8. *For* $T \in \Omega^{<\omega}$*, we have* $\{\mathrm{type}_T(x) : x \in X_T\} = T^+ \cup [T]$*.*

Proof. If $\mathrm{type}_T(x)$ is infinite, then $\mathrm{type}_T(x) \in [T]$. It is easy to construct x of given infinite type. If $\mathrm{type}_T(x) = \rho^\frown t$ as in Definition 19, suppose for contradiction that $\rho^\frown t \in T$. Then $t = (n, n+1)$ for some n, so x contains both an n-square and an $(n, n+1)$-square on alphabet $\{A_\rho, B_\rho\}$. Since $\rho^\frown t \in T$, the restrictions $R_{\rho^\frown t}$ are present and apply to x. Therefore x is $\rho^\frown t$-aligned, contradicting the initial choice of ρ. On the other hand it is easy to construct an $x \in X_T$ with $\mathrm{type}_T(x) = \rho^\frown t \in T^+$, by constructing any configuration of type t on alphabet $\{A_\rho, B_\rho\}$. □

We put the following natural linear order on $\omega \cup \Omega \cup \infty$:

$$0 < (0,1) < 1 < (1,2) < 2 < \cdots < \infty.$$

This induces a lexicographic order on types.

Definition 22. *Let* L_T *be the linear order* $(\{\mathrm{type}_T(x) : x \in X_T\}, \leq_{lex})$*.*

Finally, it will be convenient to consider only those T where the branches of T are spread out a little bit. Define $2\Omega = \{(2n, 2n+1) : n \in \omega\}$. If $T \in (2\Omega)^{<\omega}$, we still have $\{\mathrm{type}_T(x) : x \in X_T\} \subseteq (\omega \cup \Omega \cup \{\infty\})^{\leq\omega}$ because the "odd" elements of Ω can still appear at the end of $\mathrm{type}_T(x)$.

Recall the *Hausdorff derivative* is the following operation on linear orders. Given a linear order L, define an equivalence relation H on L by aHb if there are only finitely many c with $a \leq c \leq b$ or $b \leq c \leq a$. The equivalence classes are well-behaved with respect to the order. Output the linear order L' whose elements are the H-equivalence classes.

The α-th Hausdorff derivative is then defined by transfinite iteration of the Hausdorff derivative. (At limit stages, take the union of all equivalence relations defined so far.)

Recall that a linear order is *scattered* if $\mathbb{Q}$ cannot be order-embedded into it. It is well-known that repeated applications of the Hausdorff derivative stabilize to the equivalence relation L^2 if and only if L is scattered. The *Hausdorff rank* of a scattered linear order is the least α at which this occurs.

4.3. **TCPE ranks of these shifts.** For $T \in (2\Omega)^{<\omega}$, we show that X_T has TCPE if and only if $\{\text{type}_T(x) : x \in X_T\}$ is scattered when considered as a linear order with the lexicographical ordering. Furthermore, there is a precise level-by-level correspondence between the TCPE rank of X_T and the Hausdorff rank of L_T.

Theorem 10. *Given $T \subseteq (2\Omega)^{<\omega}$, the shift X_T has TCPE if and only if L_T is scattered. Furthermore, if X_T has TCPE rank α and L_T has Hausdorff rank β, then $\alpha \in \{2\beta - 1, 2\beta\}$.*

For the rest of this section, $T \subseteq (2\Omega)^{<\omega}$ is some tree. The lexicographic order on types can be lifted to X_T.

Definition 23. *Define a total pre-order on X_T by $x \leq_{lex} y$ if $\text{type}_T(x) \leq_{lex} \text{type}_T(y)$.*

The lifted pre-order relates to the topology of X_T in the following sense.

Lemma 7. *If $x_n \to x \in X_T$ in its topology, and $\rho_1 <_{lex} \text{type}_T(x) <_{lex} \rho_2$ for some types ρ_1 and ρ_2, then for sufficiently large m, we have $\rho_1 \leq_{lex} \text{type}_T(x_m) \leq_{lex} \rho_2$.*

Proof. First, let $\rho \in T$ be longest such that $\rho \prec \rho_1$ and $\rho \prec \text{type}_T(x)$. Let t_1 and t be the next symbols of each type, that is $\rho^\frown t_1 \preceq \rho_1$ and $\rho^\frown t \preceq \text{type}_T(x)$. We know that $t_1 < t$. If $t = (n, n+1)$, then x contains both an n-square and an $(n+1)$-square on alphabet $\{A_\rho, B_\rho\}$, so for sufficiently large m, each x_m also contains these features. So for sufficiently large m, we have $\rho^\frown t \prec \text{type}_T(x_m)$, and thus $\rho_1 <_{\text{lex}} \text{type}_T(x_m)$. If $t = n$ for some $n \in \omega$, then $n \neq 0$ (otherwise ρ_1 could not be less), and for sufficiently large m, x_m must also contain an n-square on alphabet $\{A_\rho, B_\rho\}$. Therefore, we eventually have $\rho^\frown(n-1, n) \leq_{\text{lex}} \text{type}_T(x_m)$. If $\rho_1 = \rho^\frown(n-1, n)$, we are done with this case. If ρ_1 is longer, that means that $\rho^\frown(n-1, n) \in T$. We know that x is not $\rho^\frown(n-1, n)$-aligned (if it were, we would have chosen a longer ρ). Therefore, there is some pattern in x that contains n-squares which are arranged in a way incompatible with such an alignment. For sufficiently large m, the configurations x_m also contain such an arrangement. Thus for sufficiently large m, these x_m cannot contain an $(n-1)$-square, so $\rho^\frown(n-1, n) <_{\text{lex}} \text{type}_T(x_m)$. Finally, if $t = \infty$, then t_1 is finite, but x contains arbitrarily large blocks of B_ρ. For sufficiently large m, each x_m contains blocks of B_ρ large enough to ensure that $\rho^\frown t_1 <_{\text{lex}} \text{type}_T(x_m)$.

The argument for the upper bound is similar. Let $\rho \in T$ be longest such that $\rho \prec \rho_2$ and $\rho \prec \text{type}_T(x)$, and let t, t_2 be such that $\rho^\frown t \prec \text{type}_T(x)$ and $\rho^\frown t_2 \prec \rho_2$. If $t \in \Omega$ the argument is as above. If $t = n \in \omega$ then now $n = 0$ is possible, but in any case the argument is as above. And $t = \infty$ is not possible because then there is nothing that ρ_2 could be. □

Let $H_\alpha \subseteq X_T^2$ be the associated liftings of the Hausdorff derivatives L_T^α of L_T. That is, if we let $[\rho]_\alpha$ denote the equivalence class of the type ρ in L_T^α, we have

$$H_\alpha = \{(x, y) : \text{type}_T(x) \in [\text{type}_T(y)]_\alpha\}.$$

In particular, H_1 is the set of all (x, y) for which there are only finitely many types between $\text{type}_T(x)$ and $\text{type}_T(y)$.

Lemma 8. *For $x, y \in X_T$, if there are infinitely many types between* $\mathrm{type}_T(x)$ *and* $\mathrm{type}_T(y)$*, then x and y are not a transitivity pair.*

Proof. We can assume that $x <_{\mathrm{lex}} y$. Supposing that x and y are a transitivity pair, we show that there are not infinitely many types between them. Let $\rho \in T$ be longest such that $\rho \prec \mathrm{type}_T(x)$ and $\rho \prec \mathrm{type}_T(y)$. Let $t_x, t_y \in \omega \cup \Omega \cup \{\infty\}$ such that $\rho^\frown t_x \prec \mathrm{type}_T(x)$ and $\rho^\frown t_y \prec \mathrm{type}_T(y)$. Then $t_x < t_y$. Thus $t_x \neq \infty$, so x contains a copy of A_ρ, and $t_y \neq 0$, so y contains a copy of B_ρ. Therefore, any configuration in which large patterns from x and y coexist contains both an A_ρ and a B_ρ, and is thus subject to the restrictions R_ρ and F_ρ. Relative to the alphabet $\{A_\rho, B_\rho\}$, if x contains an n-square, then y cannot contain any square larger than $n+1$ (including infinite partial squares).

So if $t_x = n$, then $t_y \in \{(n, n+1), n+1\}$. If $(n, n+1) \notin T$, then there are only finitely many types between x and y, so we are done. If $(n, n+1) \in T$, then y cannot contain $(n+1)$-square. If it did, x and y would not be a transitivity pair, because an element with sufficiently large patterns from x contains a pattern which has n-squares but is not $\rho^\frown(n, n+1)$-aligned; if such an element also contains $(n+1)$-square, it would be forbidden. Therefore, if $(n, n+1) \in T$, the only possibility is $t_y = (n, n+1)$, but y contains only n-squares, that is, y is an array of $A_{\rho^\frown(n,n+1)}$, and $\mathrm{type}_T(y) = \rho^\frown(n, n+1)^\frown 0$. Thus x and y have successor types in this case.

Since $t_x \neq \infty$, the last case to consider is if $t_x = (n, n+1)$ for some n. Then the only option for t_y is $t_y = n+1$. If $(n, n+1) \notin T$, we are done, so suppose that $(n, n+1) \in T$. Then x cannot contain an n-square. If it did, then any configuration containing large patterns from x and y would contain an n-square from x, an $(n+1)$-square from y, and an arrangment of $(n+1)$-squares from y that is incompatible with a $\rho^\frown(n, n+1)$-alignment; this is forbidden. Since x contains no n-squares, it must be that x is an infinite array of $B_{\rho^\frown(n,n+1)}$, and thus $\mathrm{type}_T(x) = \rho^\frown(n, n+1)^\frown\infty$. Therefore, x and y have successor types. □

It follows that if x and y do not satisfy xH_1y, then x and y are not an entropy pair, and furthermore there is no finite chain $x = z_0, z_1, \ldots, z_n = y$ in which each z_i, z_{i+1} is an entropy pair (between some z_i and z_{i+1} there will be infinitely many types). This shows that $E_2 \subseteq H_1$.

The relations H_β have the following nice property.

Definition 24. *A equivalence relation $F \subseteq X_T^2$ is* interval-like *if for all $x \leq_{lex} y \leq_{lex} z$ in X_T, if xFz then xFy and yFz.*

Note that if F is interval-like, there is a natural ordering on the equivalence classes of F defined by $[x]_F < [y]_F$ if and only if for all $x' \in [x]_F$ and all $y' \in [y]_F$, we have $x' <_{\mathrm{lex}} y'$.

To show the first half of Theorem 10, we take advantage of the fact that the subshift X_T was designed to have topological connectedness roughly corresponding to the order topology on L_T. Since the topological closure operation cannot do much more than connect elements from successor equivalence classes,[5] the combined double-operation of the TCPE process cannot connect things faster than the Hausdorff derivative.

Lemma 9. *For all $\beta \geq 1$, we have $E_{2\beta} \subseteq H_\beta$.*

[5]Technically a connection is possible from $[x]_\beta$ to $[y]_\beta$ if there is at most one equivalence classes between $[x]_\beta$ and $[y]_\beta$, but this is good enough.

Proof. By induction. The case $\beta = 1$ was dealt with above. The limit case is clear. For the successor case, suppose that $E_{2\beta} \subseteq H_\beta$. Since $E_{2\beta+2}$ is obtained by taking two closure operations on $E_{2\beta}$, we are done if we can show that applying the same two closure operations to H_β yields a subset of $H_{\beta+1}$.

Let H'_β denote the topological closure of H_β in X_T^2. If $(x, y) \in H'_\beta$, then there is a sequence $(x_n, y_n)_{n\in\omega}$ of elements of H_β whose limit is (x, y). Without loss of generality, assume that $[x]_\beta < [y]_\beta$. We would like to conclude that there are only finitely many H_β-equivalence classes between $[x]_\beta$ and $[y]_\beta$. Suppose for contradiction that there are $z, w \in X_T$ such that $[x]_\beta < [z]_\beta < [w]_\beta < [y]_\beta$. Because H_β is interval-like, we have $x <_{\text{lex}} z$, so by Lemma 7, eventually $x_n \leq_{\text{lex}} z$. Similarly, eventually we have $w \leq_{\text{lex}} y_n$. However, this is a contradiction, because each $(x_n, y_n) \in H_\beta$ and H_β is interval-like.

Let H''_β denote the symmetric and transitive closure of H'_β. If $(x, y) \in H''_\beta$ then there is a sequence $x = z_1, \ldots, z_n = y$ such that each $(z_i, z_{i+1}) \in H'_\beta$. Therefore there are at most finitely many H_β-equivalence classes between $[x]_\beta$ and $[y]_\beta$. So $(x, y) \in H_{\beta+1}$. □

Now we turn our attention to showing that the TCPE process connects things as quickly as the Hausdorff derivative allows, subject to a small caveat about alignment.

Take as an example the case $T = \{\lambda, (1, 2)\}$ which was discussed at the start of this section. It is easy to construct configurations $x, y \in X_T$ where for example each may have type $(1, 2)^\frown 3$, and yet x and y are not an entropy pair, because the $\{A_{(1,2)}, B_{(1,2)}\}$ grid of x may be off by a single pixel shift from the corresponding grid of y. Thus patterns of x and y cannot be independently swapped. Now, x and y are a transitivity pair, so we could solve this issue by putting X_T on fabric as in Theorem 7. However, we have not yet added in the computation part, and if we make X_T all wiggly before adding the computation, the required algorithm becomes very messy. It is simpler instead to notice that all we need to prove the theorem is to show that the TCPE derivation process already proceeds fast enough when its domain is restricted to a subset of X_T on which all alignments are compatible. If the derivation process goes fast enough on each restricted domain, its progress cannot get slower when all the domains are considered together.

Definition 25. *An* alignment family *is a function $a : T \to \mathbb{N}^2$ such that for every $\rho \in T$, there is an $x \in X_T$ such that $\rho \prec \text{type}_T(x)$ and $a(\tau)$ is the τ-alignment of x for all $\tau \preceq \rho$.*

If you imagine building such a in layers starting with defining $a(\lambda)$ (which has to be (0,0)), then defining a on all strings of length 1, etc, at each ρ the values of $a(\tau)$ for $\tau \prec \rho$, as well as the value at $\rho(|\rho| - 1)$, place some simple and deterministic restrictions on what $a(\rho)$ can be, and other than that you have free choice of $a(\rho)$. The important point, which the reader can verify, is that for every $x \in X_T$, there is an alignment family a such that for all $\rho \in T$ with $\rho \prec \text{type}_T(x)$, we have $a(\rho)$ is the ρ-alignment of x. Just fill in a in layers, but when there is a choice at some $\rho \prec \text{type}_T(x)$, copy the alignment of x.

Given an alignment family a, let X_T^a denote the subset of X_T consisting of all those x for which the ρ-alignment of x is $a(\rho)$ for every ρ for which x is ρ-regular. We see that $X_T = \cup_a X_T^a$, however any given x is in multiple of these sets, and some x, such as for example $1^{\mathbb{Z}^2}$, are in all of them. For each ordinal α and alignment family a, let H^a_α and E^a_α denote denote the restrictions of H_α and E_α to X^a_T respectively.

Lemma 10. *Suppose $x, y \in X_T$ such that x and y have the same ρ-alignment for any ρ such that x and y are both ρ-regular. Then*

1. *If $\mathrm{type}_T(x) = \mathrm{type}_T(y)$ then x and y are an entropy-or-equal pair.*
2. *If there are no types strictly between $\mathrm{type}_T(x)$ and $\mathrm{type}_T(y)$, then there are $x' \equiv_{lex} x$ and $y' \equiv_{lex} y$, with the same alignments as x and y, such that x' and y' are an entropy-or-equal pair.*

Proof. Without loss of generality assume that $x \leq_{\mathrm{lex}} y$. Let $\rho \in T \cup [T]$ be longest such that $\rho \prec \mathrm{type}_T(x)$ and $\rho \prec \mathrm{type}_T(y)$. If $\rho \in T$ then x and y can both be understood as configurations on the alphabet $\{A_\rho, B_\rho\}$. In the following cases we will refer several times to n-squares and appeal several times to Lemma **??**. In all cases, these references should be understood with respect to the alphabet $\{A_\rho, B_\rho\}$.

Case 1. Suppose $\mathrm{type}_T(x) = \rho^\frown n$ where $n \in \omega$. If $\mathrm{type}_T(y)$ is either $\rho^\frown n$ or $\rho^\frown(n, n+1)$, then x and y are an entropy pair by Lemma **??**. If $\rho^\frown(n, n+1) \in T$ then the L_T-successor of $\rho^\frown n$ is $\rho^\frown(n, n+1)^\frown 0$. If this is $\mathrm{type}_T(y)$, let y' be a configuration which consists of tiling the plane with $A_{\rho^\frown(n,n+1)}$, using A_ρ and B_ρ in the appropriate alignment. This configuration counts as type 0 relative to the alphabet $\{A_{\rho^\frown(n,n+1)}, B_{\rho^\frown(n,n+1)}\}$ because squares are allowed to be flush next to each other, so this is the tightest possible packing of 0-squares relative to $\{A_{\rho^\frown(n,n+1)}, B_{\rho^\frown(n,n+1)}\}$. Therefore, $y' \equiv_{\mathrm{lex}} y$. (This is why we must permit squares to touch, and it is the only place where that distinction is needed). Now y', like x, contains only n-squares; it is just a coincidence that y' is $\rho^\frown(n, n+1)$-regular, and the entropy pairhood of x and y' can be realized by elements of type $\rho^\frown n$.

Case 2. Suppose $\mathrm{type}_T(x) = \rho^\frown\infty$. Since $\rho \prec \mathrm{type}_T(y)$, it must be that $\mathrm{type}_T(y)$ is also $\rho^\frown\infty$. The proof of Lemma **??** for the case where both configurations have type ∞ shows that x and y will be an entropy-or-equal pair if there are infinitely many t such that $\rho^\frown(n, n+1) \notin T$. This condition is satisfied because $T \subseteq (2\Omega)^{<\omega}$. This is one of two places where it is used that $T \subseteq (2\Omega)^{<\omega}$ rather than $\Omega^{<\omega}$.

Case 3. Suppose $\mathrm{type}_T(x) = \rho^\frown(n, n+1)$, where $(n, n+1) \notin T$. Then $\mathrm{type}_T(y)$ is one of $\rho^\frown(n, n+1)$ or $\rho^\frown(n+1)$, and x and y are an entropy pair by Lemma **??**.

Case 4. Suppose $\rho^\frown(n, n+1) \prec \mathrm{type}_T(x)$, where $\rho^\frown(n, n+1) \in T$. By the choice of ρ we know that y is not ρ-aligned, so the only way to get x and y to be $\leq_{\mathrm{lex}}$-successors is if $\mathrm{type}_T(x) = \rho^\frown(n, n+1)^\frown\infty$ and $\mathrm{type}_T(y) = \rho^\frown(n+1)$. Let x' be a configuration which consists of tiling the plane with $B_{\rho^\frown(n,n+1)}$, using A_ρ and B_ρ with the proper alignment. Then $x' \equiv_{\mathrm{lex}} x$, and x' contains only $(t+1)$-squares; it is just a coincidence that x' is $\rho^\frown(n, n+1)$-regular, and the entropy pairhood of x' and y can be realized by elements of type $\rho^\frown(n+1)$.

Case 5. Suppose that $\mathrm{type}_T(x) = \rho$ where $\rho \in [T]$. Then ρ has no $\leq_{\mathrm{lex}}$-successor, so $\mathrm{type}_T(y) = \rho$ as well. Suppose that u is a finite pattern in x and v is a finite pattern in y. Then for some $\tau \prec \rho$, u and v are each contained in a 2×2 block of the alphabet $\{A_\tau, B_\tau\}$, so assume that u and v are each just a 2×2 block on that alphabet. Based on just a 2×2 block, the potential types of u and v relative to alphabet $\{A_\tau, B_\tau\}$ are almost unrestricted (the only restriction is if all four are B_τ, in which case there must be a n-square with $n \geq 2$ relative to alphabet $\{A_\tau, B_\tau\}$). So by Lemma 1, we may make independent choices of u and v on a set of small enough positive density and fill in the gaps to produce a configuration of type $\tau^\frown(2n+1, 2n+2)$ for any $n \geq 1$. We have no alignment restrictions (beyond sticking to the alphabet $\{A_\tau, B_\tau\}$) when building a configuration of this type because $T \subseteq (2\Omega)^{<\omega}$ (the second of two places where this assumption on T is used.) Therefore, x and y are an entropy-or-equal pair. □

Therefore, we may conclude that for each a, we have $H_1^a = E_2^a$. (We have already shown previously that $E_2 \subseteq H_1$.)

Lemma 11. *Fix an alignment family a. If F is an interval-like equivalence relation on X_T^a with $H_1^a \subseteq F$, and $[x]_F, [y]_F$ are a successor pair of F-equivalence classes, then the topological closure of F contains a pair (x', y') with $x' \in [x]_F$ and $y' \in [y]_F$.*

Proof. We may assume $[x]_F < [y]_F$.

Case 1. Suppose there is a longest $\rho \in T$ such that there exist $x' \in [x]_F$ and $y' \in [y]_F$ with $\rho \prec \text{type}_T(x')$, and $\rho \prec \text{type}_T(y')$. Let $t \in \Omega \cup \omega$ be largest such that for some $x' \in [x]_F$, we have $\rho^\frown t \prec \text{type}_T(x')$, if such t exists.

If no such t exists, then $[x]_F$ has elements of type $\rho^\frown n$ for arbitrarily large $n \in \omega$ and $[y]_F$ has all the a-aligned elements of type $\rho^\frown\infty$ (since $[y]_F$ does have something whose type starts with ρ, and ∞ is all that is left). For each sufficiently large n, let $x_n \in [x]_T$ be a configuration of type $\rho^\frown n$. Every limit point y' of this sequence has type $\rho^\frown\infty$. So (x, y') is in the closure of F via a subsequence of $(x, x_n)_{n\in\omega}$.

If t exists, then $[x]_F$ contains a $\leq_{\text{lex}}$-greatest type, which is $\rho^\frown t$ if $\rho^\frown t \notin T$, and which is $\rho^\frown t^\frown\infty$ if $\rho^\frown t \in t$. Letting s be the successor of $t \in \Omega \cup \omega$, $[y]_F$ contains a $\leq_{\text{lex}}$-least type, which is $\rho^\frown s$ if $\rho^\frown s \notin T$, and $\rho^\frown s^\frown 0$ if $\rho^\frown s \in T$. But that implies that some $x' \in [x]_F$ and $y' \in [y]_F$ are a $\leq_{\text{lex}}$-successor pair, contradicting that $H_1^a \subseteq F$.

Case 2. There is some infinite $\rho \in [T]$ such that for each $\tau \prec \rho$, there are $x' \in [x]_F$ and $y' \in [y]_F$ which are τ-aligned. The elements of type ρ are either in $[x]_T$ or in $[y]_T$. If they are in $[x]_T$ then for all sufficiently large n, there is $y_n \in [y]_T$ with $\text{type}_T(y_n) = (\rho \restriction n)^\frown\infty$. Every limit point x' of this sequence has type ρ, because the larger and larger alphabets $\{A_{\rho\restriction n}, B_{\rho\restriction n}\}$ are adopted cofinally in this sequence. Similarly, if the type ρ elements are in $[y]_T$, we can define x_n to be a configuration of type $(\rho \restriction n)^\frown 0$, and any limit point y' has type ρ for the same reason. □

Proof of Theorem 10. By Lemma **??**, we know that for all $\beta \geq 1$, we have $H_\beta \supseteq E_{2\beta}$. This shows that if L_T has Hausdorff rank β, then the TCPE rank of X_T is at least $2\beta - 1$, and that if L_T is not scattered, then X_T does not have TCPE.

By Lemma **??**, we also know that for all $\beta \geq 1$ and any alignment family a, we have $H_\beta^a \supseteq E_{2\beta}^a$. In fact, these two sets are equal, which we show by induction. The base case is above and the limit case is clear. Assuming that $H_\beta^a = E_{2\beta}^a$, by Lemma 11, we see that each successor pair of equivalence classes of H_β^a gets a connection between at least one pair of representatives in the topological closure of H_β^a. Therefore, taking symmetric and transitive closure, we find that $H_{\beta+1}^a \subseteq E_{2\beta+2}^a$.

It now follows that if $H_\beta = L_T^2$, then $E_{2\beta}^a = (X_T^a)^2$ for all a. The configuration $1^{\mathbb{Z}^2}$ is an element of every X_T^a, and $E_{2\beta}$ is an equivalence relation. Therefore, $E_{2\beta} = X_T^2$. □

4.4. **Enforcing the restrictions with sofic computation.** In this section we show that if $T \in (2\Omega)^{<\omega}$ is a computable tree, then X_T is sofic, via an SFT extension which has the same generalized transitivity rank as X_T.

We have to make a couple of modifications to the algorithm developed in Sections 3.3 and 3.5. Here are the major updates required:

(1) The parameter tape of each macrotile should now keep track of the largest $\rho \in T$ for which the underlying configuration appears to be ρ-regular, and the corresponding ρ-alignment relative to the macrotile, and check that other neighbors agree about this ρ and this alignment. Then the construction can proceed as in Section 3, keeping track of sizes of squares on the alphabet $\{A_\rho, B_\rho\}$.
(2) However, we cannot simply superimpose symbols of Λ onto occurrences of A_ρ and B_ρ for all ρ (since these patterns are arbitrarily large, it does not even make sense). Instead we simulate the square-enforcing function of the alphabet Λ by having each macrotile who thinks she is inside an A_ρ pattern guess about which symbol of Λ should be superimposed. The friendly neighbor communications can be slightly expanded to make sure everyone inside of a given A_ρ agrees about the symbol, and to make sure all the 2×2 restrictions of F_ρ are satisfied.

Of course, point (1) above is a slight lie. If the alignment ρ of a particular configuration is very long, or contains very large numbers, then it is likely that small macrotiles will not have a tape long enough to record such ρ. Instead, they should record an initial segment of ρ that is long enough, and leave it to their parent, grand-parent, and so on to lengthen the alignment as warranted.

Another caveat about point (1) is that we want to make sure that entropy pair relations of X_T are preserved as transitivity pair relations in the SFT that is currently under construction. So for example, if $T = \{\lambda, (1,2)\}$, we do want there to be a transitivity pair relationship between an element x of type 1 and the special element y of type $(1,2)^\frown 0$ which consists of a regular grid of the macrosymbol $A_{(1,2)}$. The macrotiles superimposed on x will be aware that x contains only 1-squares but is not $(1,2)$-aligned, while the macrotiles superimposed on y will be aware that y is $(1,2)$-aligned but does not contain any $B_{(1,2)}$. It should be consistent for both kinds of macrotile to exist in a single configuration which combines large patterns from x and y. We can achieve this if we stick to the convention that a macrotile keeps track of what is happening in its locality only. If two neighbor macrotiles have observed different but consistent things in their individual locality, both sets of observations will be assimilated by the parents of these macrotiles, in the same way that a parent who has one child seeing only n-squares and another seeing only $(n+1)$-squares can record both of those facts. In our example with $T = \{\lambda, (1,2)\}$, a parent macrotile whose locality contains both large patterns from x and large patterns from y would have recorded that its locality is λ-regular, that there are 1-squares in that locality, and that the locality is neither $(0,1)$-regular nor $(1,2)$-regular (the former witnessed by a child in the y part and the latter witnessed by a child in the x part).

In the above paragraph we have left the notion of locality deliberately vague. That is because the macrotiles and the macrosymbols are unlikely to line up exactly, and so if a macrotile is to know about the macrosymbols which are in its vicinity, it will necessarily know about things beyond the boundary of the usual responsibility zone. This will not cause a problem, nor will it be necessary to give a precise definition to the term "locality". However, it will be necessary to be precise about how much of ρ a given macrotile should record (in the case where the type of the configuration is, or locally appears to be, much longer than what could be written on the macrotile's tape).

Clearly, a macrotile should know the largest ρ such that the pattern in its usual responsibility zone is ρ-aligned. Imagine some parent macrotile which knows this information. If A_ρ and B_ρ are much smaller than this parent macrotile, then the parent cannot do the work of figuring out whether the configuration is so far satisfying F_ρ, because the parent cannot know which macrosymbol appears at each location (too much information compared to the tape size of the parent). Therefore, we need the children to primarily do this work (although if there are any remaining partial corners or sides made out of macrosymbols $\{A_\rho, B_\rho\}$, we want the children to report this to the parent). Assuming the children have success at enforcing F_ρ, then the parent macrotile and its neighbors can use the passed-up information to proceed exactly as in Section 3.5, either recording sizes reported by children, or passing messages to figure out what kind of larger squares are being made with alphabet $\{A_\rho, B_\rho\}$.

How will the macrotile's children figure out whether the configuration is so far satisfying F_ρ? They need to make a guess, for each appearance of the macrosymbol A_ρ or B_ρ, which symbol of Λ (the auxiliary alphabet used in Section 3.3) should be superimposed upon each macrosymbol. Then, by looking at 2×2 blocks of macrosymbols, they should forbid any combinations where the superimposed Λ symbols are inappropriately placed. An issue seems to arise: what if the macrosymbols are very large compared to the children? (If they are small, we can punt the problem to the grandchildren, so let us assume they are larger than the children.) How could the children be expected to know that it is time for them to guess a symbol of Λ, and where one macrosymbol from $\{A_\rho, B_\rho\}$ ends and the next begins? This information cannot actually be deduced by looking at the part of the configuration which is in the responsibility zone of the child. The children have to guess: what is ρ, how is the alphabet $\{A_\rho, B_\rho\}$ aligned, which of these two macrosymbols am I contained in? Only then can the child also guess about the symbol from Λ. Thus we see that the desired connection between information at the child level and parent level will be possible only if the child correctly guesses ρ and its alignment. This means in general that a macrotile needs to know the longest ρ such that the pattern in the responsibility zone of its *parent* appears to be ρ-aligned. Knowing this is sufficient as well.

Now we describe in more detail the algorithm that will be run on the macrotiles. But we do not give all details, trusting that the reader who has made it this far would be more hindered than helped by an overabundance of technical elaboration.

We first give a summary of the algorithm to be performed on all macrotiles, followed by remarks which give slightly more details about how to achieve any step for which there might be a question. Let Y_T be the SFT defined by superimposing the following computation onto Y_1 (where Y_1 is the shift on alphabet Λ described in Section 3.)

On input p, c:

(1) Data format check.
  (a) The parameter tape contains

- A level number i and starting number i_0
- A string $\rho \in \omega^{<\omega}$ for which the size of A_ρ is less than L_{i+1}
- Deep coordinates, relative to the parent, indicating where a single macrosymbol from $\{A_\rho, B_\rho\}$ has a lower left corner.
- A couple bits to indicate which of A_ρ and B_ρ have been sighted, and if so, example(s) of where.
- An assertion of whether F_ρ has been followed or not (and if not, where the violation occurred).
- Up to two sizes (of squares observed in alphabet $\{A_\rho, B_\rho\}$).
- If any sizes appear above, deep coordinates for some corners of squares of those sizes (witnessing that they exist and that they have no regularity)
- Up to four deep coordinates for corners or sides of partial squares on alphabet $\{A_\rho, B_\rho\}$,
- If the size of A_ρ is larger than L_i, up to four symbols of Λ, our guesses for what is superimposed on up to four A_ρ or B_ρ patterns in which we may be participating.

(b) The colors contain
- Location, machine and wire bits
- A copy of the parent's parameter tape
- Side message-passing bits
- Friendly neighbor parameter display bits
- Diagonal neighbor parameter display bits
- Trap neighbor message-passing bits

(2) Expanding tileset construction, including the "trap zone" feature discussed in Section 3.5.

(3) Size checking. If the parent asserts anything that relates to me (for example, that I contain some n-square on some alphabet with a corner at such-and-such location), make sure that assertion is accurate. Let ν denote the string recorded by my parent (i.e. ν is my parent's ρ). If ρ is a proper initial segment of ν, and the size of A_ν is larger than L_{i+1}, my parent's parameter tape uniquely determines my entire configuration, so no further checks are necessary. If ρ is a proper initial segment of ν and the size of A_ν is smaller than L_{i+1}, then I should have said ν instead of ρ; if that happens, kill the tiling. If $\nu \preceq \rho$, that means that ν is longest such that all my siblings are ν-aligned. If the parent indicates that F_ν is violated, no further checks are necessary. If the parent indicates that F_ν has been followed, we split into two cases.

(a) Case 1: If the size of A_ν is greater than or equal to the size of me (L_i), then I intersect up to four symbols from alphabet $\{A_\nu, B_\nu\}$. For each such macrosymbol, I have guessed a symbol from Λ to superimpose. (If $\nu = \rho$, I guessed this directly, if $\nu \prec \rho$, my guess for alphabet $\{A_\rho, B_\rho\}$ uniquely determines a guess for alphabet $\{A_\nu, B_\nu\}$.) If I have guessed a border-corner symbol of Λ, and if I contain the outermost pixel associated to that corner of that A_ν, I must send and receive messages for that corner, compute the size of a square on alphabet $\{A_\nu, B_\nu\}$ when I receive matching messages, and require my parent to record either the size or the partial corner (as appropriate), with full details just as in Section 3.3.

(b) Case 2: If the size of A_ν is less than L_i, I can assume that my children have already parsed the $\{A_\nu, B_\nu\}$ macrosymbols and informed me of any partial corners or sides on that alphabet which I may have. (If $\nu \prec \rho$, the information on my tape allows me to uniquely determine what partial corners or sides on alphabet $\{A_\nu, B_\nu\}$ are located in my vicinity, and also allows me to determine if any complete n-squares on that alphabet are in my vicinity.) Proceed as in Section 3.3.

(4) Friendly and diagonal neighbor steps. Using my own parameter tape plus those of eight neighbors,
- Certify that my parent's claims are consistent with what is on all my neighbors' parameter tapes
- Let ν be longest such that all eight neighbors are ν-aligned. If the size of A_ν is greater than or equal to L_i, collect all the neighbors' guesses about what symbols of Λ to superimpose on their macrosymbols A_ν and B_ν (in some cases this may have to be inferred from their guesses about larger macrosymbols). If I and a neighbor both intersect the same macrosymbol, make sure our guesses are the same (if they are not the same, kill the tiling). If the 3×3 block which I am viewing contains a 4-way boundary of macrotiles on alphabet $\{A_\nu, B_\nu\}$, check that the 2×2 restrictions on Λ are satisfied. If they are not satisfied do not kill the tiling, but do make sure the parent records that F_ν has been violated.
- If the size of A_ν is less than L_i, we can assume our children have already checked for F_ν compliance. If our tape says that F_ν was violated, nothing more to do. If our tape says that F_ν was followed, use the parameter tapes of non-sibling neighbors to find any sizes of squares on alphabet $\{A_\nu, B_\nu\}$ that straddle the parent boundary, and report these to the parent just as in Section 3.3.

(5) Trap neighbors: Read the parameter tapes of the trapped tiles, communicate what is on those tapes to all 12 trap neighbors, and reproduce all the missing functions (message-passing and F_ν compliance checking) which the trapped tiles should have performed.

(6) Compute facts about T. Halt if you ever see either of these situations:
- If $\rho \in T$, but F_ρ has not been followed, or
- If n- and $(n+1)$-squares on alphabet $\{A_\rho, B_\rho\}$ have been reported, and $\rho^\frown(n, n+1) \in T$, but $\rho^\frown(n, n+1)$-regularity has been shown to fail.

All of the objects being computed on have size at most $\log(L_{i+2})$, and the operations are all polynomial time operations except for the final step of computing facts about T. Therefore, all but the last step of the algorithm takes $\text{poly}(\log L_{i+2})$ time, which is asymptotically much less than $N_i/2$. It follows that we can define the tileset to start from an i_0 large enough that each macrotile will finish steps (1)-(5) within half of its available time, leaving the other half for T computation. A sufficiently large macrotile superimposed on a configuration of impermissible type will be large enough to have learned the type and large enough to compute the T-facts which witness that the type is impermissible. At that point, the impermissible type will be forbidden. Therefore, X_T is a factor of Y_T.

Let $T \in (2\Omega)^{<\omega}$ be a computable tree and let Y_T be defined as above. Let $h : Y_T \to X_T$ be the obvious factor map. Then we have the following lemmas.

Lemma 12. *Y_T has a fully supported measure.*

Proof. Let w be any pattern that appears in an element y of Y_T. Without loss of generality, we can assume that w consists of a 2×2 array of macrotiles. Let $\rho = \text{type}_T(y)$. We pick an alphabet $\{A_\sigma, B_\sigma\}$ and a $t \in \omega \cup \Omega$ as follows. The goal is to end up with a finite type $\sigma^\frown t$ that is consistent with $h(w)$. If ρ is finite, let σ be all but the last symbol of ρ; then $\rho = \sigma^\frown t$ for some $t \in \omega \cup \Omega \cup \{\infty\}$. If t is finite, we are done. If $t = \infty$, let $t \in \omega$ be a number large enough that it is consistent that $h(w)$ has type t relative to alphabet $\{A_\sigma, B_\sigma\}$. If ρ is infinite, then let $\sigma \prec \rho$ be long enough that w contains parts of no more than four macrosymbols on alphabet $\{A_\sigma, B_\sigma\}$. The parameter tapes in w have made Λ-guesses for these symbols. One possibility is that all four symbols are A_σ and the Λ-guesses imply that these four symbols form a 0-square; in that case, let $t = 0$. If this possibility does not occur, then the Λ-guesses are consistent with any finite t that is sufficiently large, so we may set $t = n$ for some large n. Observe that in all cases we have arrived at a finite type $\sigma^\frown t \notin T$ which is consistent with $h(w)$ and with the information written on the four parameter tapes of w.

Relative to the alphabet $\{A_\sigma, B_\sigma\}$, we can now proceed almost exactly as in Lemma 5. The only additional detail to consider is the way in which the macrosymbol grid intersects w. If we fix a tiling structure and then place a copy of $h(w)$ in one of the trap zones at the appropriate level, this determines where the boundaries of the macrosymbols must lie throughout the entire configuration. The trap zone of a different, arbitrarily chosen macrotile may not be able to accommodate a copy of $h(w)$ because the macrosymbol boundaries may intersect each trap zone in a different way. However, there are only finitely many ways that the macrosymbol boundaries can intersect the trap zones, and each way occurs with positive density (in fact periodically). Therefore, the argument of Lemma 5 can be repeated if we just choose M large enough to ensure not only a sufficiently-central trap zone, but a sufficiently-central one with the right macrosymbol boundaries. □

Lemma 13. *If $y_n \to y \in Y_T$ in its topology, and $\rho_1 <_{lex} \text{type}_T(h(y)) <_{lex} \rho_2$ for some types ρ_1 and ρ_2, then for sufficiently large n, we have $\rho_1 \leq_{lex} \text{type}_T(h(y_n)) \leq_{lex} \rho_2$.*

Proof. Follows immediately from Lemma 7. □

Lemma 14. *For $y_1, y_2 \in Y_T$, if there are infinitely many types between $\text{type}_T(h(y_1))$ and $\text{type}_T(h(y_2))$, then y_1 and y_2 are not a transitivity pair.*

Proof. Follows immediately from Lemma 8. □

Let $H_\alpha \subseteq Y_T^2$ denote the pull-back of the Hausdorff equivalence relations on L_T. Recall that $T_\beta \subseteq Y_T^2$ refer to the relations in the generalized transitivity hierarchy of Y_T (Definition 13).

Lemma 15. *For all $\beta \geq 1$, we have $T_{2\beta} \subseteq H_\beta$.*

Proof. Identical to the proof of Lemma 9. □

Lemma 16. *For $y_1, y_2 \in Y_T$, let $\rho_1 = \mathrm{type}_T(h(y_1))$ and $\rho_2 = \mathrm{type}_T(h(y_2))$. Then*

(1) If $\rho_1 = \rho_2$, then y_1 and y_2 are a transitivity pair.
(2) If there are no types strictly between ρ_1 and ρ_2, then there exists a transitivity pair (y_1', y_2') such that $\mathrm{type}_T(h(y_1')) = \rho_1$ and $\mathrm{type}_T(h(y_2')) = \rho_2$.

Proof. This proof will be an addendum to Lemma 10. To deal with all cases at once, let us assume that $\rho_1 \leq_{\mathrm{lex}} \rho_2$, and if $\rho_1 = \sigma^\frown n$ and $\rho_2 = \sigma^\frown(n, n+1)^\frown 0$, then $h(y_2)$ is just an infinite array of $A_{\sigma^\frown(n,n+1)}$. Similarly let us assume that if $\rho_1 = \sigma^\frown(n, n+1)^\frown\infty$ and $\rho_2 = \sigma^\frown(n+1)$, then $h(y_1)$ is just an infinite array of $B_{\sigma^\frown(n,n+1)}$. Now let us show that y_1 and y_2 are a transitivity pair.

Let w_1 and w_2 be patterns of y_1 and y_2 consisting of 2×2 arrays of macrotiles of the same size. By the arguments of Lemma 10, there exists a finite type $\sigma^\frown t \notin T$ such that this type is compatible with both $h(w_1)$ and $h(w_2)$. Furthermore, in that Lemma it was essentially shown that for any two locations far enough apart, there is an element of X_T with type $\sigma^\frown t$ such that the patterns $h(w_1)$ and $h(w_2)$ appear at those locations (provided the locations agree about the alignment of the macroalphabet $\{A_\sigma, B_\sigma\}$).

Fix a macrotile grid and place $h(w_1)$ in a trap zone. This determines the $\{A_\sigma, B_\sigma\}$ macrosymbol boundaries in the entire configuration. It is likely that in w_2, the macrosymbol boundaries intersect the macrotiles in a different way than they did in w_1. We need to find a trap zone in which the macrosymbol boundaries occur in the same way as they do in w_2. Here we use a primality trick. The pixel size of a macrotile is always a power of 2. Whereas, the pixel size of a macrosymbol is $\prod_j (n_j + 3)$, where $\sigma = (n_0, n_0 + 1) \cdots (n_k, n_k + 1)$. Since $\sigma \in T$, each n_j is even. Therefore, pixel size of a macrotile and the pixel size of a macrosymbol are relatively prime. The distance between one trap zone and the next is exactly one macrotile. It follows that every possible way for macrosymbol boundaries to intersect the trap zones occurs with positive density. Therefore, there is a trap zone, sufficiently far away, where w_2 fits. Put $h(w_2)$ there. Fill in the rest of the Y_1 part of the configuration, producing an element of type $\sigma^\frown t$. Then fill up the colors for the computation part of the configuration, copying w_1 or w_2 in the target zones, just as in Lemma 5. □

This shows that $H_1 = T_2$. Finally, we can finish the analysis with the analog of Lemma 11, but it is simpler because there is no need to deal with alignment families.

Lemma 17. *If F is an interval-like equivalence relation on Y_T with $H_1 \subseteq F$, and $[x]_F, [y]_F$ are a successor pair of F-equivalence classes, then the topological closure of F contains a pair (x', y') with $x' \in [x]_F$ and $y' \in [y]_F$.*

Proof. Same as the proof of Lemma 11. □

It follows that for all β, we have $H_\beta = T_{2\beta}$. Therefore, we may conclude the main theorem of this section.

Theorem 11. *For any computable tree $T \subseteq (2\Omega)^{<\omega}$, the $\mathbb{Z}^2$-SFT Y_T is generalized transitive if and only if L_T is scattered. Furthermore, if Y_T has generalized transitivity rank α and L_T has Hausdorff rank β, then $\alpha \in \{2\beta - 1, 2\beta\}$.* □

4.5. **Main results.** To give examples of $\mathbb{Z}^2$-SFTs of various ranks, it is useful to have a class of trees for which it is easy to see the Hausdorff rank of L_T. The next definition gives such a class.

Definition 26. *A* fat tree *is a tree $T \subseteq \omega^{<\omega}$ such that for every $\sigma \in T$ with well-founded rank α and every $\beta < \alpha$, there are infinitely many n such that $\sigma^\frown n \in T$ and $\sigma^\frown n$ has rank β. We allow α or β to be ∞ in case T is ill-founded, and declare $\infty < \infty$ to be true.*

Given any tree T, there is an easy procedure to turn it into a fat tree. For example, $T \times \omega^{<\omega}$ is fat, where $(\sigma, \tau) \in S \times T$ if σ and τ have the same length, $\sigma \in S$ and $\tau \in T$. Of course, $S \times T$ is computably isomorphic to a tree on $\omega^{<\omega}$, and it is immaterial whether $T \subseteq \omega^{<\omega}$ or $T \subseteq \Omega^{<\omega}$ or $T \subseteq (2\Omega)^{<\omega}$ or anything else.

Proposition 9. *If $T \subseteq (2\Omega)^{<\omega}$ is well-founded and fat with rank α, then the Hausdorff rank of L_T is $\alpha + 1$.*

Proof. We claim that for all $\rho \in T$, all $n \in \omega$ and all α, that $[\rho^\frown n]_\alpha = [\rho^\frown(n+1)]_\alpha$ if and only if $r_T(\rho^\frown(n, n+1)) < \alpha$. This is proved by induction. If $\alpha = 1$, the distinction is only whether $\rho^\frown(n, n+1) \in T$, and the conclusion is clear from the definition of L_T.

If α is a limit, then $[\rho^\frown n]_\alpha = [\rho^\frown(n+1)]_\alpha$ if and only if the same is true for some $\beta < \alpha$, if and only if $r_T(\rho^\frown(n, n+1)) < \beta$ for some $\beta < \alpha$, if and only if $r_T(\rho^\frown(n, n+1)) < \alpha$.

Finally if $\alpha = \beta + 1$ suppose first that $r_T(\rho^\frown(n, n+1)) \leq \beta$. Sicne $\rho^\frown n$ and $\rho^\frown(n, n+1)^\frown 0$ are successors in L_T, they are in the same class in L_T^β. Similarly, $[\rho^\frown(n, n+1)^\frown\infty]_\beta = [\rho^\frown(n+1)]_\beta$. Furthermore, for each m, we know that $\rho^\frown(n, n+1)^\frown(m, m+1)$ has rank strictly less than β, and therefore $[\rho^\frown(n, n+1)^\frown m]_\beta = [\rho^\frown(n, n+1)^\frown(m+1)]_\beta$. So in fact $[\rho^\frown n]_\beta$ and $[\rho^\frown(n+1)]_\beta$ are successor classes, and the desired conclusion follows.

On the other hand, suppose that $r_T(\rho^\frown(n, n+1) \geq \alpha$. Then by fatness, there are infinitely many m such that $r_T(\rho^\frown(n, n+1)^\frown(m, m+1)) = \beta$. For such m, we have $[\rho^\frown(n, n+1)^\frown m]_\beta < [\rho^\frown(n, n+1)^\frown(m+1)]_\beta$. Therefore, $[\rho^\frown n]_\beta$ and $[\rho^\frown(n+1)]_\beta$ are separated by infinitely many L_T^β equivalence classes.

Letting $\alpha = r_T(\lambda)$, we see that $[0]_\alpha = [n]_\alpha$ for every n, so $[0]_{\alpha+1} = [\infty]_{\alpha+1}$, thus the Hausdorff rank is at most $\alpha + 1$. But for any $\beta < \alpha$, there are infinitely many m such that $[m]_\beta < [m+1]_\beta$, so $[0]_\alpha \neq [\infty]_\alpha$. ◻

Theorem 12. *For every ordinal $\alpha < \omega_1^{ck}$, there is a $\mathbb{Z}^2$-SFT with TCPE rank α.*

Proof. For any computable ordinal β, there is a computable well-founded fat tree $T \subseteq (2\Omega)^{<\omega}$ of rank β. Let Y_T be the SFT defined in the previous section. Then Y_T has a fully supported measure, and the generalized transitivity rank of Y_T is either $2\beta - 1$ or 2β. Therefore, letting f be the "put it on fabric" operation of Theorem 7, the SFT $f(Y_T)$ has TCPE rank $2\beta - 1$ or 2β.

But, we can say more, the TCPE rank of Y_T is $2\beta - 1$. By the proof of the previous proposition, when T is fat, $L_T^{\beta-1}$ has two equivalence classes, one containing type ∞, the other containing everything else. Since every element of type ∞ is the limit of a sequence of elements of finite type, the topological closure step is sufficient to connect everything, and thus the TCPE rank is $2\beta - 1$. We can also get TCPE rank 2β from a fat tree T of rank β by considering Y_S, where

$$S = \{(0, 1)^\frown \sigma : \sigma \in T\}.$$

Now $L_S^{\beta-1}$ has two equivalence classes, types less than 1 and types greater than or equal to 1. In this case, the topological closure step is not sufficient to connect the 0 type to the ∞ type; the subsequent transitive and symmetric closure step is needed, so the rank of Y_S is 2β.

Finally, we can achieve SFTs with limit TCPE rank, using a variation on Y_T for fat T, by connecting the ∞ type to the 0 type prematurely using some extra symbols. Adding symbols $*$ and $\dagger$ to Λ, we can add restrictions which ensure that if $*$ appears, then no gray symbol of Λ appears, and if $\dagger$ appears, then no white symbol of Λ appears. Then some elements of type ∞ are an entropy pair with $*^{\mathbb{Z}^2}$, $*^{\mathbb{Z}^2}$ is an entropy pair with $\dagger^{\mathbb{Z}^2}$, $\dagger^{\mathbb{Z}^2}$ is an entropy pair with some elements of type 0, and no other entropy pair relations are added. These restrictions are also easily enforced with sofic computation. ☐

The same methods also allow us to show that TCPE admits no simpler description in the special case of $\mathbb{Z}^2$-SFTs. It is not true that Y_T has TCPE if and only if T is well-founded, because some ill-founded trees still have a scattered L_T (consider for example the tree with only a single path). However, we can get around that by fattening the tree first.

Theorem 13. *The property of TCPE is Π^1_1-complete in the set of $\mathbb{Z}^2$-SFTs.*

Proof. We describe an algorithm which, given an index for a computable tree S, produces a $\mathbb{Z}^2$-SFT which has TCPE if and only if S is well-founded. Uniformly in an index for S, we can produce an index for the tree $T \subseteq (2\Omega)^{<\omega}$ which is obtained by fattening S. If S was well-founded, then T is also well-founded (the fattening does not increase the rank of any node). But if S was ill-founded, then T is not only ill-founded, but also L_T is not scattered, because $[T]$ contains a Cantor set, and so a copy of $\mathbb{Q}$ can be order-embedded into L_T using types from $[T]$. Therefore, Y_T has TCPE if and only if S was well-founded. ☐

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