THREE TOPOLOGICAL REDUCIBILITIES FOR DISCONTINUOUS FUNCTIONS

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Abstract. We define a family of three related reducibilities, $\leq_{\mathbf{T}}$, $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$, for arbitrary functions $f, g : X \to \mathbb{R}$, where X is a compact separable metric space. The $\equiv_{\mathbf{T}}$-equivalence classes mostly coincide with the proper Baire classes. We show that certain α-jump functions $j_\alpha : 2^\omega \to \mathbb{R}$ are $\leq_{\mathbf{m}}$-minimal in their Baire class. Within the Baire 1 functions, we completely characterize the degree structure associated to $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$, finding an exact match to the α hierarchy introduced by Bourgain [Bou80] and analyzed in Kechris & Louveau [KL90].

1. Introduction

1.1. **Reducibilities.** Computability theory seeks to understand the effective content of mathematics. Ever since its beginnings in the work of Gödel, Turing, Post, Kleene, Church and others, the idea of a reduction has been a central notion in this area. Turing [Tur39] formalized we now call *Turing reducibility* which can be viewed as the most general way of allowing computation of one set of natural numbers from another using oracle queries.

In the last 60 years, we have seen the introduction of a large number of reducibilities $A \leq B$, reflecting different access mechanisms for the computation of A from B. Different oracle access mechanisms give different equivalence classes calibrating computation. The measure of the efficacy of such reductions is the extent to which

(i) they give insight into computation, and
(ii) they are useful in mathematics.

Examples of (ii) above, include the use of polynomial time reductions to enable the theory of NP-completeness, but also include the use of Π^1_1-completeness to demonstrate that classical isomorphism problems like the classification of countable abelian groups cannot have

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reasonable invariants (Downey-Montalbán [DM08]), Ziegler reducibility to classify algebraic closures of finitely presented groups (see e.g. Higman-Scott [HS88]), truth-table reducibility to analyze algorithmic randomness for continuous measures (Reimann-Slaman [RS]), and enumeration reducibility for the relativised Higman embedding theorem (see [HS88]). There are many other examples.

1.2. **Reducibilities in type II computation.** The narrative above really only refers to notions of relative computability for infinite bit sequences (or objects, such as real numbers, which can be coded by such sequences). That is, the objects whose information content is being compared have function type $A : \omega \to \omega$ or similar.

What if instead we wanted to compare the information content of functions $f : [0,1] \to \mathbb{R}$? The collection $\mathcal{F}([0,1])$ of all such functions has cardinality greater than the continuum, so it is not possible to use infinite bit sequences to code all these objects. In the next section we will say a bit more about some approaches to the problem of relative computability for higher type objects, the most prominent of which is the Weihrauch computable reducibility framework.

In this paper, we introduce and analyse three notions of reduction for $\mathcal{F}(X)$, where X is a compact Polish space. Two of our notions are completely new and one has had little previous attention. We argue that that they meet the criteria (i) and (ii) above, and provide computational insight into the hierarchies previously introduced in classical analysis for the classification of the Baire classes of functions[1].

We first concentrate upon what we define to be $f \leq_{\mathbf{T}} g$. This reduction is interpreted to mean that f is continuously Weihrauch reducible to the *parallelization* of g. In the next section, we define what we mean by this, and argue that this is the most natural (continuous) analog of Turing reducibility for higher type objects. We introduce the new notions of $f \leq_{\mathbf{tt}} g$ and $f \leq_{\mathbf{m}} g$ by restricting the oracle use of the functionals in the Weihrauch reduction in an appropriate way described in Section 5.

It seems to be folklore that the $\leq_{\mathbf{T}}$ degrees of the Baire functions are linearly ordered, and these degrees correspond to the proper Baire classes. Our main results concern the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees. We show that the αth jump operator[2] j_α is $\leq_{\mathbf{m}}$-minimal in its Baire class.

Theorem 1. *If a Baire function f is not Baire α, then $f \geq_{\mathbf{m}} j_{\alpha+1}$.*

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[1]We define these terms in Section 3.

[2]We will define j_α later, but for example $j_1 : 2^\omega \to \mathbb{R}$ is $j_1(X) := \sum_{i \in X'} 2^{-(i+1)}$.

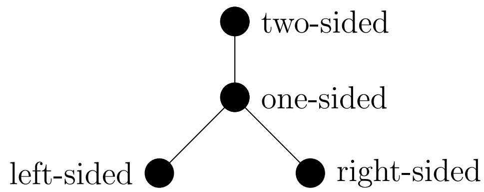

FIGURE 1. The $\leq_{\mathbf{m}}$ degrees of functions f with $|f|_\alpha$ a successor.

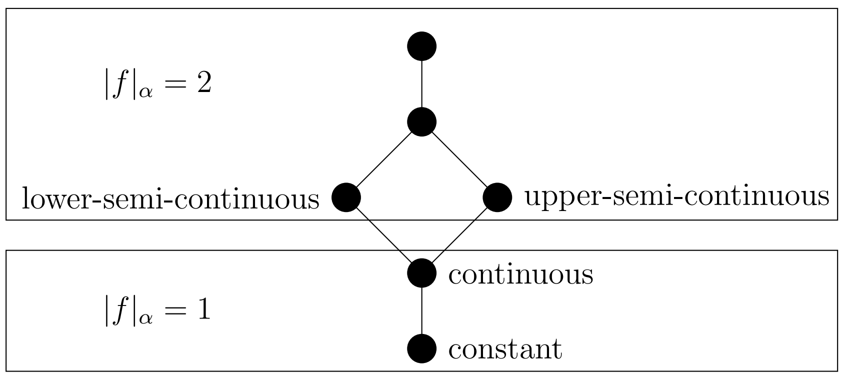

FIGURE 2. The smallest $\leq_{\mathbf{m}}$ degrees are recognizable classes.

Then we restrict attention to the Baire 1 functions. In [KL90], Kechris and Louveau consider three ranking functions α, β and γ, which take Baire 1 functions to countable ordinals. These ranks are especially robust at levels of the form ω^ξ. Letting $\xi(f)$ denote the least ξ such that $\alpha(f) \leq \omega^\xi$, in our main theorem we characterize the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees of the Baire 1 functions as follows.

Theorem 2. *For f and g discontinuous Baire 1 functions,*

(1) $f \leq_{\mathbf{tt}} g$ if and only if $\xi(f) \leq \xi(g)$.
(2) If $|f|_\alpha < |g|_\alpha$, then $f \leq_{\mathbf{m}} g$.
(3) If ν is a limit ordinal, $\{f : |f|_\alpha = \nu\}$ is an $\leq_{\mathbf{m}}$-degree.
(4) If ν is a successor, $\{f : |f|_\alpha = \nu\}$ contains exactly four $\leq_{\mathbf{m}}$-degrees arranged as in Figure 1.

The smallest $\leq_{\mathbf{m}}$-degrees are recognizable classes: constant functions, continuous functions, upper semi-continuous functions, and lower semi-continuous functions. See Figure 2.

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2. Motivations in defining $\leq_{\mathbf{T}}$

2.1. **Weihrauch/computable reducibility.** Suppose we want to define $f \leq g$ for functions f and g, with the meaning that g can compute f. The search for a natural notion of $f \leq g$ leads directly to Weihrauch reducibility. For $A, B \in 2^\omega$, it is clear what it means to "know" A. An algorithm or oracle knows A if, given input n, it outputs $A(n)$. Accordingly, a computation of A from B is a algorithm which can answer these questions about A when given query access to an oracle for B. So, what kinds of questions should we be able to answer if we claim to "know" $f : [0,1] \to \mathbb{R}$? At a minimum, an oracle for f ought to be able to produce $f(x)$ when given input x. We take this ability as the defining feature of an oracle for f.

Now, what should it mean for an algorithm to have query access to an oracle for g? Clearly, given input x, the algorithm should be able to pass it through and query $g(x)$. If $g(x)$ were the only permitted query, the algorithm could not really be said to have access to an oracle for all of g, so we should allow some other queries as well. For example, one would hope for a theory in which the functions $x \mapsto f(x)$ and $x \mapsto f(x+c)$ always compute each other, where c is a computable real. Generalizing this idea, an algorithm with query access to g should be able to ask about $g(y)$ for any $y \leq_T x$. Therefore, the notion of Weihrauch reducibility is a natural starting candidate for a notion of $f \leq g$. For simplicity of this narrative we horribly abuse some notation in the definition given below. We refer the reader to Section 3.3 for the full definition.

Definition 3. *Let $f, g : [0,1] \to \mathbb{R}$.*

(a) *f is called* computably/Weihrauch reducible to *g, $f \leq_W g$, if there are partial computable functions $H, K : 2^\omega \to 2^\omega$ with*
$$f(x) = H(x, g(K(x))).$$
(b) *If we replace "computable" by continuous in the definition above, we refer to this as* continuous *Weihrauch reducibility, and write $f \leq_W^c g$.*

The abuse of notation lies in the fact that H and K manipulate names or codes for reals rather than the reals themselves.[3]

The name "Weihrauch reducibility" was coined by Brattka and Gherardi [BG11], whereas earlier Weihrauch had called it computable reducibility.

[3]A subtle but important point is that reals have multiple names, and K is not required to have consistent behavior on two different names for the same real.

Computable/continuous Weihrauch reducibility has been studied in [Bra05], [Myl06] and [Pau10]. Brattka [Bra05] proved effective versions of classical theorems linking the Borel and Baire hierarchies using this reducibility.

2.2. **Parallelized Weihrauch reducibilty.** The above account seems to miss the feature of ordinary $\leq_T$ computation in which the algorithm may use the oracle repeatedly and interactively. We would not like to limit the reduction algorithm to a single use of the g oracle.

However, if the algorithm had access to all of $g(x)$ based on its first query, it would be able to feed this back into the g oracle, obtaining $g(g(x))$ and in general the sequence of $g^{(n)}(x)$. And if we accept some algorithm is uniformly producing the sequence $g^{(n)}(x)$, it could be simultaneously engaged in writing down a summarizing output $g^{(\omega)}(x)$, where $g^{(\omega)}(x)$ is for example defined as $\bigoplus_n g^{(n)}(x)$.[4] So we are led to accept $g^{(\omega)} \leq g$. If we accept this, and also wish our notion to be transitive, we must accept $g^{(\omega+1)} \leq g$, otherwise transitivity will be violated in the sequence $g^{(\omega+1)} \leq (g^{(\omega)} \oplus g) \leq g$. In the end, we are forced to say g computes all its iterates up to ω_1^{ck}. The notion just described, complete with all the transfinite iteration, was studied by Kleene [Kle59]. However, this reducibility is coarser than we want (for example, we would not want the jump operator on 2^ω to be able to compute the double-jump operator) and so we choose to go by another route.

Suppose instead we make the following seemingly minor adjustment to our concept of what an oracle for g should do. Instead of querying with an input x, we query with a pair (x, ε), where $\varepsilon \in \mathbb{Q}^+$. Instead of returning the entire $g(x)$, the oracle returns some $p \in \mathbb{Q}$ with $|g(x) - p| < \varepsilon$. Now an algorithm which on input x has made finitely many queries to g has only acquired a finite amount of new information, so its future queries are still restricted to those y with $y \leq_T x$. This breaks the cycle above. In order to get more and more precision on $f(x)$, such an algorithm may query $g(y)$ for many different values of y. But there are at most countably many queries to g associated to the computation of a single $f(x)$. Therefore, we can naturally express the kind of reducibility described above in the Weihrauch framework: $f \leq g$ could mean $f \leq_W \hat{g}$, where $\hat{g} : X^\omega \to Y^\omega$ is the *parallelization* of g, defined by applying g componentwise.

[4]Imagine for the purposes of this hypothetical that g is an operator on 2^ω, so that a joining operation $\oplus$ is available to us; a similar situation could be concocted for operators on the unit interval.

2.3. **What is a single bit of information about f?** Accepting parallelized Weihrauch reducibility as a higher-type notion of $\leq_T$, what should $\leq_{tt}$ and $\leq_m$ be? It is particularly informative to consider $\leq_m$. In classical computability theory, $A \leq_m B$ means that there is an algorithm which, on input n, outputs m such that $A(n) = B(m)$. For us the important features are:

(1) The oracle's response is accepted unchanged as the output, and
(2) The question is a yes/no question.

Allowing a more demanding question (such as "approximate $f(x)$ to within ε") seems unfair, ruling out m-computations between functions of disjoint ranges that are otherwise computationally identical. (Notions of m-reducibility without point (2) have been considered however, for example by Hertling [Her93], Pauly [Pau10] and Carroy [Car13].)

Our previous decision on how to finitize the oracle was, upon reflection, rather arbitrary. We could restrict ourselves to yes/no questions with the following convention about oracles, and still end up with a $\leq_T$ notion equivalent to parallelized Weihrauch reducibility. An oracle for f accepts as input a triple (x, p, ε), with $p \in \mathbb{Q}$ and $\varepsilon \in \mathbb{Q}^+$, and ε-approximately answers the question "is $f(x) < p$"? The exact version of this question would be too precise for a computable procedure, so we accept any answer as correct if $|f(x) - p| < \varepsilon$. Now that each query to the oracle yields exactly one bit of information, we can define $\leq_m$ and $\leq_{tt}$ for the higher type objects by placing corresponding restrictions on the oracle use. We do this in Section 5.

2.4. **Parameters.** Another natural question we might ask ourselves is "what parameters would be reasonable for such reductions"? For reductions between objects of type $A : \omega \to \omega$, we usually allow integer parameters in computation procedures. Therefore, for reductions between objects of type $f : [0, 1] \to \mathbb{R}$, perhaps we should allow real parameters. We take this approach, which has a substantial simplifying effect. Every continuous function is computable relative to a real parameter, so Weihrauch computability relative to a real parameter is the same as continuous Weihrauch reducibility. Therefore, our reducibilities have a more topological rather than computational character. In particular, we define $f \leq_{\mathbf{T}} g$ to mean $f \leq^c_W \hat{g}$, and make similar topological definitions for $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$ in Section 5. We plan to address the question of the lightface theory in future work.

3. Preliminaries

3.1. **Notation.** We use standard computability-theoretic notation. Brackets $\langle m, n\rangle$ denote a canonical pairing function identifying $\omega \times \omega$ with ω. The expression 0^ω refers to an ω-length string of 0's. Concatenation of finite or infinite strings σ and τ is denoted by $\sigma^\frown\tau$, which may be shortened to $\sigma\tau$ in cases where it would cause no confusion. If σ is a string with a single entry n, we also denote concatenation by $n^\frown\sigma$ or $\sigma^\frown n$. We usually use X and Y to denote compact separable metric spaces, A, B, Z, W to denote elements of 2^ω or ω^ω, C, D, P, Q to denote subsets of 2^ω, and $\mathcal{C}, \mathcal{D}$ to denote subsets of $\mathcal{P}(X)$. Usually f, g, and j are arbitrary functions from X to $\mathbb{R}$ (the ones whose complexity we seek to categorize), while h, k, u, v, H and K are typically continuous functions from ω^ω to ω^ω.

3.2. **Computability and descriptive set theory.** We assume the reader is familiar with Kleene's $\mathcal{O}$ (but without it, one could still understand the results at the finite levels of each of the hierarchies). The standard reference on this subject is Sacks [Sac90]. The nth jump of a set $A \in \omega^\omega$ is denoted $A^{(n)}$. For any $a \in \mathcal{O}^A$, if $|a| = n$ then let $A_{(a)}$ denote $A^{(n)}$, and if $|a|$ is infinite then let $A_{(a)}$ denote $H^A_{2^a}$. If $a \in \mathcal{O}$ with $|a| = \alpha$, we will often simply write α instead of a. Thus an expression like $A_{(\alpha)}$ is technically ambiguous, but since all the sets which it could refer to are one-equivalent, no problems will arise.

The reason for numbering the jumps in this lower-subscript way is to make them align correctly with the Borel hierarchy. Recall that a set is $\boldsymbol{\Sigma}^0_1$ if it is open, $\boldsymbol{\Pi}^0_\alpha$ if it is the complement of a $\boldsymbol{\Sigma}^0_\alpha$ set, and $\boldsymbol{\Sigma}^0_\alpha$ if it is of the form $\cup_{n\in\omega} C_n$ where each C_n is $\boldsymbol{\Pi}^0_{\beta_n}$ for some $\beta_n < \alpha$. Then a set $C \subseteq \omega^\omega$ is $\boldsymbol{\Sigma}^0_\alpha$ if and only if there is a parameter $Z \in \omega^\omega$ and an index i such that for all $A \in \omega^\omega$,

$$A \in C \iff i \in (A \oplus Z)_{(\alpha)}$$

(and if no parameter is needed, we say C is Σ^0_α).

Still, at least once we will want to refer to the sets H^A_a, where $|a|$ is a limit ordinal. In this case, we write $A^{(a)}$ to denote H^A_a.

3.3. **Representations.** Although our results were motivated by considering $f : [0, 1] \to \mathbb{R}$, they are also applicable in a wider context, *represented spaces*, and hence, for completeness, we will briefly give an account of such spaces. A standard reference is Weihrauch [Wei00].

In order for a machine to interact with a mathematical object, the object must be coded in a format a machine can read, such an element of 2^ω or ω^ω. For example, an element of $\mathbb{R}$ could be coded by a rapidly

Cauchy sequence of rational numbers (which is itself coded by an element of ω^ω using some fixed computable bijection $\omega \leftrightarrow \mathbb{Q}$). It is not too hard to see that a similar method will also work for any computable metric space, where the role of the rationals is taken by (codes for) a computable dense subset.

A *representation* of a space X is a partial function $\delta :\subseteq \omega^\omega \to X$, so that elements $x \in X$ have δ-names A_x (strictly a set $\{A_x \mid \delta(A_x) = x\}$). Note that x can have many names A_x, and not every element of ω^ω is a name. A representation induces a topology on X, the final topology, defined by $U \subseteq X$ is open if and only if $\delta^{-1}(U)$ is open in the subspace topology on $\operatorname{dom}\delta$. If X already has a topology, we restrict attention to representations which induce the topology of X. Then if X and Y are represented spaces and $f : X \to Y$, we say f is *computable* if there is a computable function $F : \omega^\omega \to \omega^\omega$ such that whenever A_x is a name for x, then $F(A_x)$ is a name for $f(x)$. We say that F *realizes* f. Because x and $f(x)$ each have many names, in general realizers are not unique.

Not all representations are created equal. For example, the base 10 representation for reals is a valid representation according to the above definition, but the function $f(x) = 3x$ is not computable with respect the base 10 representation on both sides (what digit should the algorithm output first when seeing input .33333...?). However, it is computable with respect to the Cauchy name representation on both sides. This difference is captured in the following definition: a representation $\delta :\subseteq \omega^\omega \to X$ is *admissible* if for every other continuous $\delta' :\subseteq \omega^\omega \to X$, there is a continuous function $G : \omega^\omega \to \omega^\omega$ such that for all $A \in \operatorname{dom}\delta'$, we have $\delta(A) = \delta'(G(A))$. That is, G transforms δ'-names to δ-names. Observe that it is possible to continuously transform a base 10 name for x into a Cauchy name for x, but not vice versa. Some definition chasing shows that the Cauchy name representation for $\mathbb{R}$ is admissible. Restricting attention to admissible representations allows continuity properties of f to be reflected in its realizers.

Theorem 4 (Kreitz and Weihrauch [KW85], Schröder [Sch02])**.** *If X and Y are admissibly represented separable T_0 spaces, then a partial function $f :\subseteq X \to Y$ has a continuous realizer if and only if f is continuous.*

All of the pain and suffering involving representations is rewarded when we want to compare functions f and g in topologically incompatible areas, like Cantor space and $\mathbb{R}$. When comparing $f : X \to Y$ and $g : U \to V$, we can so so *via their representations in ω^ω*. Given two represented spaces X and Y, a *Weihrauch problem* is a multivalued partial function $f : X \rightrightarrows Y$. The $\rightrightarrows$ indicated that this definition concerns multivalued partial functions. But in this paper, almost all problems will be total and single-valued. Accordingly, we will freely call problems "functions".

We conclude this section with the precise definition of Weihrauch reducibility on represented spaces. Let X, Y, Z and W be represented spaces with representations $\delta_X, \delta_Y, \delta_Z$, δ_W. If $f : X \to Y$ and $g : Z \to W$ are two single-valued Weihrauch problems, we say f is *Weihrauch reducible* to g, written $f \leq_W g$, if there are computable functions $H, K :\subseteq \omega^\omega \to \omega^\omega$ such that $f(\delta_X(A)) = \delta_Y H(A, B)$ for all $A \in \operatorname{dom} \delta_X$ and all B such that $\delta_W B = g(K(A))$. We say f is *strongly Weihrauch reducible* to g, written $f \leq_{sW} g$, if $H(A, B)$ can be replaced by $H(B)$ above. The notions of *continuous Weihrauch reducibility* and *continuous strong Weihrauch reducibility*, denoted $\leq^c_W$ and $\leq^c_{sW}$ respectively, are obtained by allowing H and K to be merely continuous rather than computable. The parallelization $\hat{g} : Z^\omega \to W^\omega$ is defined by $\hat{g}((z_i)_{i\in\omega}) = (g(z_i))_{i\in\omega}$.

In this paper, we will be dealing for the most part with situations where the coding is clear, and hence suppress the δ_X notation whenever possible.

3.4. **Baire functions.** Baire functions are the most tractable functions we might consider after continuous ones. Baire 1 functions are those which are defined as pointwise limits of a countable collection of continuous functions; $f(x) = \lim_s f_s(x)$ with each f_s continuous. More generally, let X be a compact separable metric space. By $\mathcal{C}(X)$, we mean the continuous functions $f : X \to \mathbb{R}$. The Baire hierarchy of functions on X is defined as follows. Let $\mathcal{B}_0(X) = \mathcal{C}(X)$. For each $\alpha > 0$, let $\mathcal{B}_\alpha(X)$ be the set of functions which are pointwise limits of sequences of functions from $\cup_{\beta<\alpha}\mathcal{B}_\beta(X)$. The functions in $\mathcal{B}_\alpha(X)$ are also referred to as the Baire α functions when X is clear.

It is well-known that a function f is Baire α if and only if the inverse image of each open set under f is $\mathbf{\Sigma}^0_{\alpha+1}$. When $X = 2^\omega$, the Baire α functions can also be characterized via the jump.

Proposition 5 (Folklore). *For each ordinal α and $f : 2^\omega \to \mathbb{R}$, $f \in \mathcal{B}_\alpha(2^\omega)$ if and only if there is a Turing functional Γ and $B \in 2^\omega$ such that*

$$f(A) = \Gamma((A \oplus B)_{(\alpha)}).^5$$

[5]Technically Γ outputs a code for $f(A)$ using some admissible representation.

Proof. When such Γ and B exist, one can readily check that the inverse images of open sets are $\mathbf{\Sigma}^0_{\alpha+1}$. Conversely, if f is Baire α, then the sets $f^{-1}((p,q))$ for $p,q \in \mathbb{Q}$ can each be written as

$$f^{-1}((p,q)) = \{A : i_{p,q} \in (A \oplus B_{p,q})_{(\alpha+1)}\}$$

Therefore, if B is an oracle containing each $B_{p,q}$ and $i_{p,q}$ in a uniformly accessible manner, one can use a $(A \oplus B)_{(\alpha)}$ oracle to enumerate the rational intervals (p,q) containing $f(A)$, which is enough to make a Cauchy name for $f(A)$. $\square$

3.5. **Ranks on Baire 1 functions.** In [KL90], Kechris and Louveau defined three ranks α, β and γ on the Baire 1 functions. These ranks had been used either explicitly or implicitly in the literature analyzing this class of functions. Given a Baire 1 function $f : X \to \mathbb{R}$, the following derivation process is used to define the α rank. Given rational numbers p,q with $p < q$ and a closed set $P \subseteq X$, let

$$P'_{p,q} = P \backslash \cup \{U \subseteq X : U \text{ is open and } f(U) \subseteq (p,\infty) \text{ or } f(U) \subseteq (-\infty, q)\}$$

For a fixed pair p,q, define an ω_1-length sequence $\{P_\nu\}_{\nu<\omega_1}$ as follows. Let $P_0 = X$, $P_{\nu+1} = (P_\nu)'_{p,q}$, and $P_\nu = \cap_{\mu<\nu} P_\mu$ if ν is a limit ordinal. Since X is separable, it has a countable basis, so the sequence must stabilize below ω_1. Let $\alpha(f,p,q)$ be the least ν such that $P_\nu = \emptyset$; one can show that such ν exists if and only if f is Baire 1.

Finally, the α rank is defined by $\alpha(f) = \sup_{p<q} \alpha(f,p,q)$. The β and γ ranks are also defined by different transfinite derivation processes. Kechris and Louveau show that the levels of the form ω^ν are especially robust in the following sense.

Theorem 6 ([KL90])**.** *For any countable ξ and any bounded Baire 1 function f,*

$$\alpha(f) \leq \omega^\xi \iff \beta(f) \leq \omega^\xi \iff \gamma(f) \leq \omega^\xi.$$

4. Topological Turing reducibility on 2^ω

First we define the topological Turing reducibility as mentioned in the introduction. First we fix $X = 2^\omega$.

Definition 7. *For $f,g : 2^\omega \to \mathbb{R}$, let $f \leq_{\mathbf{T}} g$ if $f \leq^c_W g$.*

Equivalently, $f \leq_{\mathbf{T}} g$ if and only if there is a countable sequence of continuous functions $k_i : 2^\omega \to 2^\omega$ and a continuous function $h :\subseteq 2^\omega \to 2^\omega$ such that whenever $\{B_i\}_{i<\omega}$ are Cauchy names for $\{g(k_i(A))\}_{i<\omega}$, $h(A \oplus \bigoplus_{i<\omega} B_i)$ is a Cauchy name for $f(A)$. Observe that all continuous functions are equivalent under $\leq_{\mathbf{T}}$.

The restriction of the domain to 2^ω is not essential, but helps keep the notation manageable. If X is a compact separable metrizable space and $f : X \to \mathbb{R}$, then in order to compare f with other functions, we may replace f with $fd_X : 2^\omega \to \mathbb{R}$, where $d_X : 2^\omega \to X$ is any admissible representation. This gives a well-defined extension of the notion of $\leq_{\mathbf{T}}$ because, as the following proposition makes explicit, it does not matter which admissible representation we choose.

Proposition 8. *Let X, Y be compact separable metrizable spaces and let $d_X, d'_X : 2^\omega \to X$ and $d_Y, d'_Y : 2^\omega \to Y$ be any admissible representations for X and Y respectively. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$. Then*

$$fd_X \leq_{\mathbf{T}} gd_Y \iff fd'_X \leq_{\mathbf{T}} gd'_Y$$

Proof. Suppose that $(k_i)_{i<\omega}$ and h witness that $fd_X \leq_{\mathbf{T}} gd_Y$. By admissibility, let $\phi, \psi : 2^\omega \to 2^\omega$ be continuous functions such that $d'_X = d_X\phi$ and $d_Y = d'_Y\psi$ (note the asymmetry). Then $(\psi k_i \phi)_{i<\omega}$ and h witness that $fd'_X \leq_{\mathbf{T}} gd'_Y$. The reverse implication follows by symmetry. □

Therefore, from here on we may restrict our attention to functions $f, g : 2^\omega \to \mathbb{R}$.

Note that one could also consider the notion defined by $f \leq^c_{sW} g$. However, this is almost the same notion as the one defined. If $f \leq_{\mathbf{T}} g$ via $\{k_i\}$ and h, and if g is a non-constant function, then letting B_0 and B_1 be such that $g(B_0) \neq g(B_1)$, one could additionally consider the continuous functions $\{k'_i\}$ which map A to B_0 if $A(i) = 0$ and map A to B_1 otherwise. Then A itself is continuously recoverable from $\bigoplus_i k'_i(A)$, so by adding these to the original $\{k_i\}$, a small modification to the original h will do the job in the strong Weihrauch setting.

Therefore, if g is non-constant, then $f \leq^c_{sW} g$ if and only $f \leq_{\mathbf{T}} g$. On the other hand, if g is constant, then $\{f : f \leq^c_{sW} g\}$ is just the set of constant functions. So there is no need to consider the strong variant separately.

Now let us define some jump functions to characterize the $\leq_{\mathbf{T}}$ degrees of the Baire functions. The jump functions we consider are real-valued, because of our original motivation to study functions from $[0, 1]$ to $\mathbb{R}$. But the jump operator can be represented as a real-valued function in a standard way.

Definition 9. *For $n \in \omega$, let $j_n : 2^\omega \to \mathbb{R}$ be defined by*

$$j_n(A) = \sum_{i \in A_{(n)}} 2^{-(i+1)}.$$

Because each $j_n(A)$ is irrational, its binary expansion can be continuously recovered from it. Therefore, by Proposition 5, if f is Baire n, then $f \leq_{\mathbf{T}} j_n$. We can also extend the definition to the ordinal notations. Context will make it clear whether natural number subscript should be interpreted as a natural number or as an ordinal notation.

Definition 10. *For $a \in \mathcal{O}$, let let $j_a : 2^\omega \to \mathbb{R}$ be defined by $j_a(A) = \sum_{i \in A_{(a)}} 2^{-(i+1)}$. If a is a limit notation, let $j^a : 2^\omega \to \mathbb{R}$ be defined by $j^a(A) = \sum_{i \in A^{(a)}} 2^{-(i+1)}$.*

Therefore, if f is Baire α, then $f \leq_{\mathbf{T}} j_a$ for all a with $|a| = \alpha$. The following properties are clear.

Proposition 11. *For any notations $a, b \in \mathcal{O}$,*

(1) $j_a \leq_{\mathbf{T}} j_b$ if and only if $|a| \leq |b|$.
(2) If a and b are limits with $|a| = |b|$, then $j^a \equiv_{\mathbf{T}} j^b$.
(3) If a is a limit, $j^a <_{\mathbf{T}} j_a$.

Proof. All parts of the proposition which claim that a reduction exists follow from the fact that $|a| \leq |b|$ implies $H_a^A \leq_T H_b^A$, uniformly in A. For the non-reductions, suppose for the sake of contradiction that $j_a \leq_{\mathbf{T}} j_b$ with $|a| > |b|$ or $j_a \leq_{\mathbf{T}} j^a$. Let $Z \in 2^\omega$ be an oracle strong enough to compute the continuous functions $\langle k_i \rangle$ and h used in the reduction. Then $H_{2^a}^Z \leq_T H_{2^b}^Z$ or $H_{2^a}^Z \leq_T H_a^Z$, which are not possible. □

The previous proposition justifies the use of notation j_α to refer to j_a for some unspecified $a \in \mathcal{O}$ with $|a| = \alpha$. By relativization, we can go further up the ordinals.

Definition 12. *For any $Z \in 2^\omega$ any any $a \in \mathcal{O}^Z$, define*

$$j_a^Z(A) = \sum_{i \in (A \oplus Z)_{(a)}} 2^{-(i+1)},$$

and similarly for $j^{a,Z}$.

Proposition 11 can then be generalized to replace j_a and j_b with j_a^Z and j_b^W, under the assumption that $a, b \in \mathcal{O}^Z \cap \mathcal{O}^W$. We leave both the statement and proof of this generalization to the reader, but for example, part (1) follows from the fact that $H_a^{(A \oplus Z)} \leq_T H_b^{(A \oplus Z) \oplus W}$ uniformly in A; in the generalization the forward reduction is the continuous map $A \mapsto A \oplus Z$, rather than the identity map as it was in the original. Therefore, for any $\alpha < \omega_1$, we may use j_α to refer to j_a^Z for some pair Z, a with $Z \in 2^\omega$ and $a \in \mathcal{O}^Z$ with $|a|_{\mathcal{O}}^Z = \alpha$, and it does not matter which such Z, a we use because they are all in the same $\leq_{\mathbf{T}}$ equivalence class.[6] Similar remarks apply to the expression j^α.

We conclude by showing that every Baire function in $\mathcal{F}(2^\omega, \mathbb{R})$ is topologically Turing equivalent to one of the j_α or j^α. To reduce the notational clutter, we prove the version where α is constructive, and leave the relativization to the reader.

Proposition 13. *Let α be a constructive ordinal. If a Baire function f is not Baire α, then $j_{\alpha+1} \leq_{\mathbf{T}} f$. If α is a limit and f is not Baire β for any $\beta < \alpha$, then either $f \equiv_{\mathbf{T}} j^\alpha$, or $j_\alpha \leq_{\mathbf{T}} f$.*

Proof. Since f is not Baire α, there is an open set $U \subseteq \mathbb{R}$ such that $f^{-1}(U)$ is not $\mathbf{\Sigma}^0_{\alpha+1}$. Since f is Baire, $f^{-1}(U)$ is Borel, so by Wadge determinacy $C_{\alpha+1} \leq_w f^{-1}(U)$, where $C_{\alpha+1}$ is a canonical complete $\mathbf{\Pi}^0_{\alpha+1}$ subset of 2^ω, and $\leq_w$ is Wadge reducibility. Let v be a continuous function such that for all Z,

$$Z \in C_{\alpha+1} \iff v(Z) \in f^{-1}(U).$$

We now show how to reduce $j_{\alpha+1}$ to f. It suffices to be able to compute each bit of $A_{(\alpha+1)}$ on input A. Given A and i, uniformly compute Z such that $i \notin A_{(\alpha+1)}$ if and only if $Z \in C_{\alpha+1}$. Expressing $i \in A_{(\alpha+1)}$ as the statement $\exists k[u(i,k) \in A_{(\alpha)}]$ for some computable u, compute also a sequence Z_k such that $u(i,k) \in A_{(\alpha)}$ if and only if $Z_k \in C_{\alpha+1}$. Then asking for the values of $f(v(Z))$ and $f(v(Z_k))$, wait until you see one of these enter U. This proves the first part.

Now suppose that α is a limit, $\alpha = \lim_n \alpha_n$. If f is not Baire β for any $\beta < \alpha$, then $f \geq_{\mathbf{T}} j_{\alpha_n}$ for each n. From this it is clear that $f \geq_{\mathbf{T}} j^\alpha$. Suppose that there is an open set U such that $f^{-1}(U)$ is not $\mathbf{\Sigma}^0_\alpha$. Then by the same argument as above, $f \geq_{\mathbf{T}} j_\alpha$. On the other hand, if $f^{-1}(U)$ is Σ^0_α for each open U, then $j^\alpha \geq_{\mathbf{T}} f$ as follows. Let W be an oracle such that $\{(A,p,q) : f(A) \in (p,q)\}$ is $\Sigma^0_\alpha(W)$. Given access to the oracle $j^\alpha(A \oplus W)$, we can enumerate $\{(p,q) : f(A) \in (p,q)\}$. This suffices to compute $f(A)$. ☐

So that is the complete picture for $\leq_{\mathbf{T}}$. The particularly strong way in which each Baire α function is reducible to j_α is in fact a continuous Weihrauch reduction. However, the reduction of Proposition 13 is not a continuous Weihrauch reduction since we query different values of f for each bit of $A_{(\alpha+1)}$. So the parallelization is certainly used.

[6]Given a, Z and b, W with $|a|^Z_{\mathcal{O}} = |b|^W_{\mathcal{O}}$ but $a \notin \mathcal{O}^W$ or $b \notin \mathcal{O}^Z$, first fix $a' \in \mathcal{O}^Z \cap \mathcal{O}^W$ with $|a'|^Z_{\mathcal{O}} = |a'|^W_{\mathcal{O}} = |b|^W_{\mathcal{O}}$, then observe $j^Z_a \leq_{\mathbf{T}} j^Z_{a'} \leq_{\mathbf{T}} j^W_{a'} \leq_{\mathbf{T}} j^W_b$.

Kihara has subsequently obtained a further characterization of the degree structure of $\leq_{\mathbf{T}}$ in terms of Martin reducibility on uniformly Turing order preserving operators; see [Kih].

5. Definition of topological *tt*- and *m*-reducibilities

The classical notions of *tt*- and *m*-reducibility on infinite binary sequences operate by restricting the number of bits of the oracle used and the manner in which they are used. In the case of a *tt*-reduction, in order to get the nth bit of the output, one specifies in advance, using only the number n, finitely many bits of the oracle that will be queried. For each possible way the oracle could respond, one commits to an output for the nth bit. Only then is the oracle queried and the commitment carried out. The m-reducibility is even more restrictive. In order to get the nth bit of the output, one specifies in advance a single bit of the oracle to query, and commits to copy the whatever the oracle has there as the nth bit.

As explained in the introduction, we have adopted the convention that one bit of information about f is an ε-approximate answer to the question "Is $f(A)$ greater or less than p?" where $A \in 2^\omega$ and $p \in \mathbb{Q}$.

Given $A \in 2^\omega$, $p \in \mathbb{Q}$, and $\varepsilon \in \mathbb{Q}^+$, we define the question

$$\text{“}f(A) \lesssim_\varepsilon p\text{”?}$$

so that "yes" or "1" is a correct answer if $f(A) < p + \varepsilon$ and "no" or "0" is a correct answer if $f(A) > p - \varepsilon$. Observe that either answer is considered correct if $f(A)$ is within ε of p.

We then define a representation of $\mathbb{R}$ whose domain is a subset of 2^ω, where each bit of a name for $y \in \mathbb{R}$ corresponds to a correct answer to a question of the form $y \lesssim_\varepsilon p$.

Definition 14. *We say $A \in 2^\omega$ is a* separation name *for $y \in \mathbb{R}$ if for every $p \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+$, we have $A(\langle p, \varepsilon\rangle)$ correctly answers $y \lesssim_\varepsilon p$.*

One can verify that the function mapping separation names to reals is an admissible representation. Now if we take the definition of $\leq_{\mathbf{T}}$ from the previous section, use the above representation for real numbers, and further specify that h be either an m-reduction or a *tt*-reduction respectively, we obtain the following topological definitions of $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$.

Definition 15. *We say $f \leq_{\mathbf{m}} g$ if and only if for every pair of rationals p, ε, there are rationals q, δ and a continuous function $k : 2^\omega \to 2^\omega$ such that whenever b is a correct answer to $g(h(A)) \lesssim_\delta q$, b is also a correct answer to $f(A) \lesssim_\varepsilon p$.*

Definition 16. *We say $f \leq_{\mathbf{tt}} g$ if and only if for every pair of rationals p, ε, there are*

- *finitely many rationals $(q_i, \delta_i)_{i<r}$*
- *continuous functions $k_i : 2^\omega \to 2^\omega$, and*
- *a truth table function $h : 2^r \to \{0,1\}$*

such that whenever $\sigma \in 2^r$ is a string where each $\sigma(i)$ correctly answers

$$g(k_i(A)) \lesssim_{\delta_i} q_i,$$

then $h(\sigma)$ correctly answers $f(A) \lesssim_\varepsilon p$.

It is clear that the reducibilities $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ are reflexive and transitive, and that

$$f \leq_{\mathbf{m}} g \implies f \leq_{\mathbf{tt}} g \implies f \leq_{\mathbf{T}} g.$$

Exactly as in Proposition 8, these reductions may be more generally applied to functions whose domain is any compact separable metrizable space, using admissible representations.

Finally, all these reductions are primarily suitable for comparing discontinuous functions.

Proposition 17. *If f is continuous and g is non-constant, then $f \leq_{\mathbf{m}} g$.*

Proof. Since g is non-constant, let $B_0, B_1 \in 2^\omega$ be such that $g(B_0) < g(B_1)$. Given p, ε, let k be a continuous function which is equal to B_0 on $f^{-1}((-\infty, p-\varepsilon])$ and equal to B_1 on $f^{-1}([p+\varepsilon, \infty))$. Since f is continuous, these sets are closed, so such a k exists. Let q, δ be such that $g(B_0) < q - \delta < q + \delta < g(B_1)$. Then q, δ, k satisfies the part of the $\mathbf{m}$-reduction associated to p, ε. $\square$

5.1. **Equivalent definitions.** After hearing these results, the following equivalent definitions for $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$ reducibilities were observed by Arno Pauly and Takayuki Kihara, respectively.

First some standard notation. If $g :\subseteq X \rightrightarrows Y$ is a Weihrauch problem, g^n is defined as the problem $g \times g : X^n \rightrightarrows Y^n$ where $(y_0, \dots y_{n-1}) \in g^n(x_0, \dots, x_{n-1})$ if and only if $g(x_i) = g(y_i)$ for all $i < n$. Then g^* is defined as $g^* :\subseteq \cup_n X^n \rightrightarrows \cup_n Y^n$ where $\bar{y} \in g^*(\bar{x})$ if $\bar{x}$ and $\bar{y}$ are the same length n and $\bar{y} \in g^n(\bar{x})$.

For any function $f : 2^\omega \to \mathbb{R}$, let $S_f : \omega^\omega \rightrightarrows \{0,1\}$ be defined by

$$b \in S_f((p,\varepsilon)^\frown A) \iff b \text{ correctly answers } f(A) \lesssim_\varepsilon p.$$

Proposition 18 (Pauly)**.** *For $f, g : 2^\omega \to \mathbb{R}$, $f \leq_{\mathbf{tt}} g$ if and only if $S_f \leq^c_{sW} S_g^*$.*

Proof. If g is constant, then each reducibility holds if and only if f is constant as well. So assume that $B_0, B_1 \in 2^\omega$ and $q, \delta \in \mathbb{Q}$ are inputs for which $g(B_0) < q - \varepsilon < q + \varepsilon < g(B_1)$.

If $f \leq_{\mathbf{tt}} g$, then for each p, ε, let $(q_i, \delta_i, k_i)_{i<r}$ and h be witness to this. For each p, ε, let r' be the number of bits sufficient to describe h according to some canonical self-delimiting coding. Then define a strong Weihrauch reduction from S_f to S_g^* as follows:

- Given $(p, \varepsilon)^\frown A$, determine r, r' from (p, ε) and set up a query to $S_g^{r'+r}$.
- Use r'-many queries to ask about $(q, \delta)^\frown B_0$ and $(q, \delta)^\frown B_1$ in a sequence which encodes h.
- Ask about $(q_i, \delta_i)^\frown k_i(A)$ for each $i < r$.
- Given the sequence of answers to these $r' + r$-many questions, read off h from the first r' bits and apply it to the remaining r bits.

The other direction uses the compactness of 2^ω. Suppose that $S_f \leq_{sW}^c S_g^*$ via K and H. Fix p and ε. By compactness, there are finitely many strings $(\sigma_i)_{i<\ell}$ and for each i there are finitely many rationals $(q_{ij}, \delta_{ij})_{j<r_i}$ such that the cylinders $[\sigma_i]$ cover 2^ω, and for each $A \in 2^\omega$, if $\sigma_i \prec A$, then $K(A)$ has length r_i, and its jth coordinate begins with (q_{ij}, δ_{ij}).

Let K_j be the function which computes the Cantor space part of the jth coordinate of K, when that coordinate exists. That is, K_j is defined by

$$K((p, \varepsilon)^\frown \sigma_i C)(j) = (q_{ij}, \delta_{ij})^\frown K_j(\sigma_i C).$$

Let $(k_{ij})_{j<r_i}$ be functions do the following:

$$k_{ij}(A) = \begin{cases} B_0 & \text{if } \sigma_i \not\prec A \\ K_j((p, \varepsilon)^\frown A) & \text{if } \sigma_i \prec A. \end{cases}$$

Define also $k'_i(A) = B_j$ where $j = 1$ if $\sigma_i \prec A$ and 0 otherwise, and let (q'_i, δ'_i) be all equal to (q, δ). Let r be the total number of k_{ij} and k'_i functions defined above. Let $h : 2^r \to \{0, 1\}$ be the truth table which uses the k', q, δ answers to determine which $\sigma_i \prec A$, then uses the $k_{ij}, q_{ij}, \delta_{ij}$ answers to simulate the reverse reduction H. □

Kihara has also observed an equivalent definition of $\leq_{\mathbf{m}}$ related to partial order valued Wadge reducibility. His definition and analysis also suggested a close variant of $\leq_{\mathbf{m}}$ whose theory may be even more natural than the one defined here. We refer the reader to [Kih] for details.

6. Properties of $\leq_{\mathbf{m}}$

In this section we prove our first main result concerning the $\leq_{\mathbf{m}}$ degrees of the jump functions j_α within the Baire α functions. We start with some easier facts about the structure of the $\leq_{\mathbf{m}}$ degrees. The proof of the following proposition is due to Kihara.

Proposition 19. *For all Baire functions f and g, we have either $f \leq_m g$ or $g \leq_m -f$.*

Proof. We can understand the statement $f \leq_m g$ as saying that Player II has a winning strategy in the following game. Player I plays a target bit $\langle p, \varepsilon \rangle$. Player II plays its intended oracle bit $\langle q, \delta \rangle$. Player I then starts playing bits of the input A; Player II also plays bits of a sequence B in response, but Player II can pass (however they must ultimately produce an infinite sequence in order to win.) Player II wins if any correct answer to $g(B) \lessapprox_\delta q$ is also a correct answer to $f(A) \lessapprox_\varepsilon p$. If Player II has a winning strategy, then q, δ and the continuous function k defined by $k(A) = B$ are as in the definition of $\leq_{\mathbf{m}}$. But if Player I has a winning strategy, then for any q, δ, there are p, ε (in fact, the same p and ε each time, chosen according to the winning strategy of Player I) and a continuous function k' which, following the winning strategy of Player I against Player II playing an arbitrary B, outputs $A = k'(B)$ such that either $g(B) < q + \delta$ and $f(A) \geq p + \varepsilon$ or $g(B) > q - \delta$ and $f(A) \leq p - \varepsilon$. Therefore, if $-f(A) < -p + \varepsilon$, we must be in the first case and thus $g(B) < q + \delta$. Similarly, if $-f(A) > -p - \varepsilon$ then we must be in the second case, so $g(B) > q - \delta$. This shows that $g \leq_m -f$ via k' (observe that $(-p, \varepsilon)$ is the bit of $f(A)$ actually queried). $\square$

Corollary 20. *If $f \in \mathcal{B}_\alpha$, then $f \leq_{\mathbf{m}} j_{\alpha+1}$.*

Proof. If not, then by Proposition 19 we would have $j_{\alpha+1} \leq_{\mathbf{m}} -f \leq_{\mathbf{T}} j_\alpha$, impossible as $j_{\alpha+1}$ is not $\mathcal{B}_\alpha$. $\square$

Our first theorem shows that the jump functions are the weakest functions in each Baire class.

Theorem 21. *If f is Borel and $f \notin \mathcal{B}_\alpha$, then either $j_{\alpha+1} \leq_{\mathbf{m}} f$ or $-j_{\alpha+1} \leq_{\mathbf{m}} f$.*

It is easy to see why this theorem is true when $\alpha = 0$. If f is not continuous, let $(z_n)_{n\in\omega} \to z$ be a convergent sequence of inputs for which $f(z) \neq \lim_n f(z_n)$. Without loss of generality, there is some $\delta > 0$ such that for all n, $f(z_n) > f(z) + \delta$, or for all n, $f(z_n) < f(z) - \delta$. In the first case, we have that $j_1 \leq_{\mathbf{m}} f$ via the following algorithm. On input (p, ε), choose (q, δ') so that $[q - \delta', q + \delta'] \subseteq (f(z), f(z) + \delta)$.

Then let h be the function which, on input x, outputs bits of z while computing approximations to $j_1(x)$. If it ever sees that $j_1(x) > p - \varepsilon$, it switches to outputting bits of z_n for some n large enough that z and z_n agree on all bits which were already committed to. The case where $f(z_n) < f(z) - \delta$ for all n is similar, only in that case we find that $-j_1 \leq_{\mathbf{m}} f$.

To prove this theorem in the general case we will make use of the following generalization of Borel Wadge determinacy. We provide a simple proof of this generalization using Borel determinacy, but it is interesting to note that Louveau and Saint-Raymond [LSR87, LSR88], showed that this generalization is provable in second order arithmetic via a much more intricate argument. Therefore, the use of Borel determinacy here can be avoided.

Proposition 22. *Let $D, E_0, E_1 \subseteq \omega^\omega$ be Borel. Then one of the following holds:*

(1) There is a continuous function $\varphi : \omega^\omega \to \omega^\omega$ such that $\varphi(D) \subseteq E_0$ and $\varphi(\omega^\omega \setminus D) \subseteq E_1$.
(2) There is a continuous function $\psi : \omega^\omega \to \omega^\omega$ such that $\psi(E_0) \subseteq \omega^\omega \setminus D$ and $\psi(E_1) \subseteq D$.

Proof. Define a two player game, where at turn n player I (who plays first) plays $x(n)$ and player II plays $y(n)$. At the end of the game, II wins if

$$(x \in D \wedge y \in E_0) \vee (x \notin D \wedge y \in E_1).$$

By Borel determinacy, one of the two players has a winning strategy. A winning strategy for II gives a continuous function meeting outcome (1).

If on the other hand I has a winning strategy, then for every play of the game according to I's winning strategy we have that

$$(x \notin D \vee y \notin E_0) \wedge (x \in D \vee y \notin E_1).$$

This gives a continuous function meeting outcome (2). □

We give a new corollary to this theorem.

Corollary 23. *Let $V \subseteq \omega^\omega$ be $\mathbf{\Pi}^0_\alpha$. Let $W \subseteq \omega^\omega$ be $\mathbf{\Pi}^0_\alpha$-hard and let $\{W_i\}_{i \in \mathbb{N}}$ be a partition of W into Borel sets. Then there is a continuous function $\varphi\colon \omega^\omega \to \omega^\omega$ and $i \in \mathbb{N}$ such that:*

(1) $\varphi(V) \subseteq W_i$.
(2) $\varphi(\omega^\omega \setminus V) \subseteq \omega^\omega \setminus W$.

Proof. For each i, we can apply Theorem 22 with $D = V$, $E_0 = W_i$ and $E_1 = \omega^\omega \setminus W$. Assume that for each i, the second option of the theorem holds, i.e. there is a continuous function ψ_i such that

$$\psi_i(W_i) \subseteq \omega^\omega \setminus V \text{ and } \psi_i(\omega^\omega \setminus W) \subseteq V.$$

Now take $K_i = \psi_i^{-1}(\omega^\omega \setminus V)$. Note that K_i is $\mathbf{\Sigma}^0_\alpha$ and we have that $W = \bigcup_i W_i \subseteq \bigcup_i K_i$. Further, for all i we know that $K_i \cap (\omega^\omega \setminus W) = \emptyset$. Hence $W = \bigcup_i K_i$ and so W is $\mathbf{\Sigma}^0_\alpha$. This is a contradiction as we are given that W is $\mathbf{\Pi}^0_\alpha$-hard.

Hence for some i we have that the first option of Theorem 22 holds. That is, there is a continuous φ such that $\varphi(V) \subseteq W_i$ and $\varphi(\omega^\omega \setminus V) \subseteq \omega^\omega \setminus W$. □

Proof of Theorem 21. Since f is not Baire α, let $U \subseteq \mathbb{R}$ be an open set such that $f^{-1}(U)$ is not $\mathbf{\Sigma}^0_{\alpha+1}$. Without loss of generality, U is of the form $(u, +\infty)$ or $(-\infty, u)$. If U is of the form $(-\infty, u)$, then we will have $j_{\alpha+1} \leq_{\mathbf{m}} f$, and in the other case $j_{\alpha+1} \leq_{\mathbf{m}} -f$ (or equivalently, $-j_{\alpha+1} \leq_{\mathbf{m}} f$). Replacing f with $-f$ if necessary let us assume $U = (-\infty, u)$.

Denote $f^{-1}(U)$ by W. Since f is Borel, W is Wadge determined, so it is $\mathbf{\Pi}^0_{\alpha+1}$-hard. We can partition W into the following sets $W_0 = f^{-1}((-\infty, u-1])$ and for all $i \geq 1$,

$$W_i = f^{-1}\left(\left(u - \frac{1}{i}, u - \frac{1}{i+1}\right]\right).$$

Take any $p, \epsilon \in \mathbb{Q}$ with $\epsilon > 0$. Let $V = j_{\alpha+1}^{-1}((-\infty, p-\epsilon])$. The set V is $\Pi^0_{\alpha+1}$. (We have $A \in V$ if and only if for all finite $F \subseteq \omega$ such that $\sum_{i \in F} 2^{-(i+1)} > p - \epsilon$, there is some $i \in F$ such that $i \notin A_{(\alpha+1)}$. Recall from the introduction that $\{A : i \in A_{(\eta)}\}$ is a Σ^0_η set.)

Thus by Corollary 23 there is a continuous map φ and an $i \in \mathbb{N}$ such that $\varphi(V) \subseteq W_i$ and $\varphi(2^\omega \setminus V) \subseteq f^{-1}([u, +\infty))$. Hence taking $\delta = \frac{1}{2(i+1)}$ and $q = u - \delta$ we have that for any $A \in 2^\omega$, there is only one correct answer to $f(\varphi(A)) \lesssim_\delta q$. Further, this is also a correct answer to $j_{\alpha+1}(A) \lesssim_\varepsilon p$. □

Corollary 24. *If g is Baire, $g \notin \mathcal{B}_\alpha$ and $f \in \mathcal{B}_\alpha$, then $f \leq_{\mathbf{m}} g$.*

7. The Bourgain rank on $\mathcal{B}_1$

The structure of the $\leq_{\mathbf{m}}$-degrees and $\leq_{\mathbf{tt}}$-degrees within the Baire 1 functions is related to the α rank, also known as the Bourgain rank, which was studied by Kechris and Louveau [KL90]. Here we place that rank in a slightly more general setting that will be suitable for describing both the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees, and establish some notation that will be used throughout.

Definition 25. *For any collection $\mathcal{C} \subseteq \mathcal{P}(X)$, a* derivation sequence *for $\mathcal{C}$ is defined for $\nu < \omega_1$ by*

- $P^0 = X$.
- $P^{\nu+1} \supseteq P^\nu \setminus \cup\{U \text{ open } : \text{ for some } C \in \mathcal{C}, P^\nu \cap U \subseteq C\}$
- $P^\lambda \supseteq \cap_{\nu<\lambda} P^\nu$.

By replacing $\supseteq$ with $=$ in two places, we obtain the definition for the optimal derivation sequence *for $\mathcal{C}$.*

Here are some properties of derivation sequences which will be useful and which follow directly from the definitions.

Proposition 26. *Let Q^ν be a derivation sequence for $\mathcal{C} \subseteq \mathcal{P}(X)$.*

(1) If P^ν is the optimal derivation sequence for $\mathcal{C}$, then $P^\nu \subseteq Q^\nu$ for all ν.
(2) If $k : X \to X$ is continuous, then $R^\nu := k^{-1}(Q^\nu)$ is a derivation sequence for $\{k^{-1}(C) : C \in \mathcal{C}\}$.
(3) If $\mathcal{D} \subseteq \mathcal{P}(X)$ is such that for every $C \in \mathcal{C}$, there is a $D \in \mathcal{D}$ such that $C \subseteq D$, then Q^ν is a derivation sequence for $\mathcal{D}$.

Definition 27 (Bourgain rank, also known as α rank)**.** *For $f \in \mathcal{B}_1$ and rationals p, ε, let $P^\nu_{f,p,\varepsilon}$ be the optimal derivation sequence for $\{f^{-1}((-\infty, p+\varepsilon)), f^{-1}((p-\varepsilon, \infty))\}$. Let $\alpha(f, p, \varepsilon)$ be least ordinal ν such that $P^\nu_{f,p,\varepsilon} = \emptyset$. Let the Bourgain rank of f be*

$$|f|_\alpha = \sup_{p,\varepsilon\in\mathbb{Q}} \alpha(f, p, \varepsilon).$$

If f, p, ε are clear from context, we may write P^ν or P^ν_f instead of $P^\nu_{f,p,\varepsilon}$. Observe that the compactness of X implies that $\alpha(f, p, \varepsilon)$ is always a successor, but in general $|f|_\alpha$ may be either a limit or a successor.

In the course of the optimal derivation process, individual points leave at various stages, and we would like to keep track of this.

Definition 28. *Let $A \in X$. If P^ν is the optimal derivation sequence for sets $\mathcal{C}$ and P^ν is eventually empty, let $|A|_\mathcal{C}$ denote the least ν such that $A \notin P^\nu$. Given $f \in \mathcal{B}_1$, and p, ε, let $|A|_{f,p,\varepsilon}$ be the least ν such that $A \notin P^\nu_{f,p,\varepsilon}$.*

If f, p, ε and/or $\mathcal{C}$ are clear from context, we may just write $|A|_f$ or $|A|$. Observe that $|A|$ is always a successor ordinal.

The Bourgain hierarchy can be understood as a higher type version of the Ershov hierarchy. Recall the Ershov hierarchy stratifies the Δ^0_2 subsets of ω according to the amount of mind-changes needed in an optimal limit approximation to that set. In general, ordinal-many mind-changes can be needed. For $a \in \mathcal{O}$, a function $A : \omega \to \omega$ is a-computably approximable if there is a partial computable $\varphi(n, b)$ such that $A(n) = \varphi(n, b_n)$, where b_n is the $\leq_{\mathcal{O}}$-least ordinal $b_n \leq_{\mathcal{O}} a$ for which the computation converges. We picture this process dynamically – a computable procedure makes a guess about $A(n)$ associated to a certain ordinal. If it changes its guess, it must decrease the ordinal. This limits the number of mind-changes.

We can understand each open set removed as a part of the Bourgain derivation process as a guess about the answer to the question $f(x) \lesssim_\varepsilon p$. The open sets removed later in the derivation process, have a high associated ordinal rank and correspond to early guesses; the open sets removed at the beginning of the derivation process correspond to the latest guesses. The following object, a mind-change sequence, is nothing more than a derivation sequence annotated with the guesses that justified the derivation. It can also be viewed as a higher-type analog of φ as above. To simplify the notation, we assume $\mathcal{C} = \{C_i : i < k\}$, where k could be finite or ω, and $X = 2^\omega$. Let Ord denote the ordinals.

Definition 29. *Given $\mathcal{C} = \{C_i : i < k\} \subseteq \mathcal{P}(2^\omega)$, a* mind-change sequence *for $\mathcal{C}$ is a countable subset of $M \subseteq \mathrm{Ord} \times 2^{<\omega} \times k$ for which*

(1) The sequence Q^ν defined by

$$Q^\nu = 2^\omega \setminus \left(\bigcup_{\substack{(\mu,\tau,j)\in M \\ \mu<\nu}} [\tau] \right)$$

is a derivation sequence for $\mathcal{C}$, and

(2) For all $(\nu, \sigma, i) \in M$, $[\sigma] \cap Q^\nu \subseteq C_i$.

An optimal mind-change sequence *for $\mathcal{C}$ is one in which Q^ν is the optimal derivation sequence for $\mathcal{C}$.*

Observe that an optimal mind-change sequence always exists, since it just keeps track of the open sets $[\sigma]$ which are removed at stage ν of the construction of the optimal derivation sequence, and keeps track of which set $C \in \mathcal{C}$ caused $[\sigma]$ to be removed at stage ν.

Two "mind-change" based encodings of the Baire 1 functions are suggested by this idea. One encoding of $f \in \mathcal{B}_1$, following the α rank, would consist of a countable collection of mind-change sequences $M_{p,\varepsilon}$, one for each

$$\mathcal{C}_{p,\varepsilon} = \{f^{-1}((p - \varepsilon, \infty)), f^{-1}((-\infty, p + \varepsilon))\}$$

for $p, \varepsilon \in \mathbb{Q}$. Another encoding, following the β rank, would consist of a different countable collection of mind-change sequences M_ε, one for each

$$\mathcal{C}_\varepsilon = \{f^{-1}((q-\varepsilon, q+\varepsilon)) : q \in \mathbb{Q}\}$$

for each $\varepsilon \in \mathbb{Q}^+$. We will not need to use such encodings explicitly, so we avoid further technical definitions, but this way of thinking about a Baire 1 function motivates all the arguments which follow.

A mind-change sequence can serve as evidence of an upper bound on the length of an optimal derivation sequence for a collection $\mathcal{C}$. The next notion provides evidence of a lower bound. The idea is that if $[\sigma]$ was not removed at stage ν of the derivation process, then for each $C \in \mathcal{C}$, there was some element $A_{\nu,\sigma,C}$ which witnesses that $P^\nu \cap [\sigma] \not\subseteq C$. If $\mathcal{C}$ is countable and the derivation process lasts only countably many stages, then only countably many A are needed to witness the necessity of an optimal derivation sequence being as long as it is. Below, we define a *scaffolding sequence* to be any countable collection of A's which can supply all necessary witnesses, together with a record of where in the process these A are slowing things down.

Definition 30. *Given $\mathcal{C} = \{C_i : i < k\} \subseteq \mathcal{P}(2^\omega)$, let P^ν be its optimal derivation sequence. A* scaffolding sequence *for $\mathcal{C}$ is any enumeration of a countable subset $S \subseteq 2^\omega \times \mathrm{Ord} \times 2^{<\omega} \times k$ such that*

(1) If $(A, \nu, \sigma, i) \in S$, then $A \in P^\nu \cap [\sigma] \setminus C_i$, and
(2) If $P^\nu \cap [\sigma] \not\subseteq C_i$, there is $A \in 2^\omega$ with $(A, \nu, \sigma, i) \in S$.

Letting S' be the projection of S onto its first coordinate, observe that for all $\mu < \nu$ and σ, if $P^\nu \cap [\sigma] \neq \emptyset$, then $P^\mu \cap [\sigma] \cap S' \neq \emptyset$.

8. Characterization of the $\leq_{\mathbf{m}}$ equivalence classes in $\mathcal{B}_1$

In this section we prove parts (2)-(4) of Theorem 2, characterizing the structure of the $\leq_{\mathbf{m}}$ degrees within the Baire 1 functions.

We will need to consider the case when $|f|_\alpha$ is a successor with special care. Supposing we have such an f, let ν, p, ε be defined so that $\nu + 1 = \alpha(f, p, \varepsilon) = |f|_\alpha$. Of course, we may also have $\nu + 1 = \alpha(f, p', \varepsilon')$ for some other rationals p', ε'.

Definition 31. *Given $f \in \mathcal{B}_1$ with $|f|_\alpha = \nu + 1$, and $p, \varepsilon \in \mathbb{Q}$, say (p, ε) is* maximal *if $f(P^\nu_{f,p,\varepsilon}) \setminus (p-\varepsilon, p+\varepsilon) \neq \emptyset$ and $\alpha(f, p, \varepsilon) = \nu + 1$.*

Observe that maximal (p, ε) always exist. If $P^\nu_{f,p,\varepsilon} \neq \emptyset$, but $f(P^\nu_{f,p,\varepsilon}) \setminus (p-\varepsilon, p+\varepsilon) = \emptyset$, then by decreasing ε, one may shrink $(p-\varepsilon, p+\varepsilon)$ to include an element of $f(P^\nu_{f,p,\varepsilon})$ (which grows in size).

Definition 32. *Let $f \in \mathcal{B}_1$ with $|f|_\alpha = \nu + 1$. We say f is*

- two-sided *if there is a maximal* (p, ε) *such that* $f(P^\nu) \nsubseteq (p - \varepsilon, \infty)$ *and* $f(P^\nu) \nsubseteq (-\infty, p + \varepsilon)$*;*
- one-sided *otherwise;*
- left-sided *if for every maximal* (p, ε)*,* $f(P^\nu) \subseteq (-\infty, p + \varepsilon)$*;*
- right-sided *if for every maximal* (p, ε)*,* $f(P^\nu) \subseteq (p - \varepsilon, \infty)$.

For example j_1 is left-sided, as is any discontinuous lower semi-continuous function. If f is left-sided, then $-f$ is right-sided, and vice versa. However, there are one-sided f which are neither right-sided nor left-sided. For example, consider

$$f(X) = \begin{cases} 1 & \text{if } X \in [0] \setminus \{01^\omega\} \\ -1 & \text{if } X \in [1] \setminus \{10^\omega\} \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to prove the parts (2)-(4) of our second main theorem.

Theorem 33. *For* $f, g \in B_1$*,* $|f|_\alpha < |g|_\alpha$ *implies* $f \leq_{\mathbf{m}} g$*. If* $|f|_\alpha = |g|_\alpha$*, then* $f \leq_{\mathbf{m}} g$ *if and only if at least one of the following holds:*

(1) $|f|_\alpha$ *is a limit ordinal.*
(2) g *is two-sided.*
(3) f *is one-sided and* g *is neither right-sided nor left-sided.*
(4) f *and* g *are either both right-sided or both left-sided.*

Proof. We begin with a general observation. Suppose that $p, \varepsilon, q, \delta \in \mathbb{Q}$ and $k : 2^\omega \to 2^\omega$ is a continuous function such that any correct answer to $g(k(A)) \lesssim_\delta q$ is also a correct answer to $f(A) \lesssim_\varepsilon p$. Then for any A, $g(k(A)) < q + \delta$ implies $f(A) < p + \varepsilon$, so $k^{-1}(g^{-1}((-\infty, q + \delta)) \subseteq f^{-1}((-\infty, p + \varepsilon))$. Similarly, $k^{-1}(g^{-1}((q - \delta, \infty)) \subseteq f^{-1}((p - \varepsilon, \infty))$. Therefore, the sets Q^μ defined by

$$Q^\mu = k^{-1}(P^\mu_{g,q,\delta})$$

are a derivation sequence for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$. Therefore $P^\mu_{f,p,\varepsilon} \subseteq Q^\mu$ for each μ, so $\alpha(f, p, \varepsilon) \leq \alpha(g, q, \delta)$. Furthermore, for all $A \in 2^\omega$, we have $|A|_{f,p,\varepsilon} \leq |k(A)|_{g,q,\delta}$.

Now suppose $f \leq_{\mathbf{m}} g$. Then for any p, ε, there are q, δ and k as above, so $|f|_\alpha \leq |g|_\alpha$. The first statement of the theorem now follows by Proposition 19 and the observation that $|g|_\alpha = |-g|_\alpha$ for all g.

From now on we consider the case where $|f|_\alpha = |g|_\alpha$.

Suppose that $f \leq_{\mathbf{m}} g$. We claim that if $|f|_\alpha = \nu + 1$ (a successor) then one of (2)-(4) in the statement of the theorem holds.

Let (p, ε) be maximal for f. Since $f \leq_{\mathbf{m}} g$, let q, δ and k be as in the first paragraph. By the choice of p and ε, there is an $A \in 2^\omega$ with

$|A|_{f,p,\varepsilon} = |k(A)|_{g,q,\delta} = \nu + 1$ and $f(A) < p - \varepsilon$, or there is a $B \in 2^\omega$ with $|B|_{f,p,\varepsilon} = |k(B)|_{g,q,\delta} = \nu + 1$ and $f(B) > p + \varepsilon$, or perhaps both occur. If such A exists, then $g(k(A)) < q - \delta$ and if such B exists, then $g(k(B)) > q + \delta$.

Therefore, if g is not two-sided, then f is not two-sided; in that case, if g is right-sided or left-sided, then f must match. This completes the proof that $f \leq_{\mathbf{m}} g$ implies the disjunction of (1)-(4).

Assuming now the disjunction of (1)-(4), let p, ε be given. First we choose a pair q, δ which gives us enough room to work. If $|f|_\alpha = \nu + 1$, choose (q, δ) to be maximal for g. Additionally, if g is two-sided, make sure q and δ witness the two-sidedness of g. Or, if f is one-sided and g is neither left-sided nor right-sided, then if $f(P^\nu_{f,p,\varepsilon}) \subseteq (p - \varepsilon, \infty)$ (respectively $f(P^\nu_{f,p,\varepsilon}) \subseteq (-\infty, p + \varepsilon)$) make sure $g(P^\nu_{g,q,\delta}) \subseteq (q - \delta, \infty)$ (respectively $g(P^\nu_{g,q,\delta}) \subseteq (-\infty, q + \delta)$). If f and g are both right- or both left-sided, a maximal choice of q and δ suffices without further restrictions. If $|f|_\alpha$ is a limit, choose q, δ so that $\alpha(f, p, \varepsilon) < \alpha(g, q, \delta)$ and $g(P^\nu_{g,q,\delta}) \setminus (q - \delta, q + \delta) \neq \emptyset$ (decreasing δ if necessary to achieve the latter). In this case, define ν so that $\alpha(g, q, \delta) = \nu + 1$.

We now define a continuous function k such that any correct answer to $g(k(A)) \lesssim_\delta q$ also correctly answers $f(A) \lesssim_\varepsilon p$. Given A, its image $k(A)$ will be defined in stages according to an algorithm which uses oracle information about a mind-change sequence related to f and a scaffolding sequence related to g. By defining $k(A)$ in stages, we guarantee k is continuous.

Let $\mathcal{C} = \{C_0, C_1\}$, where $C_0 = f^{-1}((-\infty, p + \varepsilon))$ and $C_1 = f^{-1}((p - \varepsilon, \infty))$. Let $\mathcal{D} = \{D_0, D_1\}$, where $D_0 = g^{-1}((-\infty, q + \delta))$ and $D_1 = g^{-1}((q - \delta, \infty))$. Let Z be an oracle which contains the following information:

- A well-order W long enough that ν has a code in $\mathcal{O}^W$ (a technical point which allows us to use $\mathcal{O}^W$ in place of Ord in the mind-change and scaffolding sequences).
- An optimal mind-change sequence M for $\mathcal{C}$.
- A scaffolding sequence S for $\mathcal{D}$.

Letting A denote the input, at each stage s, we will have defined an initial segment τ_s of $k(A)$. We will be keeping track of an ordinal μ_s, an index $i_s \in \{0, 1\}$, and an element $B_s \in 2^\omega$, where $\tau_s \prec B_s$. We will always maintain the following:

(i) that $|A|_{\mathcal{C}} \leq \mu_s + 1 \leq |B_s|_{\mathcal{D}}$,
(ii) that i_s is the only correct answer to $g(B_s) \lesssim_\delta q$, and
(iii) if $|A|_{\mathcal{C}} = \mu_s + 1$, then i_s correctly answers $f(A) \lesssim_\varepsilon p$.

The idea always is that as long as it seems like $|A|_{\mathcal{C}} = \mu_s + 1$, we are working towards making $k(A) = B_s$. If we later see the bound on $|A|_{\mathcal{C}}$ drop, and i_s no longer looks like a suitable answer, then because $|B_s|_{\mathcal{D}}$ is large, no matter how much of B_s has been copied, we can switch to a nearby B_t for which $i_t = 1 - i_s$ is the only correct answer to $g(B_t) \lesssim_\delta q$, and $|B_t|_{\mathcal{D}}$ is still large.

Let λ denote the empty string. Let $\tau_0 = \lambda$. We begin differently depending on whether g is two-sided. In both of the following cases, the reader can verify that conditions (i)-(iii) are satisfied at stage $s = 0$.

If g is two-sided, we first wait until we see A leave $P^{\mu}_{f,p,\varepsilon}$ for some $\mu \leq \nu$. That is, we see (μ, σ, i) in M with $\sigma \prec A$. Let $\mu_0 = \mu$ and $i_0 = i$. Now, since g is two-sided, regardless of i, $P^{\nu}_{g,q,\delta} \setminus D_{1-i}$ is non-empty, and we can find an element B in this set (by looking in S for something of the form $(B, \nu, \lambda, 1-i)$). Let $B_0 = B$.

If g is not two-sided, then for some j, $P^{\nu}_{g,q,\delta} \setminus D_j$ is non-empty, so first we wait until we see an element B and a j to witness this (by looking in S for something of the form (B, ν, λ, j)). Let $\mu_0 = \nu$, $i_0 = 1 - j$, and $B_0 = B$. By the choice of q and δ, if $|A|_{\mathcal{C}} = \nu + 1$, then i_0 correctly answers $f(A) \lesssim_\varepsilon p$.[7]

At stage $s + 1$, set μ_{s+1} to be the least μ for which we have seen A leave $P^{\mu}_{f,p,\varepsilon}$. If $\mu_{s+1} < \mu_s$, that is because (μ_{s+1}, σ, i) just entered M for some $\sigma \prec A$. If $i = i_s$, let $B_{s+1} = B_s$ and $i_{s+1} = i_s$. But if $i \neq i_s$, then set $i_{s+1} = i$, and look through S to find a B so that

$$B \in P^{\mu_{s+1}}_{g,q,\delta} \cap [\tau_s] \setminus D_{i_s}.$$

Such a B must exist because B_s witnesses that $P^{\mu_s}_{g,q,\delta} \cap [\tau_s]$ is non-empty. Let $B_{s+1} = B$. Finally, let $\tau_{s+1} = B_{s+1} \upharpoonright |\tau_s| + 1$. That completes the construction.

At each stage the properties (i)-(iii) are maintained. Now if $|A|_{\mathcal{C}} = \mu + 1$, there is a stage s at which it is seen that A leaves $P^{\mu}_{f,p,\varepsilon}$. The μ_s, i_s and B_s defined at that stage never change again. Then $k(A) = B_s$, and the only correct answer to $g(k(A)) \lesssim_\delta q$ is i_s, which also correctly answers $f(A) \lesssim_\varepsilon p$, as desired. $\square$

The initial segment of the $\leq_{\mathbf{m}}$-degrees contains some naturally recognizable classes which are blurred together by the α rank. The lowest $\leq_{\mathbf{m}}$ degree consists of the constant functions; right above that is the

[7] In case (1), by the choice of ν, $P^{\nu}_{f,p,\varepsilon} = \emptyset$, so $|A|_{\mathcal{C}} < \nu + 1$. In case (3), f is one-sided, so $P^{\nu}_{f,p,\varepsilon} \subseteq C_i$ for some i. Note that in this case, we have chosen q, δ specifically to make sure that $j = 1 - i$. In case (4), we also have $P^{\nu}_{f,p,\varepsilon} \subseteq C_{1-j}$ (note that $j = 1$ if f and g are both left-sided and $j = 0$ if f and g are both right-sided).

degree of the continuous non-constant functions. Next above that are two incomparable $\leq_{\mathbf{m}}$-degrees: the upper semi-continuous functions and the lower semi-continuous functions.

Proposition 34. *Let g be a lower semi-continuous, discontinuous function (for example, $g = j_1$). The following are equivalent for $f \in \mathcal{B}_1$:*

(1) $f \leq_{\mathbf{m}} g$
(2) f is lower semi-continuous.
(3) For some e and some parameter Z, $f(A) = \ell(W_e^{A\oplus Z})$, where ℓ is the representation which maps separation names to real numbers.

Proof. ($1 \implies 2$). Given $a \in \mathbb{R}$, we wish to show that $f^{-1}((a,\infty))$ is open. Let $(p_i, \varepsilon_i)_{i<\omega}$ be an infinite sequence of rationals such that $a < p_i - \varepsilon_i$ and $\lim p_i = a$. Let q_i, δ_i and k_i witness the defining property of $f \leq_{\mathbf{m}} j_1$ for each i. Now suppose that $f(A) > a$. For some i, $f(A) > p_i + \varepsilon_i$. Then the only correct answer to $f(A) \lesssim_{\varepsilon_i} p_i$ is 1, so it must be that $g(k_i(A)) > q_i - \delta_i$. The set $C := \{B : g(k_i(B)) > q_i - \delta_i\}$ is open by the lower semi-continuity of g, and since 1 is a correct answer to $f(B) \lesssim_{\varepsilon_i} p_i$ for every $B \in C$, we have $C \subseteq f^{-1}((p_i - \varepsilon_i, \infty)) \subseteq f^{-1}((a,\infty))$.

($2 \implies 3$) Assume Z is an oracle which lists, for each p, the collection of rational balls contained in $f^{-1}((p,\infty))$. To define $W_e^{A\oplus Z}(\langle p,\varepsilon\rangle)$, wait to see if A enters $f^{-1}((p-\varepsilon,\infty))$. If it does, enumerate the bit. The result is a separation name of $f(A)$ which has the additional property that it always answers 1 when 1 is a permissible answer.

($3 \implies 2$) If $f(A) = \ell(W_e^{A\oplus Z})$, then $f(A) > a$ if and only if for some p, ε, $a < p - \varepsilon$ and $\langle p,\varepsilon\rangle \in W_e^{A\oplus Z}$, which is an open condition.

($2 \implies 1$) This follows from Theorem 33 because g has rank 2 and is left-sided, and f is either discontinuous and shares these properties, or f is continuous, in which case $f \leq_{\mathbf{m}} g$ by Proposition 17. $\square$

The authors observed to Kihara that if the lattice structure of the Baire 1 $\leq_{\mathbf{m}}$-degrees would continue to higher Baire classes in the same pattern described in Theorem 33, the $\leq_{\mathbf{m}}$ reducibility could be used to extend the definition of the α rank into higher Baire classes. After seeing these results, Kihara used a different method to fully describe the structure of the $\leq_{\mathbf{m}}$-degrees beyond the Baire 1 functions [Kih], and confirmed that the pattern does continue.

Separately and independently of this, Elekes, Kiss and Vidnyansky defined a generalization of the α, β and γ ranks into the higher Baire classes [EKV16]. Interestingly, they were able to apply their extension of the β rank to solve a problem in cardinal characteristics, but an extension of the α rank was not suitable for that problem. It does not seem easy to modify our work to get a generalization of the β rank. We leave a more detailed discussion of the relation between the various generalizations to future work.

9. A reducibility between $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$

There is a reducibility notion which captures the α rank precisely. Consider a truth table reduction $f \leq_{\mathbf{tt}} g$ which looks at only one bit of g, but may use finitely many bits of A.

Definition 35. *We say $f \leq_{\mathbf{tt1}} g$ if for all rationals p, ε, there is a continuous $k : 2^\omega \to 2^\omega$, rationals q, δ, a number r, and a truth table $h : 2^{r+1} \to \{0,1\}$ such that for every $A \in 2^\omega$, if b is a correct answer to $g(k(A)) \lesssim_\delta q$, then $h(A \upharpoonright r, b)$ is a correct answer to $f(A) \lesssim_\varepsilon p$.*

Proposition 36. *The relation $f \leq_{\mathbf{tt1}} g$ is transitive.*

Proof. Suppose $f_1 \leq_{\mathbf{tt1}} f_2$ and $f_2 \leq_{\mathbf{tt1}} f_3$. Given p, ε, let δ, q, k, r and h be as guaranteed by the fact that $f_1 \leq_{\mathbf{tt1}} f_2$. Given $p' = q, \varepsilon' = \delta$, let k', q', δ', r' and h' be as guaranteed by the fact that $f_2 \leq_{\mathbf{tt1}} f_3$. Let $r'' > r'$ be also large enough that r'' bits of any input A are enough to compute r' bits of $k(A)$ (using compactness). Define

$$h''(\tau, b) = h(k(\tau) \upharpoonright r', h'(\tau \upharpoonright r'', b))$$

Then the reader can verify that $k' \circ k, q', \delta', r''$ and h'' witness $f_1 \leq_{\mathbf{tt1}} f_3$. □

Theorem 37. *If $f, g \in \mathcal{B}_1$, then $f \leq_{\mathbf{tt1}} g$ if and only if $|f|_\alpha \leq |g|_\alpha$.*

Proof. Suppose that $f \leq_{\mathbf{tt1}} g$. Given p, ε, let k, q, δ, r and h witness $f \leq_{\mathbf{tt1}} g$. We claim that $\alpha(f, p, \varepsilon) \leq \alpha(g, q, \delta)$. The proof is very similar to the $\leq_{\mathbf{m}}$ case. Let $Q^\nu = k^{-1}(P^\nu_{g,q,\delta})$, we claim that Q^ν is a derivation sequence for $\{f^{-1}((-\infty, p+\varepsilon)), f^{-1}((p-\varepsilon, \infty))\}$. If $A \in Q^\nu \setminus Q^{\nu+1}$, then $k(A) \in P^\nu \setminus P^{n+1}$, so for some $\tau \prec k(A)$, either $g(P^\nu \cap [\tau]) \subseteq (-\infty, q+\delta)$, or it is a subset of $(q-\delta, \infty)$. Without loss of generality, assume the former. Let $\sigma \prec A$ be long enough that $k([\sigma]) \subseteq [\tau]$ and $|\sigma| \geq r$. Then for all $A' \in [\sigma] \cap Q^\nu$, we have 0 correctly answers $g(k(A')) \lesssim_\delta q$, and $h(\sigma \upharpoonright r, 0)$ correctly answers $f(A) \lesssim_\varepsilon p$. So $f(Q^\nu \cap [\sigma]) \subseteq (-\infty, p+\varepsilon)$ or $(p-\varepsilon, \infty)$.

In the other direction, suppose $|f|_\alpha \leq |g|_\alpha$. Since an $\leq_{\mathbf{m}}$ reduction is a $\leq_{\mathbf{tt1}}$ reduction, Theorem 33 implies that it suffices to consider the successor case. Let ν be such that $|f|_\alpha = |g|_\alpha = \nu + 1$. It suffices to show that $f \leq_{\mathbf{tt1}} g$ while assuming that g is left-sided. (The case where g is right-sided is similar.)

Given p, ε, let q, δ be maximal for g. Let $\mathcal{C} = \{C_0, C_1\}$ and $\mathcal{D} = \{D_0, D_1\}$ be as in the proof of Theorem 33. Exactly as there, let Z be an oracle which contains a well-order long enough to code ν, an optimal mind-change sequence M for $\mathcal{C}$, and a scaffolding sequence S for $\mathcal{D}$.

Let r be long enough that r bits of any input A are enough to see when A first leaves some $P^{\mu}_{f,p,\varepsilon}$ for some $\mu \leq \nu$. This uses compactness. Equivalently, r is long enough that for some finite initial segment $(\eta_j, \sigma_j, b_j)_{j<\ell}$ from M, $\cup_j[\sigma_j] = 2^\omega$, and each $|\sigma_j| \leq r$. Without loss of generality, we can assume that the σ_j partition the space.

Define k as follows. At stage 0, on input A, let j be the index for which $\sigma_j \prec A$. Let $\mu_0 = \eta_j$ and $i_0 = b_j$ and $\tau_0 = \lambda$. Now if $b_j = 0$ (matching the natural left-sidedness of g), search through S to find $B \in P^{\nu}_{g,q,\delta} \setminus D_1$, let $B_0 = B$, and proceed exactly as in the proof of Theorem 33. But if $b_j = 1$, then unfortunately $P^{\nu}_{g,q,\delta} \setminus D_0$ is empty. So in this case also let $B_0 = B$ (the same one found above), but this means i_0 is an incorrect answer to $g(B_0) \lesssim_\delta q$. We will correct this later using h. So if $b_j = 1$, proceed almost exactly as in the proof of Theorem 33, except instead of maintaining that i_s is the only correct answer to $g(B_s) \lesssim_\delta q$, now maintain that i_s is incorrect for that question.

The same arguments as in Theorem 33 now guarantee that when μ_s, i_s and B_s stabilize, then $|A|_{\mathcal{C}} = \mu_\infty + 1$, $k(A) = B_\infty$, and i_∞ correctly answers $f(A) \lesssim_\varepsilon p$. If $b_j = 0$, i_∞ is the only correct answer to $g(B_\infty) \lesssim_\delta q$. If $b_j = 1$, then $1 - i_\infty$ is the only correct answer to $g(B_\infty) \lesssim_\delta q$.

Define $h(\sigma, b)$ as follows. Let j be the unique index such that $\sigma_j \prec \sigma$. If $b_j = 0$, let $h(\sigma, b) = b$ (letting the doubly correct answer through). If $b_j = 1$, let $h(\sigma, b) = 1 - b$ (changing the only correct answer for $g(k(A)) \lesssim_\delta q$ into a correct answer for $f(A) \lesssim_\varepsilon p$.) □

Pauly has alerted us that this notion is also quite natural in the Weihrauch framework. Using the notation of Section 5.1, he asked us whether $f \leq_{\mathbf{tt1}} g$ if and only if $S_f \leq^c_W S_g$. One direction is immediate; below we prove the other using Theorem 37. At a first glance, the problem with going directly from a Weihrauch reduction to a $\leq_{\mathbf{tt1}}$ reduction is that a Weihrauch reduction, when restricted to inputs starting with p, ε, might use several different choices of q, δ for different parts of the domain. A more subtle point is that in a Weihrauch reduction, the reverse function H does not need to be defined on all of $2^\omega \times \{0,1\}$, just on the collection of values that it could receive as input. Therefore, we cannot use compactness to automatically transform H into a truth table of the kind used in a $\leq_{\mathbf{tt1}}$ reduction.

Proposition 38. *For all $f, g \in \mathcal{B}_1$, we have $f \leq_{\mathbf{tt1}} g$ if and only if $S_f \leq^c_W S_g$.*

Proof. A $tt1$ reduction is also a Weihrauch reduction, so one direction is immediate. Suppose that $S_f \leq_W^c S_g$. We claim that then $|f|_\alpha \leq |g|_\alpha$. Let K and H be the continuous functions witnessing the Weihrauch reduction. Note that H takes two arguments, the original input A, and one bit of output representing a correct answer to $S_g(K(A))$. Given p, ε, by compactness there are finitely many strings $(\sigma_i)_{i<\ell}$, and for each i rationals (q_i, δ_i) such that $\cup_i[\sigma_i] = 2^\omega$, and $\sigma_i \prec A$ implies that $K(A)$ starts with (q_i, δ_i). Let K_1 be defined so that

$$K((p,\varepsilon)^\frown\sigma_i C) = (q_i,\delta_i)^\frown K_1(\sigma_i C)$$

For each i, let $P_i^\nu = P_{g,q_i,\delta_i}^\nu$, the optimal derivation sequence for g, q_i, δ_i. Define

$$Q_i^\nu = [\sigma_i] \cap K_1^{-1}(P_i^\nu),$$

and $Q^\nu = \cup_{i<\ell} Q_i^\nu$. We claim that Q^ν is a derivation sequence for $\{f^{-1}((-\infty, p+\varepsilon)), f^{-1}((p-\varepsilon, \infty))\}$. It suffices to check this on the restriction to each $[\sigma_i]$ separately, as these are clopen sets.

Fix one $i < \ell$. Suppose that $A \in Q_i^\nu \setminus Q_i^{n+1}$. Then $\sigma_i \prec A$ and $K_1(A) \in P_i^\nu \setminus P_i^{\nu+1}$. So for some $\tau \prec K_1(A)$, either $g(P_i^\nu \cap [tau]) \subseteq (-\infty, q_i+\delta_i)$, or it is a subset of $(q_i-\delta_i, \infty)$. Without loss of generality, assume the former. Then $(A, 0)$ must be in the domain of H. Let $b = H(A, 0)$. Let $\sigma \prec A$ be long enough that $H(A', 0) = b$ whenever $\sigma \prec A'$, and long enough that $K_1([\sigma]) \subseteq [\tau]$. It is a matter of definition chasing to verify that $f(Q_i^\nu \cap [\sigma]) \subseteq C_b$, where $C_0 = f^{-1}((-\infty, p+\varepsilon))$ and $C_1 = f^{-1}((p-\varepsilon, \infty))$. This shows that Q_i^ν is a derivation sequence on $[\sigma_i]$, and thus Q^ν is a derivation sequence.

It follows that $\alpha(f, p, \varepsilon) \leq \max_{i<\ell} \alpha(g, q_i, \delta_i)$, and therefore $|f|_\alpha \leq |g|_\alpha$. □

10. Properties of $\leq_{\mathbf{tt}}$

In this section we characterize the $\leq_{\mathbf{tt}}$ degrees inside $\mathcal{B}_1$ in terms of the Bourgain rank, proving part (1) of Theorem 2. Define a coarsening of the order on the ordinals as follows:

Definition 39. *Let $\alpha \lesssim \beta$ if for every $\gamma < \alpha$, there is $\delta < \beta$ and $n \in \omega$ such that $\gamma < \delta \cdot n$.*

This coarsening is quite robust. Recall Cantor normal form for ordinals: every ordinal α can be written uniquely as a sum of the form $\alpha = \omega^{\eta_1} \cdot k_1 + \cdots + \omega^{\eta_n} \cdot k_n$, where $\eta_1 > \cdots > \eta_n$ and $k_i \in \mathbb{N}^+$. Considering the existence of Cantor normal form, one can see that $\alpha \lesssim \beta$ if for all η, $\beta \leq \omega^\eta$ implies $\alpha \leq \omega^\eta$.

The natural sum $\alpha \# \beta$ is defined by $\alpha \# \beta = \omega^{\xi_1} \cdot k_1 + \dots \omega^{\xi_r} \cdot k_r$, where $\xi_1 > \cdots > \xi_r$ are exactly the exponents in the Cantor normal forms of α and β, and k_i is the sum of the coefficients of ω^{ξ_i} in α and β. One sees also that $\alpha \lesssim \beta$ if for every $\gamma < \alpha$, there is $\delta < \beta$ and $n \in \omega$ such that

$$\gamma < \underbrace{\delta\#\delta\#\dots\#\delta}_{n}.$$

We will show that the $\leq_{\mathbf{tt}}$ degrees inside $\mathcal{B}_1$ correspond to functions whose ranks are equivalent according to this relation. The next lemma describes the length of combined derivation sequences.

Lemma 40. *Let X be a compact metric space and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(X)$. Let $P^\nu_{\mathcal{C}}$ and $P^\nu_{\mathcal{D}}$ be the optimal derivation sequences for $\mathcal{C}$ and $\mathcal{D}$. Let Q^ν be the optimal derivation sequence for*

$$\{C \cap D : C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}.$$

Then for all ν and μ,

$$Q^{\nu\#\mu} \subseteq P^\nu_{\mathcal{C}} \cup P^\mu_{\mathcal{D}}.$$

Proof. By induction on $\nu\#\mu$. If $\nu\#\mu = 0$, the statement is immediate. Suppose the statement holds for all pairs of ordinals with natural sum less than $\nu\#\mu$. Let $A \notin P^\nu_{\mathcal{C}} \cup P^\mu_{\mathcal{D}}$. Then there are ordinals $\eta < \nu$ and $\xi < \mu$, a neighborhood U of A, and sets $C \in \mathcal{C}$ and $D \in \mathcal{D}$ such that $P^\eta_{\mathcal{C}} \cap U \subseteq C$ and $P^\xi_{\mathcal{D}} \cap U \subseteq D$.

Let $\zeta = \max(\eta\#\mu, \nu\#\xi)$. Then since $\eta < \nu$ and $\xi < \mu$, we have $\zeta < \nu\#\mu$. So by induction,

$$Q^\zeta \subseteq Q^{\eta\#\mu} \cap Q^{\nu\#\xi} \subseteq (P^\eta_{\mathcal{C}} \cup P^\mu_{\mathcal{D}}) \cap (P^\nu_{\mathcal{C}} \cup P^\xi_{\mathcal{D}})$$

Rearranging the right hand side, we have

$$Q^\zeta \subseteq P^\nu_{\mathcal{C}} \cup P^\mu_{\mathcal{D}} \cup (P^\eta_{\mathcal{C}} \cap P^\xi_{\mathcal{D}}).$$

Because $P^\nu_{\mathcal{C}} \cap U = P^\mu_{\mathcal{D}} \cap U = \emptyset$ and $P^\eta_{\mathcal{C}} \cap P^\xi_{\mathcal{D}} \cap U \subseteq C \cap D$, we have $Q^{\zeta+1} \cap U = \emptyset$. So $A \notin Q^{\nu\#\mu}$, because $Q^{\nu\#\mu} \subseteq Q^{\zeta+1}$. □

The following is then immediate by induction.

Lemma 41. *Let X be a compact metric space and let $C_i \subseteq \mathcal{P}(X)$ for all $i < r$. Let P^ν_i be the optimal derivation sequences for $\mathcal{C}_i$, and let Q^ν be the optimal derivation sequence for*

$$\{\cap_{i<r} C_i : C_i \in \mathcal{C}_i\}.$$

Then for all $(\nu_i)_{i<r}$,

$$Q^{\#_{i<r}\nu_i} \subseteq \cup_{i<r} P^{\nu_i}_i.$$

Theorem 42. *If $f, g \in \mathcal{B}_1 \setminus \mathcal{B}_0$, then $f \leq_{\mathbf{tt}} g$ if and only if $|f|_\alpha \lesssim |g|_\alpha$.*

Proof. Suppose $f \leq_{\mathbf{tt}} g$. Given p, ε, let $(k_i, q_i, \delta_i)_{i<r}$ and h be as in the definition of $\leq_{\mathbf{tt}}$. For each i, define

$$\mathcal{C}_i = \{k_i^{-1}(g^{-1}((-\infty, q_i + \delta_i))), k_i^{-1}(g^{-1}((q_i - \delta_i, \infty)))\}.$$

Let

$$\mathcal{C} = \{\cap_{i<r} C_i : C_i \in \mathcal{C}_i\}.$$

We claim that any derivation sequence for $\mathcal{C}$ is also a derivation sequence for

$$\mathcal{D} := \{f^{-1}((-\infty, p+\varepsilon)), f^{-1}((p-\varepsilon, \infty))\}.$$

This follows because for every $\cap_{i<r} C_i \in \mathcal{C}$, there is a $\sigma \in 2^r$ such that $\sigma(i)$ correctly answers $g(k(A)) \lesssim_{\delta_i} q_i$, for every $i < r$ and $A \in \cap_{i<r} C_i$. Therefore, for each $A \in \cap_{i<r} C_i$, $h(\sigma)$ is a correct answer to $f(A) \lesssim_{\varepsilon} p$. Therefore, for some $D \in \mathcal{D}$, we have $\cap_{i<r} C_i \subseteq D$, and the claim follows by Proposition 26.

Define $Q_i^\nu = k_i^{-1}(P^\nu_{g,q_i,\delta_i})$. By Proposition 26, Q_i^ν is a derivation sequence for $\mathcal{C}_i$. Let $\nu_i = \alpha(g, q_i, \delta_i)$, so that $Q_i^{\nu_i} = \emptyset$. Let Q^ν be the optimal derivation sequence for $\mathcal{C}$. By Lemma 41,

$$Q^{\#_{i<r}\nu_i} \subseteq \cup_{i<r} Q_i^{\nu_i}.$$

Therefore, as Q^ν is also a derivation sequence for $\mathcal{D}$, we have

$$\alpha(f, p, \varepsilon) \leq \#_{i<r}\nu_i \leq \underbrace{\nu \# \ldots \# \nu}_{r},$$

where $\nu = \max_i \alpha(g, q_i, \delta_i)$. Therefore, $|f|_\alpha \lesssim |g|_\alpha$.

Now suppose that $|f|_\alpha \lesssim |g|_\alpha$. We run a daisy-chain of the kind of argument used in the $\leq_{\mathbf{tt1}}$ case. Given p, ε, let q, δ and n be such that $\alpha(f, p, \varepsilon) < \alpha(g, q, \delta) \cdot n$, and $\alpha(g, q, \delta) \geq 2$. Letting $\nu = \alpha(g, q, \delta)$, we may also guarantee that $P^{\nu-1}_{g,q,\delta} \not\subseteq (q - \delta, q + \delta)$, by decreasing δ if necessary.

We will define $3n$ functions k_i, all of them associated to this same pair q, δ. The functions are defined computably relative to an oracle which contains enough information to compute notations up to ν (and thus up to $\nu \cdot n$), a mind-change sequence M for $\{f^{-1}((-\infty, p+\varepsilon)), f^{-1}((p-\varepsilon, \infty))\}$, and a scaffolding sequence S for $\{g^{-1}((-\infty, q+\delta)), g^{-1}((q-\delta, \infty))\}$.

Fix $B_0 \in P^{\nu-1}_{g,q,\delta}$ with $g(B_0) \notin (q - \delta, q + \delta)$, and let b_0 be the unique correct answer to $g(B_0) \lesssim_\delta q$. Since $\nu \geq 2$, $|B_0|_{g,q,\delta} \geq 2$.

Given input A, the first n functions $\{k_i\}_{i<n}$ are used to figure out in which interval

$$I_i = [\nu \cdot i + 1, \nu \cdot (i+1)]_{i<n}$$

$|A|_{f,p,\varepsilon}$ lies. Define $k_i(A)$ as follows. Copy B_0 until such a time as you see $A \notin P^{\nu\cdot(i+1)}_{f,p,\varepsilon}$. If this occurs, switch to copying a nearby input B_1 with $|B_1|_{g,q,\delta} < |W|_{g,q,\delta}$ and where the unique correct answer to $g(B_1) \lesssim_\delta q$ is $1 - b_0$. That completes the description of the first n functions k_i. By observing the answers for $g(k_i(A)) \lesssim_\delta q$ for $i < n$, one can determine the uniqe $i < n$ such that $A \in I_i$.

The next n functions $\{k_{n+i}\}_{i<n}$ track the mind-changes of $f(A) \lesssim_\varepsilon p$ under the assumption that $A \in I_i$. Given input A, and letting B_0 and b_0 be as above, first copy B_0 into the output until such a time as you see $A \notin P^{\nu\cdot(i+1)}_{f,p,\varepsilon}$. If this occurs, then we also know a rank $\mu_0 < \nu$ and bit i_0 such that if $|A|_{f,p,\varepsilon} = (\nu \cdot i) + \mu_0 + 1$, then i_0 correctly answers $f(A) \lesssim_\varepsilon p$. Let τ_0 be whatever amount of B_0 has been copied so far. Now proceed similarly as in Theorem 33, but maintain the following at each stage:

(i) that $|A|_\mathcal{C} \le (\nu \cdot i) + \mu_s + 1 \le (\nu \cdot i) + |B_s|_\mathcal{D}$,
(ii) that the only correct answer to $g(B_s) \lesssim_\delta q$ is i_s if $i_0 = b_0$, and the only correct answer is $1 - i_s$ if $i_0 \ne b_0$.
(iii) if $|A|_\mathcal{C} = (\nu \cdot i) + \mu_s + 1$, then i_s correctly answers $f(A) \lesssim_\varepsilon p$.

Proceeding now just as in Theorem 33, the above can be maintained unless A leaves $P^{\nu\cdot i}_{f,p,\varepsilon}$. In that case, the output of this computation will not be used, so one can continue to copy whatever B_s is active at the moment this is discovered. But if μ_s, i_s and B_s stabilize to values μ_∞, i_∞ and B_∞, then if $A \in I_i$, we have $|A|_{f,p,\varepsilon} = (\nu \cdot i) + \mu_\infty + 1$, $k_{n+1}(A) = B_\infty$, i_∞ is a correct answer to $f(A) \lesssim_\varepsilon p$, and the only correct answer to $g(B_\infty) \lesssim_\delta q$ is either i_s or $1-i_s$ depending on whether $i_0 = b_0$ or not.

The last n functions $\{k_{2n+i}\}_{i<n}$ are simple indicator functions, with k_{2n+1} copying B_0 and silently carrying out the same computation as k_{n+i} until that computation finds an i_0 and a b_0. If k_{n+i} finds $i_0 \ne b_0$, switch to a nearby B_1 with $|B_1|_{g,q,\delta} < |B_0|_{g,q,\delta}$ and where the unique correct answer to $g(B_1) \lesssim_\delta q$ is $1 - b_0$. Otherwise (including if i_0 is never defined), continue copying B_0.

Putting this all together, given A, a truth table which has access to separating bits for each $g(k_i(A))$ can correctly answer $f(A) \lesssim_\varepsilon p$ as follows. First use the separating bits of $g(k_i(A))$ for $i < n$ to find the unique i such that $|A|_{f,p,\varepsilon} \in I_i$. Then query $g(h_{2n+i}(A))$ to learn whether $i_0 = b_0$ in the computation of $k_{n+i}(A)$. Finally, query $g(k_{n+i}(A))$ to obtain a bit b which correctly answers $f(A) \lesssim_\varepsilon p$ if $i_0 = b_0$. If $i_0 \ne b_0$, then $1 - b$ will do for a correct answer. $\square$

As a corollary we can give a short algorithmic proof of the following result of Kechris and Louveau, which is a consequence of their Lemma 5 and Theorem 8, and which allows them to conclude that their "small Baire classes" $\mathcal{B}_1^\xi$ are Banach algebras.

Corollary 43. [KL90] *If $f, g \in \mathcal{B}_1$ are bounded, then*

$$|f+g|_\alpha, |fg|_\alpha \lesssim \max(|f|_\alpha, |g|_\alpha).$$

Proof. Without loss of generality we can assume that $|f|_\alpha \leq |g|_\alpha$, so $f \leq_{\mathbf{tt}} g$. Also, let $M \in \mathbb{R}$ be chosen so that all outputs of f and g lie in $[-M, M]$.

Then $f+g \leq_{\mathbf{tt}} g$ via the following algorithm. Given A, p, ε, first ask finitely many questions of f and g to determine both $f(A)$ and $g(A)$ to within precision $\varepsilon/2$ (by asking each function $2M/(\varepsilon/2)$ questions of the form $f(A) \lesssim_{\varepsilon/2} q_i$, where the q_i are evenly spaced at intervals of $\varepsilon/2$ in $[-M, M]$.) Adding the two approximations gives an approximation to $(f+g)(A)$ which is correct to within ε. Use this approximation to answer $f(A) \lesssim_\varepsilon p$.

Similarly, $fg \leq_{\mathbf{tt}} g$ as follows. Given A, p, ε, first use finitely many questions to approximate $f(A)$ and $g(A)$ to within precision $\varepsilon/(2M)$. Multiplying the results gives an approximation to $(fg)(A)$ that is correct to within ε. ☐

11. Further directions and open questions

11.1. **A road not taken.** Recall that we used admissible representations to allow our results about functions on 2^ω to extend to arbitrary compact separable metrizable spaces. Another option for extending these reducibilities would be to transfer the definitions literally to the new spaces, without using representations. For example, one could define $f \leq'_{\mathbf{m}} g$ to mean that for every p, ε, there is a continuous function k and rationals q, δ such that for all x, we have any correct answer to $g(k(x)) \lesssim_\delta q$ is a correct answer to $f(x) \lesssim_\varepsilon p$.

This option behaves very differently from the one we chose, for if X is very connected, then there are not enough continuous functions $k : X \to Y$ to get the same results. For example, we can define two left-sided, rank 3 functions in $\mathcal{B}_1([0,1])$ are not $\leq'_{\mathbf{m}}$-equivalent under this alternate definition. Let $f_1 = \chi_{\{1/n : n \in \omega\}}$. And let $f_2 = \chi_S$ where

$$S = \{x_I^* : I \text{ is a middle third}\}$$

where I is a middle third means that I belongs to the sequence $(1/3, 2/3)$, $(1/9, 2/9), (7/9, 8/9), \ldots$ of intervals removed to create the Cantor set in $[0,1]$, and x_I^* denotes the midpoint of I.

To see that $f_2 \not\leq'_{\mathbf{m}} f_1$ under this less robust definition of $\leq'_{\mathbf{m}}$, fix $p = 1/2$ and $\varepsilon = 1/3$; q and δ will have to be similarly assigned since we are working with characteristic sets. Then any continuous k that would work for the reduction would have to send the Cantor subset of $[0, 1]$ to 0. For if any z from the Cantor subset of I satisfied $k(z) \in (1/(n+1), 1/n)$, then by pulling back $(1/(n+1), 1/n)$ via k, we'd find a whole neighborhood of z mapped to $(1/(n+1), 1/n)$, impossible since every neighborhood of z includes an element of S. So $h(1/3) = h(2/3) = 0$. Now, what is $k(1/2)$. It must be equal to $1/n$ for some n or the reduction fails. So $k([1/3, 1/2])$ includes both 0 and some $1/n$. Since k is continuous and $[1/3, 1/2]$ is connected, its image is connected so also includes $1/m$ for all $m > n$. But who are getting mapped to $1/m$? The purported reduction is wrong on $k^{-1}(1/m)$ for such m.

In fact f_2 is not even $\leq'_{\mathbf{m}}$ the characteristic function of the rationals, for a similar reason: if $k(1/3)$ is irrational and $k(1/2)$ is rational, then $k([1/3, 1/2])$ contains many rationals.

Since the characteristic function of the rationals is Baire 2, this alternate generalization produces a very different theory, which we did not pursue further.

11.2. **Computable reducibilities for discontinuous functions.** The original motivation for this work was to devise a notion of *computable* reducibility between arbitrary (especially discontinuous) functions. There is a well-established notion of computable reducibility between continuous functions due to Miller [Mil04], based on the notion of computable function due to Grzegorczyk [Grz55, Grz57] and Lacombe [Lac55a, Lac55b]. A truly satisfying notion of computable reducibility for arbitrary functions would have its restriction to continuous functions agree with with this established notion. Unfortunately, the computable/lightface versions of our reducibilities do not have this property. The reason for this, roughly speaking, is that the Weihrauch-based reductions operate pointwise, whereas the established computable reducibility on continuous functions makes essential use of global information in the form of the modulus of continuity. Therefore, the following question remains of interest, where of course satisfaction lies in the eye of the beholder.

Question 44. *Is there a satisfying notion of computable reducibility for arbitrary functions, whose restriction to the continuous functions is exactly continuous reducibility in the sense of Miller?*

And of course, it would still be interesting to know more about the structure of arbitrary functions under the computable versions of these reducibilities.

Question 45. *What can be said about the degree structure of $\mathcal{F}(X, \mathbb{R})$ under the computable versions of $\leq_{\mathbf{T}}, \leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$?*

We will address further details and progress on these questions in a forthcoming paper.

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