Ginzburg-Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices

Leonid Berlyand[(1)], **Petru Mironescu**[(2)]

May 30, 2004

Abstract. Let Ω be a 2D simply connected domain, ω be a simply connected subdomain of Ω and set $A = \Omega \setminus \omega$. In the annular type domain A, we consider the class $\mathcal{J}$ of complex valued maps having degrees 1 on $\partial\Omega$ and on $\partial\omega$. We investigate whether the minimum of the Ginzburg-Landau energy E_λ is attained in $\mathcal{J}$, as well as the asymptotic behavior of minimizers as the coherency length $\lambda^{-1/2}$ tends to 0. We show that the answer to these questions is determined by the value of the H^1-capacity cap(A) of the domain A. This is due to the degree boundary conditions; by contrast, when Dirichlet conditions are prescribed, it is known that the behavior of minimizers does not depend on A. If cap$(A) > \pi$ (A is a "thin" or "subcritical" domain), minimizers exist for each λ. As $\lambda \to \infty$, they converge in $H^1(A)$ (and even better) to an S^1-valued harmonic map we identify. Furthermore, these minimizers are vortexless for large λ. The same properties hold when cap$(A) = \pi$ ("critical" domain), but the proof is more involved. When cap$(A) < \pi$ ("thick" or "supercritical" domain), we prove that either (i) minimizers cease to exist for large λ, or (ii) that they exist for each λ. For large λ, minimizing sequences (in case (i)) or minimizers (in case (ii)) develop exactly two vortices, one of degree 1 near $\partial\Omega$, the other one of degree -1 near $\partial\omega$. We conjecture that case (ii) never occurs.

1 Introduction

Consider the following problem

$$m_\lambda = \mathrm{Inf}\left\{ E_\lambda(u) = \frac{1}{2}\int_A |\nabla u|^2 + \frac{\lambda}{4}\int_A (1-|u|^2)^2 \ ; \ u \in \mathcal{J} \right\}. \tag{1.1}$$

Here, E_λ is a Ginzburg-Landau (GL, hereafter) type energy, A is a 2D annular type domain, i.e., $A = \Omega \setminus \omega$, $\overline{\omega} \subset \Omega$, with Ω, ω, simply connected bounded smooth domains. The class $\mathcal{J}$ of testing maps is

$$\mathcal{J} = \{u \in H^1(A\,;\mathbb{R}^2) \ ; \ |u| = 1 \text{ a.e. on } \partial A, \ \deg(u, \partial\Omega) = \deg(u, \partial\omega) = 1\}. \tag{1.2}$$

The definition of $\mathcal{J}$ is meaningful. Indeed, let Γ be $\partial\Omega$ or $\partial\omega$ (counterclockwise oriented) and set $X = H^{1/2}(\Gamma; S^1)$. If $u \in H^1(A\ ;\mathbb{R}^2)$ and $|u| = 1$ a.e. on ∂A, then $g := u_{|\Gamma} \in X$ (here, the restiriction is to be understood in the sense of traces). Maps in X have a well-defined topological degree (winding number), see [?]. This degree is defined as follows: every map $g \in X$ is the strong $H^{1/2}$ limit of a sequence $(g_n) \subset C^\infty(\Gamma; S^1)$. Each g_n has a degree (with respect to the counterclockwise orientation on Γ) given, e.g., by the classical formula

$$\deg g = \frac{1}{2\pi} \int_\Gamma g \wedge g_\tau. \tag{1.3}$$

Then $\lim_n \deg g_n$ exists; see [?] for the details. This allows to define $\deg g = \lim_n \deg g_n$. Formula (??) is still valid for arbitrary maps in X, provided we interpret the integral as an $H^{1/2} - H^{-1/2}$ duality.

We may now address a first natural question concerning the minimization problem (??)-(??)

Question 1. Is m_λ attained ?

Before discussing this question, we start by recalling the most intensively studied minimization problem for the Ginzburg-Landau functional, namely

$$e_\lambda = \mathrm{Inf}\{E_\lambda(u)\,;\, u_{|\partial G} = g\}, \tag{1.4}$$

see [?]. Here, G is a smooth bounded domain in $\mathbb{R}^2$ and $g \in H^{1/2}(\partial G; S^1)$ is fixed. In this case, e_λ is obviously attained, since the class $\{u \in H^1(G)\,;\, u_{|\partial G} = g\}$ is closed with respect to weak H^1 convergence.

The situation is more delicate when we do not prescribe a Dirichlet boundary condition, but only degrees, as shown by the following

Example 1. (Inf is not attained) [?] Let

$$n_\lambda = \mathrm{Inf}\{E_\lambda(u)\,;\, u \in \mathcal{M}\}, \tag{1.5}$$

where

$$\mathcal{M} = \{u \in H^1(\mathbb{D})\,;\, |u| = 1 \text{ a.e. on } S^1\,,\ \deg(u, S^1) = 1\}. \tag{1.6}$$

Here, $\mathbb{D}$ is the unit disc and we consider the counterclockwise orientation on S^1.
Then, for each $\lambda > 0$, $n_\lambda = \pi$ and n_λ is not attained.

In particular, this example implies that the class $\mathcal{M}$ is not closed with respect to weak H^1 convergence. Here is an explicit example of a sequence in $\mathcal{M}$ weakly converging in H^1 to a map which is not in $\mathcal{M}$:

Example 2. [?] Let $(a_n) \subset (0,1)$ be such that $a_n \to 1$. Set $u_n(z) = \dfrac{z - a_n}{1 - a_n z}$, $z \in \mathbb{D}$. Then $u_n \rightharpoonup -1$ weakly in H^1.

Example 2 can be easily extended to $\mathcal{J}$:

Proposition 1. [?] *The class $\mathcal{J}$ is not closed with respect to weak H^1 convergence.*

This implies that the existence of minimizers of (**??**)-(**??**) does not follow immediately from the direct method of Calculus of Variations.

Before discussing further Question 1, we mention some useful a priori bounds on m_λ. Recall that in the case of a prescribed Dirichlet data with non zero degree (thoroughly studied in [?]) the GL energy tends to infinity as $\lambda \to \infty$. However, a straightforward calculation shows that the energy remains bounded (with a bound independent of A and λ) when we only prescribe degrees on the boundary:

$$m_\lambda \leq 2\pi, \tag{1.7}$$

see [?].

There is yet another upper bound, which is obtained by considering all S^1-valued maps in $\mathcal{J}$. Set

$$\mathcal{K} = \{u \in \mathcal{J}\,;\, |u| = 1 \text{ a.e. in } A\}. \tag{1.8}$$

$\mathcal{K}$ is not empty: if $a \in \omega$, then $(x-a)/|x-a| \in \mathcal{K}$. It is known that, in $\mathcal{K}$, Min E_λ is attained, see [?]. Define

$$I_0 = \text{Min}\,\{E_\lambda(u)\,;\, u \in \mathcal{K}\} = \text{Min}\,\left\{\frac{1}{2}\int_A |\nabla u|^2\,;\, u \in \mathcal{K}\right\}. \tag{1.9}$$

Proposition 2. *We have*

$$m_\lambda < I_0. \tag{1.10}$$

Clearly, (**??**) and (**??**) imply that $m_\lambda \leq \text{Min}\,\{I_0, 2\pi\}$. This bound is close to optimal when λ is large:

Proposition 3. *We have*

$$\lim_{\lambda\to\infty} m_\lambda = \text{Min}\,\{I_0, 2\pi\}. \tag{1.11}$$

It turns out that I_0 has a simple geometrical interpretation via capacity:

Proposition 4. [?] *I_0 and the H^1-capacity* cap(A) *of the domain A are related by*

$$I_0 = \frac{2\pi^2}{\text{cap}(A)}. \tag{1.12}$$

Recall that, if $A = \{x \; ; \; r < |x| < R\}$, then $\text{cap}(A) = \pi \ln(R/r)$. In general, one may think of the capacity as a measure of "thickness" of A.

Formula (**??**), the discussion on capacity, and our results on existence of minimizers suggest disinguishing three types of domains :
a) "subcritical" or "thick", when $\text{cap}(A) > \pi$ (or, equivalently, $I_0 < 2\pi$) ;
b) "critical", when $\text{cap}(A) = \pi$ (or, equivalently, $I_0 = 2\pi$) ;
c) "supercritical" or "thin", when $\text{cap}(A) < \pi$ (or, equivalently, $I_0 > 2\pi$).

We now return to the existence of minimizers. The main tool in proving existence is the following

Proposition 5. *Assume that $m_\lambda < 2\pi$. Then m_λ is attained.*

The first result of this type was established for the Yamabe problem by Th. Aubin in [**?**]. Such results subsequently proved to be extremely useful in minimization problems with possible lack of compactness of minimizing sequences; see [**?**], [**?**], [**?**], [**?**] and the more recent papers [**?**], [**?**] and [**?**].
The proof of Proposition **??** relies on the following

Lemma 1. (Price lemma) *Let (u_n) be a bounded sequence in $\mathcal{J}$ such that $u_n \rightharpoonup u$ in $H^1(A)$. Then :*

$$\liminf_n \frac{1}{2} \int_A |\nabla u_n|^2 \geq \frac{1}{2} \int_A |\nabla u|^2 + \pi(|1 - \deg(u, \partial\Omega)| + |1 - \deg(u, \partial\omega_0)|). \tag{1.13}$$

In addition,

$$\frac{1}{2} \int_A |\nabla u|^2 \geq \pi |\deg(u, \partial\Omega) - \deg(u, \partial\omega)|. \tag{1.14}$$

The argument we use works for arbitrary fixed degrees instead of 1 and 1, see [**?**]; the general form of the estimate (**??**) shows that the minimal energy needed to jump, on a component of ∂A, from degree d (for the maps u_n) to degree δ (for u), is $\pi|d - \delta|$, see [**?**].

As an immediate consequence of Proposition **??** and of the upper bound (**??**), we obtain the following

Theorem 1. *Asssume that A is subcritical or critical. Then m_λ is attained for each $\lambda \geq 0$.*

In the subcritical and critical case, we further address the following natural

Question 2. What is the behavior of minimizers u_λ of (**??**)-(**??**) as $\lambda \to \infty$?

The answer is given by

Theorem 2. *Let* $\text{cap}(A) \geq \pi$*, i.e., A is subcritical or critical. Let* u_λ *be a minimizer of (??)-(??). Then* $|u_\lambda| \to 1$ *uniformly in* $\overline{A}$*. In addition, up to some subsequence,* $u_\lambda \to u_\infty$ *in* $H^1(A)$*, where* u_∞ *is a minimizer of (??)-(??).*

Theorem 2 combined with the method developed in [?] yield the stronger convergence $u_\lambda \to u_\infty \in C^{1,\alpha}(\overline{A})$, $0 < \alpha < 1$; see [?]. We also prove in [?] that, for large λ, minimizers are unique modulo multiplication with a constant in S^1, and, in addition, symmetric, if the domain is symmetric.

Whenever minimizers u_λ exist, they are smooth, see [?]. This requires some proof, since the boundary conditions satisfied by the u_λ's are of mixed type, Dirichlet for the modulus $|u_\lambda|$, Neumann for the phase $\arg u_\lambda$.

We now turn to the supercritical case $\text{cap}(A) < \pi$. Here, unlike in the subcritical/critical case, we prove that, for large λ, minimizing sequences must have vortices (zeroes of non-zero degree). Concerning existence of minimizers, we prove that there are exactly two possible behaviors (see Fig. 1)

Theorem 3. *Let* $\text{cap}(A) < \pi$*, i.e., A is supercritical. Then either*
a) m_λ *is attained for all* λ*;*
or
b) there exists a critical value $\lambda_1 \in (0, \infty)$ *such that: if* $\lambda < \lambda_1$*, then* m_λ *is attained, while, if* $\lambda > \lambda_1$*, then* m_λ *is not attained.*

Theorem 4. (Rise of vortices) *Let A be supercritical.*
In case a), let u_λ *be a minimizer of (??)-(??). Then, for large* λ*,* u_λ *has exactly two simple zeroes,* ζ_λ *of degree* 1 *and* ξ_λ *of degree* -1*, such that* $\zeta_\lambda \to \partial\Omega$ *and* $\xi_\lambda \to \partial\omega$ *as* $\lambda \to \infty$*.*
In case b), let $\lambda > \lambda_1$ *and let* (u_k) *be a minimizing sequence for (??)-(??). Then* $u_k = v_k + w_k$*, where* $w_k \to 0$ *in* $H^1(A)$ *as* $k \to \infty$ *and* v_k *has exactly two simple zeroes,* ζ_k *of degree* 1*, and* ξ_k *of degree* -1*, such that* $\zeta_k \to \partial\Omega$ *and* $\xi_k \to \partial\omega$ *as* $n \to \infty$*.*

We further prove that, in case b), near ζ_k (ξ_k respectively), u_k essentially behaves like a conformal representation of Ω into $\mathbb{D}$ vanishing at ζ_k (anti-conformal representation of $\mathbb{C} \setminus \omega$ into $\mathbb{D}$ vanishing at ξ_k, respectively); see Step 5 in the proof of Theorem 4 in Section 4 for precise statements. A similar analysis holds in case a).

We believe that case a) **never** occurs, which led us to the following

Conjecture. *In the supercritical case, there exists a* **finite** *constant* $\lambda_1 > 0$ *such that, if* $\lambda > \lambda_1$*, then* m_λ *is never attained.*

The heuristics in support of this conjecture is the following: assume case a) holds. For large λ, let (with the notations in Theorem 4) $d = \text{dist}(\{\zeta_\lambda, \xi_\lambda\}, \partial A)$. It is easy to check that

$\lambda/4 \int\limits_A (1-|u_\lambda|^2)^2 \geq C_1 \lambda d^2$. On the other hand, examples suggest that $1/2 \int\limits_A |\nabla u_\lambda|^2 \geq 2\pi - C_2 d^2$; here, C_1, C_2 do not depend on λ or d. If this inequality holds, then the upper bound (**??**) contradicts existence of minimizers for large λ.

Finally, we discuss specific features of the critical case. It is known that, in variational problems with lack of compactness, the critical case could inherit the properties of either the supercritical or the subcritical case (see, e.g., [**?**], [**?**], [**?**], [**?**]). In our problem, the results are the same in critical and subcritical case, the supercritical case being qualitatively different. However, while the proof of the existence is the same in the subcritical and critical cases, the argument that leads to H^1-convergence of the minimizers u_λ as $\lambda \to \infty$ does not apply to the critical case; a more subtle argument is required at criticality.

Acknowledgments. The authors thank H. Brezis for very useful discussions. They also thank D. Golovaty for a careful reading of the manuscript. The work of L.B. was supported by NSF grant DMS-0204637. The work of P.M. is part of the RTN Program "Fronts-Singularities". This work was initiated while both authors were visiting the Rutgers University; part of the work was done while L. B. was visiting Université Paris-Sud and P. M. was visiting the Penn State University. They thank the Mathematics Departments in these universities for their hospitality.

2 Existence of minimizers

The following simple remark will be repeatedly used in the sequel. Let (u_n) be a bounded sequence in $H^1(A)$ such that $|u_n| = 1$ a.e. on ∂A for each n. If $u_n \rightharpoonup u$ in H^1, then clearly $|u| = 1$ a.e. on ∂A. Thus $\deg(u, \partial\Omega)$ and $\deg(u, \partial\omega)$ are well-defined.

Proof of the Price lemma: Set $v_n = u_n - u$. We have, as $n \to \infty$,

$$\int\limits_A |\nabla u_n|^2 = \int\limits_A |\nabla u|^2 + \int\limits_A |\nabla v_n|^2 + o(1). \tag{2.1}$$

Let $f \in C^\infty(\overline{A}\ ; [-1,1])$ to be determined later. Integrating by parts the pointwise inequality $|\nabla v_n|^2 \geq 2f$ Jac v_n, we find

$$\int\limits_A |\nabla v_n|^2 \geq \int\limits_{\partial A} f v_n \wedge \frac{\partial v_n}{\partial \tau} + \int\limits_A (f_x (v_n)_y \wedge v_n - f_y (v_n)_x \wedge v_n); \tag{2.2}$$

here, ∂A is directly oriented. The above equality is clear when v_n is smooth; it relies on the identity

$$2\text{Jac} v_n = (v_n \wedge (v_n)_y)_x + ((v_n)_x \wedge (v_n))_y.$$

The case of an arbitrary v_n follows by approximation. Since $v_n \rightharpoonup 0$ in H^1, (**??**) and (**??**) yield

$$\int_A |\nabla u_n|^2 \geq \int_A |\nabla u|^2 + \int_{\partial A} f v_n \wedge \frac{\partial v_n}{\partial \tau} + o(1). \tag{2.3}$$

On the other hand, we claim that, if Γ is any connected component of ∂A, then

$$\int_\Gamma v_n \wedge \frac{\partial v_n}{\partial \tau} = \int_\Gamma u_n \wedge \frac{\partial u_n}{\partial \tau} - \int_\Gamma u \wedge \frac{\partial u}{\partial \tau} + o(1). \tag{2.4}$$

Indeed, if $g_n \rightharpoonup g$ in $H^{1/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$, then clearly

$$\int_\Gamma g_n \wedge \frac{\partial h}{\partial \tau} = \int_\Gamma g \wedge \frac{\partial h}{\partial \tau} + o(1) \quad \text{and} \quad \int_\Gamma h \wedge \frac{\partial g_n}{\partial \tau} = \int_\Gamma h \wedge \frac{\partial g}{\partial \tau} + o(1). \tag{2.5}$$

Equality (**??**) follows easily from (**??**) and the fact that $u_{n|\Gamma} \rightharpoonup u_{|\Gamma}$ in $H^{1/2}(\Gamma)$.
Pick now f such that $f = \text{sgn}(1-\deg(u, \partial\Omega))$ on $\partial\Omega$, $f = -\text{sgn}(1-\deg(u, \partial\omega))$ on $\partial\omega$ and $-1 \leq f \leq 1$. By combining (**??**), (**??**), (**??**) and the degree formula (**??**), we obtain (**??**).

As for (**??**), it relies on the pointwise inequality $|\nabla u|^2 \geq 2|\text{Jac } u|$, which yields, after integration by parts and use of (**??**),

$$\int_A |\nabla u|^2 \geq 2 \int_A |\text{Jac u}| \geq 2 \left| \int_A \text{Jac u} \right| = \left| \int_{\partial A} u \wedge \frac{\partial u}{\partial \tau} \right| = 2\pi |\deg(u, \partial\Omega) - \deg(u, \partial\omega)|. \tag{2.6}$$

Proof of Proposition ??: Let (u_n) be a minimizing sequence for E_λ in $\mathcal{J}$. Up to some subsequence, we may assume that $u_n \rightharpoonup u$ for some u. Set $D = \deg(u, \partial\Omega)$, $d = \deg(u, \partial\omega)$. If $d = D = 1$, then $u \in \mathcal{J}$ and u is a minimizer of (**??**)-(**??**). If $D \neq 1$ and $d \neq 1$, (**??**) implies that

$$2\pi > m_\lambda = \liminf_n E_\lambda(u_n) \geq \liminf_n \frac{1}{2} \int_A |\nabla u_n|^2 \geq \pi(|1-d| + |1-D|) \geq 2\pi, \tag{2.7}$$

which is a contradiction. Finally, if exactly one among d and D equals 1, then $|d - D| \geq 1$ and $|1-d| + |1-D| \geq 1$. By combining (**??**) and (**??**) we obtain as above $m_\lambda \geq 2\pi$, which is impossible.

Proof of Proposition ??: Let u be a minimizer of (**??**)-(**??**) and set $g = u_{|\partial A}$. If v minimizes E_λ among all the maps $w \in H^1(A)$ such that $w_{|\partial A} = g$, then $v \in \mathcal{J}$ and $m_\lambda \leq E_\lambda(v) \leq E_\lambda(u) = I_0$. We claim that the last inequality is strict. Argue by contradiction and assume that $E_\lambda(v) = E_\lambda(u)$. Then u minimizes E_λ with respect to its own boundary condition; in particular, u satisfies the GL equation $-\Delta u = \lambda u(1-|u|^2)$. Since $|u| = 1$ a.e., we find that u is harmonic and of modulus 1. Thus u has to be a constant, which contradicts the fact that $u \in \mathcal{K}$.

Proof of Theorem ??: Clearly, $\lambda \mapsto m_\lambda$ is not decreasing and continuous. In view of the upper bound (**??**), there is some $\lambda_1 \in [0, \infty]$ such that $m_\lambda < 2\pi$ if $\lambda < \lambda_1$ and $m_\lambda = 2\pi$ if $\lambda \geq \lambda_1$. We first claim that m_λ is not attained if $\lambda > \lambda_1$. Argue by contradiction and assume that there are some $\lambda > \lambda_1$ and $u \in \mathcal{J}$ such that $E_\lambda(u) = m_\lambda = 2\pi$. As in the proof of Proposition **??**, we cannot have $|u| = 1$ a.e. Thus $\displaystyle\int_A (1-|u|^2)^2 > 0$ and therefore $E_{\lambda'}(u) < E_\lambda(u)$ if $\lambda' < \lambda$. For any λ' such that $\lambda_1 < \lambda' < \lambda$, this implies that $m_{\lambda'} \leq E_{\lambda'}(u) < 2\pi$, which is impossible.

In view of Proposition **??**, m_λ is attained for $\lambda < \lambda_1$. In order to complete the proof of Theorem **??**, it remains to rule out the possibility $\lambda_1 = 0$. This amounts to proving the following

Lemma 2. *We have $m_0 < 2\pi$.*

Proof of Lemma ??: We start with the case of a circular annulus, $A = \{z \in \mathbb{R}^2 \; ; \; r < |z| < R\}$. Set $u(z) = \dfrac{z}{R+r} + \dfrac{rR}{(R+r)\overline{z}}$. It is easy to check that $u(z) = \dfrac{z}{|z|}$ on ∂A, so that $u \in \mathcal{J}$. On the other hand, it is straightforward that $E_0(u) = 2\pi \dfrac{R-r}{R+r} < 2\pi$; thus $m_0 < 2\pi$.

Consider now a general A. Recall that there is a conformal representation Φ of A into some circular annulus C; moreover, Φ extends to a C^1-diffeomorphism of $\overline{A}$ into $\overline{C}$ and we may choose Φ in order to preserve the orientation of curves, see [**?**]. Let $F : H^1(C) \to H^1(A)$, $F(u) = u \circ \Phi$. If $\mathcal{J}(A)$ and $\mathcal{J}(C)$ stand for the corresponding classes of testing maps, we claim that F is a bijection of $\mathcal{J}(C)$ into $\mathcal{J}(A)$. Indeed, let Γ be a connected component of ∂A and let $\gamma = \Phi(\Gamma)$. Since Φ is orientation preserving, we have

$$\deg(g, \gamma) = \deg(g, \Gamma) \tag{2.8}$$

for $g \in C^\infty(\gamma; S^1)$. Using the density of $C^\infty(\gamma; S^1)$ into $H^{1/2}(\gamma; S^1)$ and the continuity of the map $g \mapsto g \circ \Phi$ from $H^{1/2}(\gamma; S^1)$ into $H^{1/2}(\Gamma; S^1)$, we find that (**??**) is still valid for $g \in H^{1/2}(\gamma; S^1)$. Thus F maps $\mathcal{J}(C)$ into $\mathcal{J}(A)$. Similarly, F^{-1} maps $\mathcal{J}(A)$ into $\mathcal{J}(C)$, which completes the proof of the claim.

Using the conformal invariance of the Dirichlet integral, we find that m_0 has the same value for A and for C. In view of our discussion on circular annuli, the proof of Lemma **??** is complete.

3 Proof of Theorem ??

Let, for $\lambda \geq 0$, u_λ be a minimizer of (**??**)-(**??**). We start by noting that (u_λ) is bounded in $H^1(A)$. Indeed, the upper bound (**??**) implies that (∇u_λ) is bounded in $L^2(A)$. Thus, by a Poincaré type inequality, $(u_\lambda - a_\lambda)$ is bounded in $H^1(A)$, where $a_\lambda = \dfrac{1}{|\partial\Omega|}\displaystyle\int_{\partial\Omega} u_\lambda$. Since $|u_\lambda| = 1$ a.e. on $\partial\Omega$, a_λ is bounded, so that u_λ is bounded in $H^1(A)$.

Let $u_\infty \in H^1(A)$ be such that, up to some subsequence, $u_{\lambda_n} \rightharpoonup u_\infty$ in $H^1(A)$. In view of (**??**), we have $\int_A (1-|u_\lambda|^2)^2 \to 0$, and thus $u_\infty \in H^1(A; S^1)$.

In the subcritical case, we will identify u_∞ with the help of the Price lemma and of the following simple

Lemma 3. *Let* $u \in H^1(A; S^1)$. *Then* $\deg(u, \partial\Omega) = \deg(u, \partial\omega)$.

Proof of Lemma ??: Differentiating the equality $|u|^2 = 1$ a.e. we find that $u \cdot u_x = u \cdot u_y = 0$ a.e., so that Jac $u = 0$ a.e. On the other hand, an integration by parts used in conjunction with the degree formula (**??**) yields

$$0 = \int_A \text{Jac } u = \frac{1}{2}\int_{\partial A} u \wedge \frac{\partial u}{\partial \tau} = \pi(\deg(u, \partial\Omega) - \deg(u, \partial\omega)). \tag{3.1}$$

For the convenience of the reader, we split the remaining part of the proof of Theorem **??** into 5 steps.

Step 1. Identification of u_∞ and strong $H^1(A)$ convergence in the subcritical case

By combinining the Price Lemma, Proposition **??**, Lemma **??** and the upper bound (**??**), we have, in the subcritical case $I_0 < 2\pi$,

$$2\pi > I_0 \geq \liminf_n m_{\lambda_n} \geq \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geq \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi|1 - \deg(u_\infty, \partial\Omega)|. \tag{3.2}$$

On the one hand, the above inequality implies that $\deg(u_\infty, \partial\Omega) = \deg(u_\infty, \partial\omega) = 1$, that is $u_\infty \in \mathcal{K}$. On the other hand, we have $I_0 \geq \dfrac{1}{2}\displaystyle\int_A |\nabla u_\infty|^2$. Recalling the definition of I_0, we find that u_∞ minimizes (**??**)-(**??**). Turning back to (**??**), we then obtain

$$I_0 \geq \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geq \frac{1}{2}\int_A |\nabla u_\infty|^2 = I_0, \tag{3.3}$$

which implies that $u_{\lambda_n} \to u_\infty$ in $H^1(A)$.

Step 2. An improved upper bound for m_λ

The following result is a slight improvement of the upper bound (**??**).

Lemma 4. *There are some $C>0$, $\lambda_0>0$ such that $m_\lambda \le I_0 - \dfrac{C}{\lambda}$ for $\lambda > \lambda_0$.*

Proof of Lemma ??: Let u minimize (**??**)-(**??**). Then $u \in C^\infty(\overline{A})$, see [?]. Let $f \in C_0^\infty(A;\mathbb{R})$ to be determined later. Set $v_\lambda = (1-f/\lambda)u$, which agrees with u on ∂A and thus belongs to $\mathcal{J}$. It is easy to see that, u being S^1-valued, we have $|\nabla v_\lambda|^2 = (1-f/\lambda)^2|\nabla u|^2 + |\nabla f|^2/\lambda^2$. Thus

$$m_\lambda \le E_\lambda(v_\lambda) = \frac{1}{2}\int_A |\nabla u|^2 - \frac{1}{\lambda}\int_A f(|\nabla u|^2 - f) + O\Big(\frac{1}{\lambda^2}\Big). \tag{3.4}$$

The conclusion of Lemma **??** follows from (**??**); it suffices to consider f such that $0 \le f \le |\nabla u|^2$ in A and $0 < f < |\nabla u|^2$ in some nonempty open subset of A.

Step 3. Candidates for u_∞ in the critical case

Lemma 5. *Assume A critical. Then either u_∞ minimizes (**??**)-(**??**), or u_∞ is a constant of modulus* 1.

Proof of Lemma ??: We rely on the Price Lemma, Lemma **??** and the upper bound (**??**). As in (**??**), we have

$$2\pi = I_0 \ge \liminf_n m_{\lambda_n} \ge \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi|1 - \deg(u_\infty, \partial\Omega)|. \tag{3.5}$$

If $\deg(u_\infty,\partial\Omega)$=$\deg(u_\infty,\partial\omega) = 1$, then, as in Step 1, we find that u_∞ minimizes (**??**)-(**??**). On the other hand, if $\deg(u_\infty,\partial\Omega)$=$\deg(u_\infty,\partial\omega) \ne 1$, then (**??**) implies that u_∞ must be a constant. Since $|u_\infty| = 1$ a.e. on ∂A, this constant is of modulus 1.

Step 4. Identification of u_∞ and strong $H^1(A)$ convergence in the critical case

We rely on the following

Lemma 6. [?] *Let (v_λ) be a family of solutions of the GL equation $-\Delta v_\lambda = \lambda v_\lambda(1-|v_\lambda|^2)$ in A. Assume that $|v_\lambda| \le 1$ and $E_\lambda(v_\lambda) \le C$. Then (v_λ) is bounded in $C^\infty_{\rm loc}(A)$. In addition, the following pointwise estimates hold:*

$$1 - |v_\lambda(z)|^2 \le \frac{D}{\lambda d^2(z)}, \qquad z \in A \tag{3.6}$$

and

$$|D^k v_\lambda(z)| \le \frac{D_k}{d^k(z)}, \qquad z \in A,\, k \in \mathbb{N}; \tag{3.7}$$

here, $d(z) = \mathrm{dist}(z, \partial A)$ *and the constants* D, D_k *depend only on* C.

In order to identify u_∞, we rule out the possibility that u_∞ is a constant. We argue by contradiction. Let Γ be a simple curve in A enclosing $\partial\omega$. Let U be the domain enclosed by $\partial\Omega$ and Γ and set $V = A \setminus \overline{U}$. Integrating, in U, the pointwise inequality $|\nabla u_\lambda|^2 \ge 2\mathrm{Jac}\ u_\lambda$, we find, with the help of the degree formula (**??**), that

$$\frac{1}{2}\int_U |\nabla u_\lambda|^2 \ge \pi - \frac{1}{2}\int_\Gamma u_\lambda \wedge \frac{\partial u_\lambda}{\partial \tau}; \tag{3.8}$$

here, Γ is counterclockwise oriented. Similarly, the use of the inequality $|\nabla u_\lambda|^2 \ge -2\mathrm{Jac}\ u_\lambda$ yields

$$\frac{1}{2}\int_V |\nabla u_\lambda|^2 \ge \pi - \frac{1}{2}\int_\Gamma u_\lambda \wedge \frac{\partial u_\lambda}{\partial \tau}, \tag{3.9}$$

and thus

$$m_\lambda \ge \frac{1}{2}\int_A |\nabla u_\lambda|^2 \ge \pi - \int_\Gamma u_\lambda \wedge \frac{\partial u_\lambda}{\partial \tau}. \tag{3.10}$$

We next note that the u_λ's satisfy the assumption of the Lemma **??**. Indeed, any minimizer of (**??**)-(**??**) satisfies the GL equation. Since $|u_\lambda| = 1$ a.e. on ∂A, we have $|u_\lambda| \le 1$ in A, by the maximum principle, see [**?**]. Finally, we have $E_\lambda(u_\lambda) \le 2\pi$ for each λ.

Since u_∞ is a constant, for large λ we have, in view of Lemma **??**, $1/2 \le |u_\lambda| \le 1$ on Γ and $\deg(u_\lambda, \Gamma) = 0$. We may thus write, for large λ, $u_\lambda = \rho_\lambda e^{\imath\varphi_\lambda}$ on Γ, where $1/2 \le \rho_\lambda \le 1$ and φ_λ is **single-valued**. Therefore, we have

$$\int_\Gamma u_\lambda \wedge \frac{\partial u_\lambda}{\partial \tau} = \int_\Gamma \rho_\lambda^2 \frac{\partial \varphi_\lambda}{\partial \tau} = \int_\Gamma (\rho_\lambda^2 - 1)\frac{\partial \varphi_\lambda}{\partial \tau}. \tag{3.11}$$

On the other hand, Lemma **??** and the assumption that u_∞ is a constant imply that $\nabla\varphi_\lambda \to 0$ uniformly on Γ as $\lambda \to \infty$. Formula (**??**) and estimate (**??**) used in conjunction with the fact that $\nabla\varphi_\lambda \to 0$ uniformly on Γ yield

$$\int_\Gamma u_\lambda \wedge \frac{\partial u_\lambda}{\partial \tau} = o\left(\frac{1}{\lambda}\right), \tag{3.12}$$

which in turn implies, with the help of (**??**), that

$$m_\lambda \ge 2\pi - o\left(\frac{1}{\lambda}\right) \qquad \text{as } \lambda \to \infty. \tag{3.13}$$

Inequality (**??**) contradicts, for large λ, the conclusion of Lemma **??**. In conclusion, u_∞ is not a constant. In view of Step 3, u_∞ minimizes (**??**)-(**??**). As in Step 1, this implies the strong H^1 convergence $u_{\lambda_n} \to u_\infty$.

Step 5. $|u_\lambda| \to 1$ uniformly in $\overline{A}$ as $\lambda \to \infty$

As we have already noted, the family (u_λ) is bounded in $H^1(A)$. Moreover, if $u_{\lambda_n} \to u_\infty$ weakly in H^1, we know, from Step 1 and Step 4, that $u_{\lambda_n} \to u_\infty$ strongly in H^1, and that u_∞ minimizes (**??**)-(**??**). It is easy to see that it suffices to prove that, for such a sequence (u_{λ_n}), we have $|u_{\lambda_n}| \to 1$ uniformly in $\overline{A}$ as $n \to \infty$.

Fix some $a \in (0,1)$. We have to establish the inequality

$$|u_{\lambda_n}(z)| \geq a \qquad \text{in } A \text{ for large } n. \tag{3.14}$$

We recall the following

Lemma 7. [?] *Let $g_n, g \in$VMO$(\partial A; S^1)$ be such that $g_n \to g$ in* VMO. *Let $\widetilde{g_n}$, $\widetilde{g}$ be the corresponding harmonic extensions to A. Then, for each $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ (independent of n) such that*

$$|\widetilde{g_n}(z)| \geq 1 - \varepsilon \qquad \text{if } d(z) \leq \delta. \tag{3.15}$$

Lemma 8. [?] *Let $v \in H_0^1(A)$ be such that $\Delta v \in L^\infty$. Then, for some C depending only on A, we have*

$$\|\nabla v\|_{L^\infty} \leq C \|v\|_{L^\infty}^{1/2} \|\Delta v\|_{L^\infty}^{1/2}. \tag{3.16}$$

Set $g_n = u_{\lambda_n|\partial A}$, $g = u_{\infty|\partial A}$. Since $H^{1/2}(\partial A) \subset$VMO$(\partial A)$ and $u_{\lambda_n} \to u_\infty$ in $H^1(A)$, we find that $g_n \to g$ in VMO. We split $u_{\lambda_n} = \widetilde{g_n} + v_{\lambda_n}$, where $v_{\lambda_n} \in H_0^1(A)$ is the solution of $-\Delta v_{\lambda_n} = \lambda u_{\lambda_n}(1 - |u_{\lambda_n}|^2)$. We note that

$$|v_{\lambda_n}| \leq |\widetilde{g_n}| + |u_{\lambda_n}| \leq 2; \tag{3.17}$$

here we rely on the inequality $|u_{\lambda_n}| \leq 1$ and on the fact that, $\widetilde{g_n}$ being the harmonic extension of a map of modulus 1, has modulus lesser or equal to 1. Using Lemma **??** in conjunction with (**??**), we find that

$$|\nabla v_{\lambda_n}| \leq C\sqrt{2\lambda_n}. \tag{3.18}$$

Since $v_{\lambda_n} = 0$ on ∂A, we obtain that

$$|v_{\lambda_n}(z)| \leq C_1 \sqrt{\lambda_n} d(z) \tag{3.19}$$

for some C_1 independent of n. By combining (**??**) with Lemma **??** it follows that that, for some $C_2 = C_2(a)$ and $n_0 = n_0(a)$, we have

$$|u_{\lambda_n}(z)| \geq a \qquad \text{if } d(z) \leq \frac{C_2}{\sqrt{\lambda_n}} \text{ and } n \geq n_0. \tag{3.20}$$

Returning to the proof of (**??**), we proceed as in [**?**]. We argue by contradiction: we assume that, possibly after passing to a further subsequence, there are points $z_n \in A$ such that $|u_{\lambda_n}(z_n)| \leq a$. In view of (**??**), we have

$$d(z_n) \geq \frac{C_2}{\sqrt{\lambda_n}} \qquad \text{for large } n. \tag{3.21}$$

Let $C_3 \in (0, C_2)$ to be determined later. By (**??**), we have $|\nabla u_{\lambda_n}(z)| \leq \dfrac{C_4}{\sqrt{\lambda_n}}$ if $|z - z_n| \leq \dfrac{C_3}{\sqrt{\lambda_n}}$. Since $|u_{\lambda_n}(z_n)| \leq a$, we thus have

$$|u_{\lambda_n}(z)| \leq \frac{1+a}{2} \qquad \text{if } |z - z_n| \leq \frac{C_3}{\sqrt{\lambda_n}} \text{ and } n \text{ is large,} \tag{3.22}$$

provided we choose C_3 sufficiently small. For such a C_3 and for sufficiently large n, we find that

$$\lambda_n \int_A (1 - |u_{\lambda_n}|^2)^2 \geq \lambda_n \int_{\{z;\, |z - z_n| \leq C_3/\sqrt{\lambda_n}\}} (1 - |u_{\lambda_n}|^2)^2 \geq C_4; \tag{3.23}$$

here, C_4 is independent of n.

On the other hand, the upper bound (**??**), the strong H^1 convergence $u_{\lambda_n} \to u_\infty$ together with the fact that u_∞ minimizes (**??**)-(**??**) yield

$$I_0 \geq \lim_n \left(\frac{1}{2} \int_A |\nabla u_{\lambda_n}|^2 + \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda_n}|^2)^2 \right) = I_0 + \lim_n \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda_n}|^2)^2. \tag{3.24}$$

Thus we must have

$$\lim_n \frac{\lambda_n}{4} \int_A (1 - |u_{\lambda_n}|^2)^2 = 0. \tag{3.25}$$

For large n, (**??**) and (**??**) contradict each other. Therefore, (**??**) holds. The proof of Theorem 2 is complete.

4 Rise of vortices

Throughout this section, we consider a supercritical domain A. Assume first that A obeys case a) in Theorem **??**. As noted at the beginning of the proof of Theorem **??**, the family (u_λ) is bounded in $H^1(A)$, and thus, up to some subsequence, $u_{\lambda_n} \rightharpoonup u_\infty$; moreover, $u_\infty \in H^1(A; S^1)$. Assume next that A obeys case b). If we consider, for a fixed $\lambda > \lambda_1$, a minimizing sequence (u_k), then the argument employed for the family (u_λ) shows that (u_k) is bounded in $H^1(A)$, and thus, up to some subsequence, $u_{k_n} \rightharpoonup u_\infty$; here, $u_\infty \in H^1(A; \mathbb{C})$. We start by identifying u_∞.

Lemma 9. *In both cases a) and b), u_∞ is a constant of modulus* 1.

Proof of Lemma ??: Assume first case a). By combining the Price Lemma, the upper bound (**??**) and Lemma **??**, we find that

$$2\pi \geq \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geq \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi|1-\deg(u_\infty,\partial\Omega)|. \tag{4.1}$$

If $\deg(u_\infty,\partial\Omega)\neq 1$, then u_∞ has to be a constant; this constant is of modulus 1, since $|u_\infty|=1$ a.e. on ∂A. If $\deg(u_\infty,\partial\Omega)=1$, then $u_\infty\in\mathcal{K}$, and thus (**??**) yields

$$2\pi \geq \frac{1}{2}\int_A |\nabla u_\infty|^2 \geq I_0; \tag{4.2}$$

this is impossible, since we are in the supercritical case. Thus u_∞ is a constant of modulus 1.

Assume next case b); the proof of Theorem **??** shows that $m_\lambda = 2\pi$ for $\lambda>\lambda_1$. The Price Lemma implies that

$$2\pi = m_\lambda = \lim_n E_\lambda(u_{k_n}) \geq E_\lambda(u_\infty) + \pi(|1-\deg(u_\infty,\partial\Omega)| + |1-\deg(u_\infty,\partial\omega)|). \tag{4.3}$$

If $\deg(u_\infty,\partial\Omega)=\deg(u_\infty,\partial\omega)=1$, then $u_\infty\in\mathcal{J}$ and thus, by (**??**), u_∞ minimizes (**??**)-(**??**); this is impossible, since m_λ is not attained for $\lambda>\lambda_1$. If $\deg(u_\infty,\partial\Omega)\neq 1$ and $\deg(u_\infty,\partial\omega)\neq 1$, then u_∞ has to be a constant (of modulus 1). Finally, if exactly one among $\deg(u_\infty,\partial\Omega)$ and $\deg(u_\infty,\partial\omega)$ equals 1, then (**??**) combined with (**??**) yields

$$2\pi \geq 2\pi + \frac{\lambda}{4}\int_A (1-|u_\infty|^2)^2; \tag{4.4}$$

therefore, u_∞ is a constant of modulus 1, which is in contradiction with the degrees assumption on u_∞. In conclusion, u_∞ is a constant of modulus 1.

As a byproduct of the above lemma, it is easy to establish Proposition **??**.

Proof of Proposition ??: Since m_λ is not decreasing, for each sequence $\lambda_n\to\infty$ we have $\lim\limits_{\lambda\to\infty} m_\lambda = \lim\limits_n m_{\lambda_n}$.

Assume first A subcritical/critical. Consider a sequence (λ_n) such that $u_{\lambda_n}\to u_\infty$ strongly in $H^1(A)$, where u_∞ minimizes (**??**)-(**??**). By combining the upper bound (**??**) with the definition of I_0, we find that

$$I_0 \geq \lim_{\lambda\to\infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geq \frac{1}{2}\int_A |\nabla u_\infty|^2 = I_0. \tag{4.5}$$

Thus $\lim_{\lambda\to\infty} m_\lambda = I_0$, which is the desired conclusion.

Assume next A supercritical. If A obeys case b), then $m_\lambda = 2\pi$ for large λ, and (**??**) follows. If A obeys case a), consider a sequence (λ_n) such that $u_{\lambda_n} \rightharpoonup u_\infty$ weakly in $H^1(A)$, where u_∞ is a constant of modulus 1. Using the Price lemma and the upper bound (**??**), we obtain

$$2\pi \geq \lim_{\lambda\to\infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geq 2\pi, \tag{4.6}$$

which yields $\lim_{\lambda\to\infty} m_\lambda = 2\pi$, as stated.

Proof of Theorem ?? in case b): We consider, for $\lambda > \lambda_1$, a minimizing sequence (u_k), whose behavior we will describe below. For the convenience of the reader, we divide the proof into 6 steps.

Step 1. Splitting u_k

Let v_k minimize the GL energy E_λ among all the maps $v \in H^1(A)$ such that $v = u_k$ on ∂A. Clearly, (i) v_k satisfies the GL equation $-\Delta v_k = \lambda v_k(1 - |v_k|^2)$, (ii) $|v_k| \leq 1$ (by the maximum principle), (iii) $v_k \in \mathcal{J}$, and (iv) the sequence (v_k) is still a minimizing sequence (since $E_\lambda(v_k) \leq E_\lambda(u_k)$). Set $w_k = u_k - v_k \in H_0^1(A)$.

Lemma 10. *We have $w_k \to 0$ in $H^1(A)$ as $k \to \infty$.*

Proof of Lemma ??: In view of Lemma **??**, we may assume that, up to some subsequence, $u_{kn} \rightharpoonup u$ and $v_{kn} \rightharpoonup v$ weakly in $H^1(A)$, where u, v are constants of modulus 1. Since $u_k = v_k$ on ∂A, we find that $u = v$; in particular, $w_{kn} \rightharpoonup 0$. It is easy to see that, in fact, the stronger property $w_k \rightharpoonup 0$ holds. Inserting the equality $u_k = v_k + w_k$ into the formula of $E_\lambda(u_k)$ and using the fact that $w_k \rightharpoonup 0$, we find that

$$E_\lambda(u_k) = E_\lambda(v_k) + \frac{1}{2}\int_A |\nabla w_k|^2 + \int_A \nabla v_k \cdot \nabla w_k + o(1). \tag{4.7}$$

Since both (u_k) and (v_k) are minimizing sequences, we obtain

$$\frac{1}{2}\int_A |\nabla w_k|^2 + \int_A \nabla v_k \cdot \nabla w_k \to 0 \qquad \text{as } k \to \infty. \tag{4.8}$$

On the other hand, if we multiply by w_k the GL equation satisfied by v_k and integrate, we find that

$$\left|\int_A \nabla v_k \cdot \nabla w_k\right| = \left|\int_A \lambda v_k \cdot w_k(1 - |v_k|^2)\right| \leq \lambda \int_A |w_k| \to 0 \qquad \text{as } k \to \infty. \tag{4.9}$$

(**??**) used in conjunction with (**??**) yields $\lim_k \int_A |\nabla w_k|^2 = 0$; since $w_k = 0$ on ∂A, we find that $w_k \to 0$ in $H^1(A)$, as stated.

In conclusion, modulo a small reminder in $H^1(A)$, we may replace a minimizing sequence (u_k) by another one, (v_k), having the additional properties (i) and (ii). In the remaining part of the proof, we will examine the behavior of the sequence (v_k).

Step 2. Concentration of the energy near ∂A

We fix two simple curves in A, γ and Γ, such that γ encloses $\partial\omega$ and Γ encloses γ. Let U be the domain enclosed by $\partial\Omega$ and Γ, V be the domain enclosed by γ and $\partial\omega$ and set $W = A \setminus (\overline{U} \cup \overline{V})$.

Lemma 11. *We have, as $k \to \infty$,*

$$\int_A (1 - |v_k|^2)^2 \to 0, \tag{4.10}$$

$$\|\nabla v_k\|_{L^\infty(W)} \to 0, \tag{4.11}$$

$$\|\partial_{\overline{z}} v_k\|_{L^2(U)} \to 0 \qquad \text{and} \qquad \|\partial_z v_k\|_{L^2(V)} \to 0 \tag{4.12}$$

$$\frac{1}{2}\int_U |\nabla v_k|^2 \to \pi \qquad \text{and} \qquad \int_U \text{Jac } v_k \to \pi, \tag{4.13}$$

$$\frac{1}{2}\int_V |\nabla v_k|^2 \to \pi \qquad \text{and} \qquad \int_V \text{Jac } v_k \to -\pi. \tag{4.14}$$

Proof of Lemma ??: We integrate over U (V, respectively) the identity $\frac{1}{2}|\nabla v_k|^2 = \text{Jac } v_k + 2|\partial_{\overline{z}} v_k|^2$ ($\frac{1}{2}|\nabla v_k|^2 = -\text{Jac } v_k + 2|\partial_z v_k|^2$, respectively). We find that

$$E_\lambda(v_k) = \int_U \text{Jac } v_k - \int_V \text{Jac } v_k + 2\int_U |\partial_{\overline{z}} v_k|^2 + 2\int_V |\partial_z v_k|^2 + \frac{1}{2}\int_W |\nabla v_k|^2 + \frac{\lambda}{4}\int_A (1 - |v_k|^2)^2. \tag{4.15}$$

An integration by parts combined with the degree formula (**??**) yields, for the counterclockwise orientation on γ and Γ,

$$\int_U \text{Jac } v_k = \pi - \frac{1}{2}\int_\Gamma v_k \wedge \frac{\partial v_k}{\partial \tau} \qquad \text{and} \qquad -\int_U \text{Jac } v_k = \pi - \frac{1}{2}\int_\gamma v_k \wedge \frac{\partial v_k}{\partial \tau}. \tag{4.16}$$

We claim that, as $k \to \infty$,

$$\nabla v_k \to 0 \quad \text{in } C^0_{\text{loc}}(A); \tag{4.17}$$

clearly, the conclusions of Lemma **??** follow by combining (**??**)-(**??**) with the inequality $|v_k| \leq 1$ and the fact that $\lim_k E_\lambda(v_k) = 2\pi$.

It remains to establish (**??**). Since $|v_k| \leq 1$, we find that $|\Delta v_k| \leq \lambda$. The sequence (v_k) being bounded in H^1, it follows, from standard elliptic estimates [**?**], that (v_k) is bounded in $W^{2,p}_{\text{loc}}(A)$, $1 < p < \infty$, and thus relatively compact in $C^1_{\text{loc}}(A)$, via the Sobolev embeddings. In view of Lemma **??**, each subsequence of (v_k) contains a further subsequence converging weakly in H^1 to a constant of modulus 1; it is easy to see that this property, combined with the fact that (v_k) is relatively compact in $C^1_{\text{loc}}(A)$, implies (**??**). For further use, we note that the same argument implies that $|v_k| \to 1$ in $C^1_{\text{loc}}(A)$.

Step 3. Existence of zeroes

Lemma 12. *There is some k_0 such that, for $k \geq k_0$, v_k has at least a zero ζ_k in U, at least a zero ξ_k in V and no zeroes in $\overline{W}$. In addition, for any zero ${\zeta_k}'$ in U (${\xi_k}'$ in V, respectively) we have* $\text{dist}({\zeta_k}', \partial\Omega) \to 0$ *as $k \to \infty$ ($\text{dist}(\xi_k', \partial\omega) \to 0$ as $k \to \infty$, respectively).*

Proof of Lemma ??: Non-existence of zeroes in $\overline{W}$ for large λ and the last assertion follow from the fact that $|v_k| \to 1$ in $C^1_{\text{loc}}(A)$. It remains to establish existence of zeroes in U and in V for large λ. We argue by contradiction and assume, e.g., that, possibly up to some subsequence, $v_k \neq 0$ in U. We claim that, for a fixed k, there is some $C = C_k > 0$ such that $C \leq |v_k| \leq 1$ in U. Indeed, Lemma **??** applied to $g = v_{k|\partial A}$, $g_n \equiv g$, implies that there is some $\delta_1 > 0$ such that $\tilde{g}(z) \geq 3/4$ if $d(z) < \delta_1$. On the other hand, if we set $w_k = v_k - \tilde{g}(z) \in H^1_0(A)$, then $\Delta w_k \in L^\infty(A)$ and thus $w_k \in C^1_0(\overline{A})$. Therefore, there is some $\delta_2 > 0$ such that $|w_k(z)| \leq 1/4$ if $d(z) < \delta_2$. We find that $|v_k(z)| \geq 1/2$ if $d(z) <\text{Min}(\delta_1, \delta_2)$; v_k being smooth in A as a solution of GL and non vanishing in $\overline{U}$ according to our hypothesis, this implies the existence of C, as claimed.

Set $y_k = v_k/|v_k|$; this map belongs to $H^1(U; S^1)$, since $C \leq |v_k| \leq 1$ in U. Lemma **??** yields $\deg(y_k, \Gamma) = \deg(y_k, \partial\Omega) = 1$; the last inequality follows from the fact that $y_k = v_k$ on $\partial\Omega$. Thus $\deg(v_k, \Gamma) = \deg(y_k, \Gamma) = 1$. This is impossible since, up to a subsequence, $v_k \to v$ in $C^1(\Gamma)$, where v is a constant of modulus 1. The proof of Lemma **??** is complete.

Step 4. Rescaling v_k

We recall that $\nabla v_k \to 0$ and $|v_k| \to 1$ in $C^1(\Gamma)$; therefore, we may extend $v_{k|U}$ to Ω such that the extension v_k^1 satisfies $\|\nabla v_k^1\|_{L^\infty(\Omega \setminus U)} \to 0$ and $1/2 \leq |v_k^1| \leq 1$ in $\Omega \setminus U$ for large k. Similarly, $v_{k|V}$ has an extension v_k^2 to $\mathbb{C} \setminus \overline{\omega}$ satisfying $\|\nabla v_k^2\|_{L^\infty(\mathbb{C} \setminus V)} \to 0$ and $1/2 \leq |v_k^2| \leq 1$ in $\mathbb{C} \setminus V$ for large k.

Let Φ be a fixed conformal representation of Ω into $\mathbb{D}$. It is well-known that all the conformal representations Φ_k of Ω into $\mathbb{D}$ satisfying the property $\Phi_k(\zeta_k) = 0$ are given by $\Phi_k(z) = \alpha \dfrac{\Phi(z) - \Phi(\zeta_k)}{1 - \overline{\Phi(\zeta_k)}\Phi(z)}$, where $\alpha \in S^1$. Set $y_k = v_k^1 \circ \Phi_k^{-1}$. By construction, y_k maps $\mathbb{D}$ into $\mathbb{D}$ and vanishes at the origin; moreover, the trace of y_k on S^1 has modulus 1 and degree 1 (since Φ_k preserves the orientation of curves). It is easy to see that, for an appropriate choice of α, we may assume that $\partial_z y_k(0) \geq 0$. Similarly, we may construct a conformal representation Ψ_k of $\mathbb{C} \setminus \overline{\omega}$ onto $\mathbb{D}$ vanishing at ξ_k and such that $z_k = \overline{v_k^2 \circ \Psi_k^{-1}}$ has the same properties as y_k.

In the remaining part of the proof, we study the asymptotic properties of y_k and z_k and relate these properties to the asymptotic behavior of v_k. The reason we prefer to deal with y_k, z_k instead of v_k is strong H^1 convergence: as we have already seen, up to a subsequence, $g_{k_n} \rightharpoonup v$, where v is some constant of modulus 1; in particular, (g_{k_n}) is not strongly convergent in H^1, since the degree constraints are lost in the limit. However, we will establish below that y_k and z_k do strongly converge in H^1. We focus ourselves on the behavior of y_k; the analysis is the same for z_k.

To start with, we collect some elementary properties of the Φ_k's.

Lemma 13. [?] *For each $r \in (0,1)$, there are constants $C_j = C_j(r)$ independent of k such that:*
i) $\Phi_k^{-1}(\mathbb{D}_r) \subset \{z \in \Omega\ ;\ |z - \zeta_k| \leq C_1 d(\zeta_k, \partial\Omega) \text{ and } d(z, \partial\Omega) \geq C_2 d(\zeta_k, \partial\Omega)\}$;
ii) $|\nabla \Phi_k^{-1}| \leq C_3 d(\zeta_k, \partial\Omega)$ *in* $\mathbb{D}_r$.
For each R_1, $R_2 > 0$, there is some $r \in (0,1)$ independent of k such that
iii) $\Phi_k(\{z \in \Omega\ ;\ |z - \zeta_k| \leq R_1 d(\zeta_k, \partial\Omega) \text{ and } d(z, \partial\Omega) \geq R_2 d(\zeta_k, \partial\Omega)\}) \subset \mathbb{D}_r$.

Lemma 14. *We have $y_k \to \mathrm{id}$ and $z_k \to \mathrm{id}$ strongly in $H^1(\mathbb{D})$ and in $C^1_{\mathrm{loc}}(\mathbb{D})$.*

Proof of Lemma ??: Since the Dirichlet integral is conformally invariant, we have

$$\int_{\mathbb{D}} |\nabla y_k|^2 = \int_{\Omega} |\nabla v_k^1|^2 = \int_{U} |\nabla v_k|^2 + \int_{\Omega \setminus U} |\nabla v_k^1|^2 = 2\pi + o(1) \qquad \text{as } k \to \infty; \tag{4.18}$$

here, we use Lemma **??**. Similarly, we have

$$\int_{\mathbb{D}} (|\nabla y_k|^2 - 2\mathrm{Jac}\ y_k) = o(1) \qquad \text{as } k \to \infty. \tag{4.19}$$

The fact that $|y_k| \leq 1$ combined with (**??**) implies that (y_k) is bounded in $H^1(\mathbb{D})$. Let $y \in H^1(\mathbb{D})$ be such that, up to some subsequence, $y_{k_n} \rightharpoonup y$; thus $|y| = 1$ a.e. on S^1.

The map $H^1(\mathbb{D}) \ni u \mapsto \int_{\mathbb{D}} (|\nabla u|^2 - 2\text{Jac } u)$ being convex and continuous (and thus weakly l.s.c.), (??) and the fact that $y_{k_n} \rightharpoonup y$ imply

$$\int_{\mathbb{D}} (|\nabla y|^2 - 2\text{Jac } y) = 4 \int_{\mathbb{D}} |\partial_{\overline{z}} y|^2 \leq 0. \tag{4.20}$$

Thus $\partial_{\overline{z}} y = 0$ a.e. in $\mathbb{D}$, i.e., y is holomorphic in $\mathbb{D}$. Set $g = y_{|S^1} \in H^{1/2}(S^1; S^1)$, whose Fourier expansion is of the form $g = \sum_{l=0}^{\infty} a_l e^{\imath l\theta}$. Then $\deg g = \sum_{l=0}^{\infty} l|a_l|^2$ (when g is smooth, this equality is equivalent to the degree formula (??); equality still holds for a general $g \in H^{1/2}(S^1; S^1)$, see [?]). On the other hand, y, being holomorphic, is the harmonic extension g, and thus

$$\int_{\mathbb{D}} |\nabla y|^2 = 2\pi \sum_{l=0}^{\infty} l|a_l|^2 = 2\pi \deg g \leq 2\pi; \tag{4.21}$$

the last inequality follows from (??). In conclusion, either $\deg g = 0$, in which case y is a constant of modulus 1, or $\deg g = 1$.

We first rule out the possibility that y is a constant. For large k, the set

$$M_k = \{z \ ; \ |z - \zeta_k| \leq C_1 d(\zeta_k, \partial\Omega) \ \text{ and } d(z, \partial\Omega) \geq C_2 d(\zeta_k, \partial\Omega)\}$$

is contained in U, and thus $|\Delta v_k^1| = \lambda |v_k(1 - |v_k|^2)| \leq \lambda$ in M_k. Using Lemma **??** *ii)*, we find that

$$|\Delta y_k| = \frac{1}{2} |\nabla \Phi_k^{-1}|^2 |(\Delta v_k^1) \circ \Phi_k^{-1}| \to 0 \qquad \text{uniformly in } \mathbb{D}_r \text{ as } k \to \infty. \tag{4.22}$$

Since y_k is bounded in H^1, it follows, from standard elliptic estimates, that y_k is relatively compact in $C^1_{\text{loc}}(\mathbb{D})$. In particular, $y_{k_n} \to y$ uniformly in $\mathbb{D}_{1/2}$. Recalling that $y_k(0) = 0$, we find that $y(0) = 0$, that is, y can not be a constant of modulus 1.

We next identify y. Lemma **??** applied to $g_n \equiv g$ implies that $|y(z)| \to 1$ uniformly as $|z| \to 1$. We recall that a holomorphic map y in $\mathbb{D}$ satisfying $|y(z)| \to 1$ uniformly as $|z| \to 1$ is a Blaschke product, i.e., $y(z) = \alpha \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}$ for some $\alpha \in S^1$ and $a_1, \ldots, a_d \in \mathbb{D}$; see [?]. Here, d is the degree of $y_{|S^1}$. In our case, $d = 1$ and $y(0) = 0$; thus $y = \alpha$ id with $\alpha \in S^1$. Since $\partial_z y_k(0) \geq 0$, we have $\alpha = \partial_z y(0) \geq 0$, and thus $\alpha = 1$; therefore, $y =$id.

The uniqueness of the weak limit implies that $y_k \rightharpoonup \text{id}$ in H^1. (**??**) combined with the fact that $\int_{\mathbb{D}} |\nabla \text{ id}|^2 = 2\pi$ yields $y_k \to \text{id}$ in H^1; the sequence (y_k) being relatively compact in in $C^1_{\text{loc}}(\mathbb{D})$, it follows that $y_k \to \text{id}$ in $C^1_{\text{loc}}(\mathbb{D})$.

Step 5. Holomorphic/anti-holomorphic behavior of v_k near $\partial\Omega/\partial\omega$

As an immediate consequence of Lemma **??**, we obtain the following

Lemma 15. *We have* $v_k - \Phi_k \to 0$ *in* $L^2_{\text{loc}}(\overline{A} \setminus \partial\omega)$ *and* $v_k - \overline{\Psi_k} \to 0$ *in* $L^2_{\text{loc}}(\overline{A} \setminus \partial\Omega)$.

Proof of Lemma ??: We prove, e.g., the first assertion. Fix a compact $K \subset \overline{A} \setminus \partial\omega$. The curves γ, Γ introduced in Step 2 being arbitrary, we have, thanks to Lemma **??**,

$$\int_{K \setminus U} |\nabla v_k|^2 \to 0 \qquad \text{as } k \to \infty. \tag{4.23}$$

On the other hand, Lemma **??** *i)* and the fact that $d(\zeta_k, \partial\Omega) \to 0$ imply that $\Phi_k(K \setminus U) \subset \mathbb{D} \setminus \mathbb{D}_{r_k}$ for some sequence $r_k \to 1$. The conformal invariance of the Dirichlet integral yields

$$\int_{K \setminus U} |\nabla \Phi_k|^2 = \int_{\Phi_k(K \setminus U)} |\nabla \text{ id}|^2 \leq \int_{\mathbb{D} \setminus \mathbb{D}_{r_k}} |\nabla \text{ id}|^2 \to 0 \qquad \text{as } k \to \infty. \tag{4.24}$$

Finally,

$$\int_{K \cap U} |\nabla \Phi_k - \nabla v_k|^2 \leq \int_{U} |\nabla \Phi_k - \nabla v_k|^2 = \int_{\Phi_k(U)} |\nabla \text{ id} - \nabla y_k|^2 \qquad \text{as } k \to \infty, \tag{4.25}$$

by Lemma **??** and conformal invariance. The conclusion of Lemma **??** follows by combining (**??**)-(**??**).

Step 6. Uniqueness of ζ_k, ξ_k for large k

We argue by contradiction and assume that, possibly up to some subsequence, v_k has two distinct zeroes in U, ζ_k and $\widetilde{\zeta_k}$. Without loss of generality, we may further assume that $d(\zeta_k, \partial\Omega) \geq d(\widetilde{\zeta_k}, \partial\Omega)$. Let Φ_k and $\widetilde{\Phi_k}$ be the corresponding conformal representations. We claim that, for each $r \in (0,1)$, we have $\Phi_k^{-1}(\mathbb{D}_r) \cap \widetilde{\Phi_k}^{-1}(\mathbb{D}_r) = \emptyset$ for large k. Indeed, if $z \in \Phi_k^{-1}(\mathbb{D}_r) \cap \widetilde{\Phi_k}^{-1}(\mathbb{D}_r)$, then, with C_1 as in Lemma **??**, we have

$$|z - \zeta_k| \leq C_1 d(\zeta_k, \partial\Omega), \qquad |z - \widetilde{\zeta_k}| \leq C_1 d(\widetilde{\zeta_k}, \partial\Omega), \tag{4.26}$$

by Lemma **??** *i)*, and therefore

$$d(\zeta_k, \partial\Omega) \geq d(\widetilde{\zeta_k}, \partial\Omega), \qquad |\widetilde{\zeta_k} - \zeta_k| \leq 2C_1 d(\zeta_k, \partial\Omega). \tag{4.27}$$

Lemma **??** *iii)* combined with (**??**) implies the existence of some fixed $\rho \in (0,1)$ such that $\Phi_k(\widetilde{\zeta_k}) \in \overline{\mathbb{D}_\rho}$ for each k. This is impossible for large k, since on the one hand $y_k = v_k \circ \Phi_k^{-1} \to$id in $C^1(\overline{\mathbb{D}_\rho})$ (and thus, for large k, $y_{k|\overline{\mathbb{D}_r}}$ is into), on the other hand $y_k(\Phi_k(\zeta_k)) = y_k(\Phi_k(\widetilde{\zeta_k})) = 0$ for each k. The claim is proved.

Fix now $r \in (1/\sqrt{2}, 1)$, so that $\int\limits_{\mathbb{D}_r} |\nabla \text{ id}|^2 = 2\pi r^2 > \pi$. With $\widetilde{y_k} = v_k \circ \widetilde{\Phi_k^{-1}}$, we have, as $k \to \infty$,

$$\frac{1}{2}\int\limits_U |\nabla v_k|^2 \geq \frac{1}{2}\int\limits_{\Phi_k^{-1}(\mathbb{D}_r)\cup\widetilde{\Phi_k}^{-1}(\mathbb{D}_r)} |\nabla v_k|^2 = \frac{1}{2}\int\limits_{\mathbb{D}_r} |\nabla y_k|^2 + \frac{1}{2}\int\limits_{\mathbb{D}_r} |\nabla \widetilde{y_k}|^2 \to 2\pi r^2, \tag{4.28}$$

by Lemma **??**. With our choice of r, (**??**) contradicts (**??**). This contradiction proves the uniqueness of ζ_k.

Proof of Theorem ?? in case a): Our purpose is to describe the behavior, as $\lambda \to \infty$, of a family (u_λ) of minimizers of (**??**)-(**??**). The proof follows essentially the same lines as the one in case a). We point out the changes to be made. Step 1 is not needed here, since the minimizers already satisfy the GL equation and the property $|u_\lambda| \leq 1$. The analogs of (**??**)-(**??**) in Step 2 are

$$\lambda \int\limits_A (1 - |u_\lambda|^2)^2 \to 0, \tag{4.29}$$

$$\|\nabla u_\lambda\|_{L^\infty(W)} \to 0, \tag{4.30}$$

$$\|\partial_{\overline{z}} u_\lambda\|_{L^2(U)} \to 0 \qquad \text{and} \qquad \|\partial_z u_\lambda\|_{L^2(V)} \to 0 \tag{4.31}$$

$$\frac{1}{2}\int\limits_U |\nabla u_\lambda|^2 \to \pi \qquad \text{and} \qquad \int\limits_U \text{Jac } u_\lambda \to \pi, \tag{4.32}$$

$$\frac{1}{2}\int\limits_V |\nabla u_\lambda|^2 \to \pi \qquad \text{and} \qquad \int\limits_V \text{Jac } u_\lambda \to -\pi. \tag{4.33}$$

However, while (**??**)-(**??**) were obtained via (**??**), one has to use in this case the estimate (**??**) (note that, although we established (**??**) in the critical case, it is still valid in our context, since it relies only on the assumption that the only possible weak H^1 limits of sequences (u_{λ_n}) are constants).

With the same proof as in Step 3, case b), we find that, for large λ, u_λ has a zero, ζ_λ, in U, respectively a zero, ξ_λ, in V. An additional information needed is given by the following

Lemma 16. *We have $\lambda^{1/2} d(\zeta_\lambda, \partial\Omega) \to 0$ and $\lambda^{1/2} d(\xi_\lambda, \partial\omega) \to 0$ as $\lambda \to \infty$.*

Proof of Lemma ??: We establish the first assertion. By (??), we have, with some constant C independent of large λ,

$$|\nabla u_\lambda(z)| \leq \frac{C}{d(\zeta_\lambda, \partial\Omega)} \qquad \text{if } |z - \zeta_\lambda| \leq \frac{1}{2} d(\zeta_\lambda, \partial\Omega), \tag{4.34}$$

and thus, with $c_\lambda = 1/2\text{Min}\{1, 1/C\} d(\zeta_\lambda, \partial\Omega)$, we have $\mathbb{D}_{c_\lambda}(\zeta_\lambda) \subset A$ and $|u_\lambda| \leq 1/2$ in $\mathbb{D}_{c_\lambda}(\zeta_\lambda)$. Therefore,

$$\lambda \int_A (1 - |u_\lambda|^2)^2 \geq \lambda \int_{\mathbb{D}_{c_\lambda}(\zeta_\lambda)} (1 - |u_\lambda|^2)^2 \geq \frac{9\pi c_\lambda^2}{16}. \tag{4.35}$$

The conclusion of Lemma **??** follows by combining (**??**) with (**??**).

We next consider the rescaled maps $y_\lambda = u_\lambda \circ \Phi_\lambda^{-1}$, respectively $z_\lambda = u_\lambda \circ \overline{\Psi_\lambda^{-1}}$, where Φ_λ, Ψ_λ are suitable conformal representations vanishing at ζ_λ, respectively ξ_λ. Step 4 works with the same proof except when establishing the analog of (**??**), which is

$$|\Delta y_\lambda| \to 0 \qquad \text{in } C^0_{\text{loc}}(\mathbb{D}). \tag{4.36}$$

The argument that leads to (**??**) is the following: let $r \in (0, 1)$. By combining Lemma **??** *i), ii)* with Lemma **??**, we have, for large λ,

$$\|\Delta y_\lambda\|_{L^\infty(\mathbb{D}_r)} = \frac{1}{2} \| |\nabla \Phi_\lambda^{-1}|^2 |(\Delta u_\lambda) \circ \Phi_k^{-1}\|_{L^\infty(\Phi_\lambda^{-1}(\mathbb{D}_r))} \leq C_3 \lambda d(\zeta_\lambda, \partial\Omega) \to 0 \qquad \text{as } \lambda \to \infty. \tag{4.37}$$

Finally, Steps 5 and 6 are the same, and no changes are needed in the proof.

References

[1] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1966.

[2] Th. Aubin, *Equations différentieles nonlinéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. , 55(1976), 269-293.

[3] L. Berlyand and P. Mironescu, *Ginzburg-Landau minimizers with prescribed degrees : dependence on domain*, C. Rendus Acad. Sci. Paris, 337 (2003), 375-380.

[4] L. Berlyand and P. Mironescu, *Ginzburg-Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices*, preprint.

[5] L. Berlyand and K. Voss, *Symmetry breaking in annular domains for a Ginzburg-Landau superconductivity model*, Proceedings of IUTAM 99/4 Symposium (Sydney, Australia), Kluwer Academic Publishers, 1999.

[6] F. Bethuel, H. Brezis, B. D. Coleman and F. Hélein, *Bifurcation analysis of minimizing harmonic maps describing the equilibrium of nematic phases between cylinders*, Arch. Rational Mech. Anal., 118 (1992), 149-168.

[7] F. Bethuel, H. Brezis, F. Hélein, *Asymptotics for the minimization of a Ginzburg-Landau functional*, Calc. Var., 1 (1993), 123-148.

[8] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau Vortices*, Birkhäuser, 1997.

[9] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys., 142 (1991), 1-23.

[10] H. Brezis,*Metastable harmonic maps*, in *Metastability and Incompletely Posed Problems,* , S. S. Antman, J. L. Ericksen, D. Kinderlehrer, I. Mller, (eds.) 33-42, Springer-Verlag, 1987.

[11] Brezis, H. : *Degree theory : old and new* in Topological Nonlinear Analysis, II (Frascati, 1995), Prog. Nonlinear Differential Equations Appl., vol. 27. Birkhäuser, Boston, MA, 1997, pp. 87-108.

[12] H. Brezis, *Vorticité de Ginzburg-Landau*, graduate course, Université Paris 6, 2001-2002.

[13] H. Brezis and J.-M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Comm. Pure Appl. Math. 37 (1984), 149-187.

[14] H. Brezis and J.-M. Coron, *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. 92 (1983), 203-215.

[15] H. Brezis, M. Marcus and I. Shafrir, *Extremal functions for Hardy's inequality with weight*, J. Funct. Anal. 171 (2000), 177-191.

[16] H. Brezis and L. Nirenberg, *Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents*, Comm. Pure Appl. Math., 36 (1983), 437-477.

[17] H. Brezis and L. Nirenberg, *Degree Theory and BMO*, Part I: *Compact manifolds without boundaries*, Selecta Math., 1 (1995), 197-263 ; Part II: *Compact manifolds with boundaries*, Selecta Math., 2 (1996), 309-368.

[18] R. Burckel, *An introduction to classical complex analysis. Vol.1.* Pure and Applied Mathematics, **82**, Academic Press, New York 1979.

[19] R. J. Donnelly and A. L. Fetter, *Stability of superfluid flow in an annulus*, Phys. Rev. Lett., 17 (1966), 747-750.

[20] O. Druet, *Elliptic equations with critical Sobolev exponent in dimension 3*, Ann. I.H.P., Analyse non-linéaire, 19, (2002), 125-142.

[21] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1993.

[22] D. Golovaty and L. Berlyand, *On uniqueness of vector-valued minimizers of the Ginzburg-Landau functional in annular domains*, Calc. Var., 14 (2002), 213-232.

[23] P. Mironescu, *Explicit bounds for solutions to a Ginzburg- Landau type equation*, Rev. Roumaine Math Pures Appl. 41 (1996), 263-271.

[24] P. Mironescu, A. Pisante, *A variational problem with lack of compactness for* $H^{1/2}(S^1; S^1)$ *maps of prescribed degree*, to appear.

Leonid Berlyand
Department of Mathematics, The Pennsylvania State University
University Park PA 16802 , USA
berlyand@math.psu.edu

Petru Mironescu
Laboratoire d'Analyse Numérique et EDP, Université Paris-Sud 11
Bâtiment 425, 91405 Orsay Cedex, France
Petru.Mironescu@math.u-psud.fr